5. BUTLER MODULES

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Consider again Example 2.28. As described at the beginning of Chapter 4, we can think of the construction there as beginning with three incomparable types $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$ satisfying the compatibility condition that $\mathbf{t}_i \wedge \mathbf{t}_j$ is the same for all pairs $i \neq j$. We then construct a rank-two module G such that the three submodules $G(\mathbf{t}_i)$ are distinct and $G = G(\mathbf{t}_1) + G(\mathbf{t}_2) + G(\mathbf{t}_3)$. We then observed that if $\varphi \in \operatorname{End} G$ then φ must leave these three submodules invariant, and we concluded that $\operatorname{QEnd} G \approx Q$.

In Example 1.49 similar reasoning was used. Using two incomparable types $\mathbf{t}_1$ and $\mathbf{t}_2$ we constructed a module rank-two module H such that $H(\mathbf{t}_1) \oplus H(\mathbf{t}_2)$ is a proper essential submodule of H and is quasi-equal to H. Using the fact that $H(\mathbf{t}_1)$ and $H(\mathbf{t}_2)$ must be invariant under any $\varphi \in \operatorname{End} H$ we were then able to calculate $\operatorname{End} H$ (see Example 2.21.) Furthermore, we also constructed a module H_2 looking very much like H, but were able to prove that H and H_2 are not isomorphic by using the fact that an isomorphism between them would have to map $H(\mathbf{t}_1)$ isomorphically onto $H_2(\mathbf{t}_1)$ and $H(\mathbf{t}_2)$ isomorphically onto $H_2(\mathbf{t}_2)$. In fact, we could have easily continued by actually calculating $\operatorname{Hom}(H, H_2)$ in the same way that we calculated $\operatorname{End} H$.

We can easily construct more complicated examples using this same idea. Consider, for instance, the following:

EXAMPLE 5.1. Take the rank-two module G in Example 2.28, given as $G = A_1(1,0) + A_2(0,1) + A_3(1,1) \subseteq Q \oplus Q$, where A_1, A_2, A_3 are rank-one modules with $\mathbf{t}_i = \mathbf{t}(A_i)$ and $A_i \cap A_j = A_1 \cap A_2$ for all pairs $i \neq j$. Let A be a rank-one module such that $\mathbf{t}(A)$ is not comparable to $\mathbf{t}_1$, $\mathbf{t}_2$, or $\mathbf{t}_3$ and such that $A \cap A_i = A_1 \cap A_2$ for $i = 1, 2, 3$. Let w_1, w_2 be distinct elements in W, neither of which is either 0 or 1. Construct two rank-two modules G_1, G_2 containing G as follows:

$$G_1 = G + A(1, w_1) = A_1(1,0) + A_2(0,1) + A_3(1,1) + A(1, w_1)$$
$$G_2 = G + A(1, w_2) = A_1(1,0) + A_2(0,1) + A_3(1,1) + A(1, w_2) \subseteq QG.$$

Let $\mathbf{t} = \mathbf{t}(A)$. Then as shown in Proposition 2.*

$$\begin{aligned} G_1(\mathbf{t}_1) = G_2(\mathbf{t}_1) &= A_1(1,0) = G(\mathbf{t}_1) \\ G_1(\mathbf{t}_2) = G_2(\mathbf{t}_2) &= A_2(0,1) = G(\mathbf{t}_2) \\ G_1(\mathbf{t}_3) = G_2(\mathbf{t}_3) &= A_3(1,1) = G(\mathbf{t}_3) \\ G = G_i(\mathbf{t}_1) + G_i(\mathbf{t}_2) &+ G_i(\mathbf{t}_3) \qquad (i = 1,\, 2) \\ G_1(\mathbf{t}) &= A(1, w_1) \\ G_2(\mathbf{t}) &= A(1, w_2). \end{aligned}$$

It follows that $\operatorname{Hom}(G_1, G_2) = 0$.

PROOF: If $\varphi\colon G_1 \to G_2$ then $\varphi_1(G(\mathbf{t}_i)) \subseteq G_2(\mathbf{t}_i)$ for $i = 1, 2, 3$ and so it follows from the above that $\varphi(G) \subseteq G$. As was shown in Example 1.45, it then follows that φ is given by multiplication by some $q \in Q$. (In fact, if we assume further that $\mathbf{t}_1 \wedge \mathbf{t}_2 \wedge \mathbf{t}_3 = \mathbf{t}(W)$, as we did in Example 1.45, then $q \in W$.) But now it must also be true that $\varphi(G(\mathbf{t})) \subseteq G(\mathbf{t})$, i.e. that $\varphi(1, w_1) = (q, qw_1)$ is a multiple of $(1, w_2)$. This is possible only if $qw_1 = qw_2$, and since $w_1 \neq w_2$ it follows that $q = 0$. Thus $\operatorname{Hom}(G_1, G_2) = 0$. $\boxed{\checkmark}$

Since there are infinitely many choices for w_1 and w_2, this construction produces an infinite family of rank-two modules, all with the same typesets and the same p-ranks which look very much alike to the naked eye and yet are not quasi-isomorphic. Among other things, this shows what a strong relation quasi-isomorphism is.

In these examples, one is able to compute $\operatorname{QEnd} G$ and $\operatorname{QHom}(G, H)$ by using the fact that if $\varphi\colon G \to H$ is a quasi-homomorphism then for all $g \in G$, $\mathbf{t}(\varphi(g)) \geq \mathbf{t}(g)$ (c.f. Proposition 3.7). Before investigating the scope of application of this technique, we will give a final, fairly elaborate but very important example of its use.

LEMMA 5.2. *Let $A_1, \ldots, A_n$ be rank-one modules with mutually incomparable types such that $\mathbf{t}_i \wedge \mathbf{t}_j$ is the same for all pairs $i \neq j$. Let V be a finite dimensional Q-space and for each i let V_i be a subspace of V. Let $\mathbf{t}_0 = \mathbf{t}_1 \wedge \mathbf{t}_2$. Then there exists a module G such that $QG = V$ and for $i = 1, \ldots, n$, $QG(\mathbf{t}_i) = QV_i$, and such that G is generated by a finite number of rank-one submodules with types in $\{\mathbf{t}_0, \ldots, \mathbf{t}_n\}$.*

PROOF: Let C_0 be an essential $\mathbf{t}_0$-projective submodule of V and for $i = 1, \ldots, n$ let C_i be an essential $\mathbf{t}_i$-projective submodule of V_i. Let $G = \sum_0^n C_i$. Then since each C_i is a direct sum of rank-one modules of type $\mathbf{t}_i$, G is generated by a finite number of rank-one submodules with types in $\{\mathbf{t}_0, \ldots, \mathbf{t}_n\}$. Clearly $QG = V$ and for $i = 1, \ldots, n$, $C_i \subseteq G(\mathbf{t}_i)$ so that $V_i \subseteq QG(\mathbf{t}_i)$. We claim that in fact $G(\mathbf{t}_i) = C_{i*}$. Suppose by way of contradiction that there exists $g \in G$ with $\mathbf{t}(g) \geq \mathbf{t}_i$ and $g \notin C_{i*}$. Then there exists $H \lhd G$ with $C_{i*} \subseteq H$ and $g \notin H$ and $\operatorname{rank} G/H = 1$, i.e. there exists $\mu\colon G \to Q$ such that $C_i \subseteq \operatorname{Ker} \mu$ and $\mu(g) \neq 0$. Thus $\mathbf{t}(\mu(G)) \geq \mathbf{t}(g) = \mathbf{t}_i$. But since $C_i \subseteq \operatorname{Ker} \mu$, $\mu(G)$ is generated by $\mu(C_j)$ for $j \neq i$, and since C_j is $\mathbf{t}_j$-projective $\mathbf{CT}(C_j) = \{\mathbf{t}_j\}$ and so $\mathbf{t}(\mu(C_j)) = \mathbf{t}_j$. Thus $\mathbf{t}(\mu(G)) \leq \mathbf{sup}_{j \neq i}\, \mathbf{t}_j$. But by Proposition 2.*

$\mathbf{t}_i \wedge \mathbf{sup}_{j\neq i}\, \mathbf{t}_j = \mathbf{sup}_{j\neq i}\{\mathbf{t}_i \wedge \mathbf{t}_j\} = \mathbf{t}_0 < \mathbf{t}_i$, since $\mathbf{t}_i \wedge \mathbf{t}_j = \mathbf{t}_0$ for $j \neq i$. Thus $\mathbf{t}_i \not\leq \mathbf{t}(\mu(G))$, a contradiction. Therefore for all $g \in G$, $\mathbf{t}(g) \geq \mathbf{t}_i \Rightarrow g \in C_{i*}$, so that $G(\mathbf{t}_i) = C_{i*}$, and so $V_i = QG(\mathbf{t}_i)$. $\boxed{\checkmark}$

PROPOSITION 5.3. *For $n \geq 1$ let $\Lambda = Q\langle x_1, \ldots, x_n\rangle$ be the ring of polynomials in n non-commuting indeterminates over Q. Suppose that W has at least $n+3$ prime ideals. Then there exists a functor $\mathcal{B}$ from the category of finite dimensional Λ-modules into the category of finite rank torsion free W-modules such that for every pair of finite dimensional Λ-modules M and N, $\mathrm{QHom}(\mathcal{B}(M), \mathcal{B}(N)) \approx \mathrm{Hom}_\Lambda(M, N)$. In particular, $\mathcal{B}(M)$ and $\mathcal{B}(N)$ are quasi-isomorphic if and only if M and N are isomorphic Λ-modules.*

PROOF: Let $\mathbf{t}_1, \ldots, \mathbf{t}_{n+3}$ be a fixed set of mutually incomparable types such that $\mathbf{t}_i \wedge \mathbf{t}_j$ is the same for all pairs $i \neq j$. (For instance, choose $\mathbf{t}_i = \mathbf{t}(p_i^{-\infty})$, where $p_1, \ldots, p_{n+3}$ are distinct prime ideals.) Now let M be a finitely generated Λ-module. By Lemma 5.2, we can define a finite rank torsion free module G such that

$$\begin{aligned} QG &= M \oplus M \\ QG(\mathbf{t}_{n+1}) &= M \oplus 0 = \{(m, 0) \mid m \in M\} \\ QG(\mathbf{t}_{n+2}) &= 0 \oplus M = \{(0, m) \mid m \in M\} \\ QG(\mathbf{t}_{n+3}) &= \{(m, m) \mid m \in M\} \\ \text{and } QG(\mathbf{t}_i) &= \{(m, x_i m) \mid m \in M\} \text{ for } i = 1, \ldots, n. \end{aligned}$$

Write $G = \mathcal{B}(M)$.

First note that $\mathcal{B}$ is a functor on the category of Λ-modules. In fact, if $\mu\colon M \to N$ is a homomorphism of Λ-modules then μ induces $\tilde{\mu} = \mu \oplus \mu\colon M \oplus M \to N \oplus N$. If $G = \mathcal{B}(M)$, $H = \mathcal{B}(N)$, define $\mathcal{B}(\mu) = \tilde{\mu}\colon QG \to QH$. Note that $\mathcal{B}(\mu)(QG(\mathbf{t}_i)) \subseteq QH(\mathbf{t}_i)$ for $i = 1, \ldots, n+3$. In fact,

$$\begin{aligned} g \in QG(\mathbf{t}_{n+1}) &\Rightarrow (\exists m \in M)\ g = (m, 0) \Rightarrow \quad \tilde{\mu}(g) = (\mu(m), 0) \in QH(\mathbf{t}_{n+1}), \\ g \in QG(\mathbf{t}_{n+2}) &\Rightarrow (\exists m \in M)\ g = (0, m) \Rightarrow \quad \tilde{\mu}(g) = (0, \mu(m)) \in QH(\mathbf{t}_{n+2}), \\ g \in QG(\mathbf{t}_{n+3}) &\Rightarrow (\exists m \in M)\ g = (m, m) \Rightarrow \quad \tilde{\mu}(g) = (\mu(m), \mu(m)) \in QH(\mathbf{t}_{n+3}), \end{aligned}$$

and for $i = 1, \ldots, n$,

$$g \in QG(\mathbf{t}_i) \Rightarrow (\exists m \in M)\ g = (m, x_i m) \Rightarrow \quad \tilde{\mu}(g) = (\mu(m), x_i \mu(m)) \in QH(\mathbf{t}_i)$$

(using the fact that μ is Λ-linear).

Now by assumption $\mathcal{B}(M)$ is generated by a finite number of rank-one submodules with types in $\{\mathbf{t}_0, \ldots, \mathbf{t}_n\}$. If A is such a rank-one submodule of $\mathcal{B}(M)$ and $\mathbf{t}(A) = \mathbf{t}_i$, then $A \subseteq \mathcal{B}(M)(\mathbf{t}_i)$ and so by the preceding paragraph $\tilde{\mu}(A) \subseteq \mathcal{B}(N)(\mathbf{t}_i)$. Thus by Proposition 2.* there exists $w \neq 0 \in W$ such that $w\tilde{\mu}(A) \subseteq \mathcal{B}(N)(\mathbf{t}_i) \subseteq \mathcal{B}(N)$. Since $\mathcal{B}(M)$ is generated by finitely many such rank-one submodules A, we can multiply

together these w to get $w \neq 0 \in W$ such that $w\tilde{\mu}(\mathcal{B}(M)) \subseteq \mathcal{B}(N)$. Thus $\tilde{\mu}$ is a quasi-homomorphism from $\mathcal{B}(M)$ to $\mathcal{B}(N)$ and so $\mathcal{B}$ is a functor.

To finish the proof, we show that if M and N are Λ-modules then $\mu \mapsto \mathcal{B}(\mu)$ is an isomorphism from $\mathrm{Hom}_\Lambda(M,N)$ to $\mathrm{QHom}(\mathcal{B}(M),\mathcal{B}(N))$. Clearly $\mu \neq 0 \Rightarrow \mathcal{B}(\mu) \neq 0$ so it remains to see that every $\varphi \in \mathrm{QHom}(\mathcal{B}(M),\mathcal{B}(N))$ has the form $\mathcal{B}(\mu)$ for some $\mu\colon M \to N$. In fact, let $G = \mathcal{B}(M)$ and $H = \mathcal{B}(N)$. Then $\varphi\colon QG = M \oplus M \to N \oplus N = QH$. Then $\varphi(QG(\mathbf{t}_{n+1})) \subseteq QH(\mathbf{t}_{n+1})$, $\varphi(QG(\mathbf{t}_{n+2})) \subseteq QH(\mathbf{t}_{n+2})$, and $\varphi(QG(\mathbf{t}_{n+3}) \subseteq QG(\mathbf{t}_{n+3})$. For $m \in M$, let $\mu(m)$ be the first coordinate of $\varphi(m,0)$. Then for $m \in M$,

$$\begin{aligned} \varphi(m,0) &= (\mu(m),0) && \text{since } \varphi((M \oplus 0) \subseteq N \oplus 0 \\ \varphi(m,m) &= (\mu(m),\mu(m)) && \text{since } \varphi(QG(\mathbf{t}_{n+3}) \subseteq QH(\mathbf{t}_{n+3}) \\ \text{and} \quad \varphi(0,m) &= \varphi(m,m) - \varphi(m,0) = (0,\mu(m)). \end{aligned}$$

Thus $\varphi = \tilde{\mu}$. It remains to see that μ is Λ-linear. But for $i = 1,\dots n$,

$$\varphi(m, x_i m) = \tilde{\mu}(m, x_i m) = (\mu(m), \mu(x_i m)),$$

and since $\varphi(Q(\mathbf{t}_i)) \subseteq H(\mathbf{t}_i)$ this must belong to $H(\mathbf{t}_i)$, i.e. $\varphi(m, x_i m) = (\mu(m), x_i\mu(m))$, which implies that $\mu(x_i m) = x_i \mu(m)$. Since this is true for all i, μ must in fact be Λ-linear. This completes the proof. $\boxed{\checkmark}$

It is generally acknowledged for that $n \geq 2$ the task of classifying all modules over the ring Λ in Proposition 5.3 is hopeless. Thus, among other things, Proposition 5.3 shows that it is hopeless to expect to every be able to classify finite rank torsion free W-modules even up to quasi-isomorphism.

BUTLER MODULES. Based on the methodology of these examples, it would be easy to conjecture that the existence of homomorphisms, or at least quasi-homomorphisms, between finite rank torsion free modules is determined solely by the types of the elements in these modules. One might indeed look at Proposition 2.37 which states that a map from QG to QH belongs to $\mathrm{Hom}(G,H)$ if and only if it increases p-heights, and conjecture the following:

PLAUSIBLE CONJECTURE 5.4. *A map φ from QG to QH belongs to $\mathrm{QHom}(G,H)$ if and only if $\mathbf{t}(\varphi(g)) \geq \mathbf{t}(g)$ for all $g \in G$, or, equivalently, if and only if for all $\mathbf{t} \in \mathbf{T}(G)$, $\varphi(QG(\mathbf{t})) \subseteq QH(\mathbf{t})$.*

The Pontryagin module (Example 1.47), however, shows that this conjecture is false. In fact, the Pontryagin module G is homogeneous: $\mathbf{T}(G) = \{\mathbf{t}(W)\}$. Thus Conjecture 5.4 would imply that $\mathrm{QEnd}\, G = \mathrm{End}\, QG$, which by Proposition 4.6 would mean that G dominates itself. It would then follow from Proposition 4.15 that the Pontryagin module is $\mathbf{t}(W)$-projective, i.e. a projective W-module. But this is certainly not the case.

The class of modules for which Conjecture 5.4 is true is very important. In the search for this class, we can begin by noting that it includes completely decomposable modules.

The implications of this are explored in the following proposition:

PROPOSITION 5.5. (1) *If $G = \alpha(C)$ for some $\alpha \in \mathrm{Hom}(C, G)$ and $\varphi\colon QG \to QH$, then $\varphi \in \mathrm{QHom}(G, H) \iff \varphi\alpha \in \mathrm{QHom}(C, H)$.*

(2) *If H is a pure submodule of D and $\iota\colon H \to D$ is the inclusion map, and $\varphi\colon QG \to QH$, then $\varphi \in \mathrm{QHom}(G, H) \iff \iota\varphi \in \mathrm{QHom}(G, D)$.*

(3) *If A and B are rank-one modules and $\varphi\colon QA \to QB$, then $\varphi \in \mathrm{QHom}(A, B) \iff \mathbf{t}(A) \le \mathbf{t}(B)$.*

(4) *If A has rank one and $\mathbf{t} = \mathbf{t}(A)$ and $\varphi\colon QA \to QH$, then $\varphi \in \mathrm{QHom}(A, H) \iff \varphi(QA) \subseteq QH(\mathbf{t})$.*

(5) *If B has rank-one and $\mathbf{t} = \mathbf{t}(B)$ and $\varphi\colon QG \to QB$, then $\varphi \in \mathrm{QHom}(G, B) \iff \varphi(G[\mathbf{t}]) = 0$.*

(6) *If $G = \bigoplus_1^n G_i$ and $\varphi\colon QG \to QH$ then $\varphi \in \mathrm{QHom}(G, H)$ if and only if for all i the restriction of φ to G_i is a quasi-homomorphism from G_i to H.*

(7) *If $H = \bigoplus_1^n H_i$ and $\varphi\colon QG \to QH$, then $\varphi \in \mathrm{QHom}(G, H)$ if and only if for each i the composition φ_i of φ with the projection $\bigoplus H_i \to H_i$ belongs to $\mathrm{QHom}(G, H_i)$.*

PROOF: Each of these is quite obvious.

(1) ($\Rightarrow$): Clear from general principles.

($\Leftarrow$): If $\varphi\alpha \in \mathrm{QHom}(C, H)$ then $(\exists w \neq 0)\, w\varphi\alpha(C) \subseteq H$. Since $\alpha(C) = G$, thus $w\varphi(G) \subseteq H$ and $\varphi \in \mathrm{QHom}(G, H)$.

(2) Obvious.

(3) $\mathrm{QHom}(A, B)$ is a subspace of the one-dimensional vector space $\mathrm{Hom}(QA, QB)$. Hence either $\mathrm{QHom}(A, B) = \mathrm{Hom}(QA, QB)$ or $\mathrm{QHom}(A, B) = 0$, depending on whether or not $\mathbf{t}(A) \le \mathbf{t}(B)$.

(4) Assuming $\varphi \neq 0$, let $B = H \cap \varphi(QA)$. Then B is a pure rank-one submodule of H and $\varphi \in \mathrm{QHom}(A, H) \iff \varphi \in \mathrm{QHom}(A, B)$. By (1) this is true if and only if $\mathbf{t}(B) \ge \mathbf{t}(A) = \mathbf{t}$. But the latter is equivalent to saying that $B \subseteq QG(\mathbf{t})$, i.e. $\varphi(A) \subseteq QG(\mathbf{t})$.

(5) Let $K = G \cap \mathrm{Ker}\,\varphi$. Then $\mathrm{rank}\, G/K = 1$ and $\varphi \in \mathrm{QHom}(G, H) \iff$ the induced map $\bar\varphi\colon G/K \to QB$ belongs to $\mathrm{QHom}(G/K, B)$, i.e. if and only if $\mathbf{t}(G/K) \le \mathbf{t}(B) = \mathbf{t}$. By Proposition 4.10 this is true if and only if $K \subseteq G[\mathbf{t}]$, i.e. if and only if $\varphi(G[\mathbf{t}]) = 0$.

(6) ($\Leftarrow$): Clear on general principles.

($\Rightarrow$): If $\varphi \in \mathrm{QHom}(G_i, H)$ then $(\exists w_i \neq 0)\, w_i\varphi(G_i) \subseteq H$. Then $w\varphi(G) \subseteq H$, where $w = w_1 \cdots w_n$, so $\varphi \in \mathrm{QHom}(G, H)$.

(7) Analogous. $\boxed{\checkmark}$

One obvious moral that can be drawn from the sequence of observations in Proposition 5.5 is that homomorphic images of completely decomposable modules satisfy Conjecture 5.4.

DEFINITION 5.6. We say that a finite rank torsion free module G is a **Butler module** if G is a homomorphic image of a completely decomposable module.

PROPOSITION 5.7. (1) *G is a Butler module if and only if G is generated by a finite number of its rank-one submodules.*

(2) *A module quasi-isomorphic to a Butler module is a Butler module.*

PROOF: (1) If G is a Butler module then it is a homomorphic image of a completely decomposable module C, and is generated by the images of the rank-one summands of C. Conversely, if G is generated by a finite set of rank-one submodules A_i then G is a homomorphic image of $\bigoplus A_i$.

(2) Suppose G is a Butler module and $H \sim G$. We may suppose $G \subseteq H$. Now G is generated by a finite set of rank-one submodules and H is generated by G together with a finite set of elements $h_i \in H$. Then H is generated by the rank-one submodules which generate G plus the finite set of cyclic submodules generated by the h_i, hence H is a Butler module by (1). $\boxed{\checkmark}$

Now putting several items of Proposition 5.5 together yields

PROPOSITION 5.8. *Let G be a Butler module and H be a torsion free module and $\varphi\colon QG \to QH$. The following conditions are equivalent:*

(1) $\varphi \in \mathrm{QHom}(G, H)$
(2) $\varphi(QG(\mathbf{t})) \subseteq QH(\mathbf{t})$ *for all* $\mathbf{t} \in \mathbf{T}(G)$.
(3) *For all* $g \in G$, $\mathbf{t}_H(\varphi(g)) \geq \mathbf{t}_G(g)$.

Furthermore, if $\mathbf{S}$ is a subset of $\mathbf{T}(G)$ such that G is quasi-equal to $\sum_{\mathbf{S}} G(\mathbf{t})$, then for the implication (2) $\Rightarrow$ (1) it suffices to verify (2) for those $\mathbf{t} \in \mathbf{S}$.

PROOF: (1) $\Rightarrow$ (2): By Proposition 3.7.

(2) $\Rightarrow$ (3): Assume (2) and let $g \in G$ and $\mathbf{t} = \mathbf{t}(g)$. Then $g \in G(\mathbf{t})$ so by (2) $\varphi(g) \in H(\mathbf{t})$, i.e. $\mathbf{t}_H(\varphi(g)) \geq \mathbf{t} = \mathbf{t}_G(g)$.

(3) $\Rightarrow$ (1): Assume that $\mathbf{t}(\varphi(g)) \geq \mathbf{t}(g)$ for all $g \in G$. Since G is a Butler module, then there is a completely decomposable module $C = \bigoplus A_i$ and a surjection $\alpha\colon C \to G$. By Proposition 5.5, in order to prove that $\varphi \in \mathrm{QHom}(G, H)$ it suffices to show that $\varphi\alpha \in \mathrm{QHom}(C, H)$. Furthermore, for all $c \in C$, $\mathbf{t}_H(\varphi\alpha(c)) \geq \mathbf{t}_G(\alpha(c)) \geq \mathbf{t}_C(c)$. Then by Proposition 5.5 (4), the restriction of $\varphi\alpha$ to each rank-one summand A_i of C is a quasi-homomorphism. Therefore by Proposition 5.5 (6), $\varphi\alpha \in \mathrm{QHom}(C, H)$. Thus $\varphi \in \mathrm{QHom}(G, H)$. $\boxed{\checkmark}$

COROLLARY 5.9. *Let G and H be Butler modules. If $QG = QH$ then G and H are quasi-equal if and only if $QG(\mathbf{t}) = QH(\mathbf{t})$ for all $\mathbf{t} \in \mathbf{T}(G) \cup \mathbf{T}(H)$.* (If this is so, then by Proposition 3.8 $\mathbf{T}(G) = \mathbf{T}(H)$.)

PROOF: G and H are quasi-equal if and only if the identity map on QG is both a quasi-homomorphism from G to H and from H to G. By Proposition 5.8 this is equivalent to saying that $QG(\mathbf{t}) \subseteq QH(\mathbf{t})$ and $QH(\mathbf{t}) \subseteq QG(\mathbf{t})$ for all $\mathbf{t} \in \mathbf{T}(G)$ and $\mathbf{t} \in \mathbf{T}(H)$. $\boxed{\checkmark}$

BUTLER'S THEOREM. In [Butler] Butler proved the following:

THEOREM 5.10. *The class of Butler modules is the same as the class of pure submodules of completely decomposable modules.*

EXAMPLE 5.11. Consider again the module G in Example 1.45: G is a rank-two module and $G = A_1(1,0) + A_2(0,1) + A_3(1,1)$, where the A_i are submodules of Q with $1 \in A_i$ for all i and $A_i \cap A_j$ is the same for all pairs $i \neq j$. Write $A_0 = A_1 \cap A_2$. Now let $B_1 = A_2 + A_3$, $B_2 = A_1 + A_3$, and $B_3 = A_1 + A_2$. By Proposition 2.*

$$B_1 \cap B_2 = A_2 \cap A_1 + A_2 \cap A_3 + A_3 \cap A_1 + A_3 \cap A_3 = A_0 + A_0 + A_0 + A_3 = A_3.$$

Likewise

$$B_2 \cap B_3 = A_1 \quad \text{and } B_1 \cap B_3 = A_2.$$

Then G is isomorphic to the pure closure of the submodule of the completely decomposable module $B_1 \oplus B_2 \oplus B_3$ generated by $a_1 = (0,1,-1)$, $a_2 = (-1,0,1)$ and $a_3 = (-1,1,0)$.

(For simplicity, one may want to suppose, as in the original construction, that $A_i = p_i^{-\infty}$, where p_1, p_2, p_3 are distinct primes. Then $G = (p_1^{-\infty}, 0) + (0, p_2^{-\infty}) + p_3^{-\infty}(1,1)$, and, for instance, $B_1 = p_2^{-\infty} p_3^{-\infty}$.)

PROOF: Let H be the pure closure of the submodule of $B_1 \oplus B_2 \oplus B_3$ generated by a_1, a_2, amnd a_3. For $i = 1,2,3$, let A_i' be the pure closure of the submodule of $B_1 \oplus B_2 \oplus B_3$ generated by a_i. The map $(0,x,-x) \mapsto x$ is an isomorphism from A_1' to $B_2 \cap B_3$, so that $A_1' \approx B_2 \cap B_3 = A_1$. Likewise $A_2' \approx A_2$ and $A_3' \approx A_3$. Note that since $a_3 = a_1 + a_2$, QH is generated over Q by a_1 and a_2. Thus $\operatorname{rank} H = 2$. Furthermore $Q(A_1' + A_2' + A_3') = QH$ and $A_1' + A_2' + A_3' \lhd B_1 \oplus B_2 \oplus B_3$ (WHY?) so that $A_1' + A_2' + A_3' = H$. Finally this shows that there is an isomorphism φ from G to H given by $(1,0) \mapsto a_1$, $(0,1) \mapsto a_2$ and $(1,1) \mapsto a_3$. In fact, φ is monic and $\varphi(A_i) = \langle a_i \rangle_*$ for $i = 1,2,3$, and thus since $G = A_1 + A_2 + A_3$, $\varphi(G) = H$. $\boxed{\checkmark}$

After contemplating a few examples such as this, it is easy to become completely convinced of the validity of Butler's Theorem. In fact, it is not that hard to devise specific procedures for representing a Butler module as a pure submodule of a completely decomposable module, and vice versa. However the problem of actually proving that such procedures work is another matter.

BALANCED AND COBALANCED SUBMODULES. Before embarking on the proof of Butler's Theorem, we need to note that as far as representing G as a homomorphic image of a completely decomposable module or a pure submodule of a completely decomposable module, there are good surjections and good pure embeddings, and others which are less desirable. For instance, let $G \subseteq Q$ be such that $G = A + B$ where A, $B \subseteq Q$ with $\mathbf{t}(A), \mathbf{t}(B) < \mathbf{t}(G)$. Then G is trivially a Butler module, and G is a homomorphic image of $A \oplus B$. However representing the rank-one module as a homomorphic image of a rank-two module $A \oplus B$ is clearly uneconomical. Furthermore $\mathbf{T}(A \oplus B)$ contains types which are not in $\mathbf{T}(G)$, so that this way of representing G as a homomorphic image of a completely decomposable module casts less enlightenment on the properties of G than one might wish. Similarly if we have $H = A \cap B$ with A, $B \subseteq Q$

and $\mathbf{t}(H) < \mathbf{t}(A), \mathbf{t}(B)$, then H is isomorphic to a pure submodule of $A \oplus B$, but once again this is not the most desirable way of representing H as a pure submodule of a completely decomposable module.

The concept we need is the following:

DEFINITION 5.12. Let $H \lhd G$ and let $\varphi\colon G \to G/H$ be the quotient map. We say that H of G is a **balanced submodule** of G if for every type $\mathbf{t}$, $\varphi(G)(\mathbf{t}) = \varphi(G(\mathbf{t}))$, i.e. $(G/H)(\mathbf{t}) = (G(\mathbf{t}) + H)/H$. We also say that $\varphi\colon G \to G/H$ is a **balanced surjection**. If $\varphi \in \mathrm{QHom}(G, K)$ is a quasi-surjection, we say that φ is balanced if for every type $\mathbf{t}$, $\varphi(G)(\mathbf{t})$ is quasi-equal to $\varphi(G(\mathbf{t}))$.

We say that H is a **co-balanced submodule** of G if for every type $\mathbf{t}$, $H[\mathbf{t}] = H \cap G[\mathbf{t}]$. This is equivalent to the assertion that for every type $\mathbf{t}$, the map $H/H[\mathbf{t}] \to G/G[\mathbf{t}]$ induced by the inclusion $H \hookrightarrow G$ is monic.

PROPOSITION 5.13. (1) *A surjection $\varphi\colon G \to K$ is balanced if and only if every $k \in K$ has a pre-image $g \in G$ with $\mathbf{t}(g) = \mathbf{t}(k)$.*

(2) *A pure submodule H of G is cobalanced if and only if every $\mu \in \mathrm{Hom}(H, Q)$ is the restriction of a map $\mu' \in \mathrm{Hom}(G, Q)$ such that $\mathbf{t}(\mu'(G)) = \mathbf{t}(\mu(H))$.*

PROOF: (1) ($\Rightarrow$): Let φ be a balanced surjection of G onto K and let $k \in K$ and $\mathbf{t} = \mathbf{t}(k)$. Then $k \in K(\mathbf{t})$ and since φ is balanced there exists $g \in G(\mathbf{t})$ with $\varphi(g) = k$. Then $\mathbf{t}(g) \geq \mathbf{t}$. But by Proposition 2.21 $\mathbf{t}(g) \leq \mathbf{t}$, so $\mathbf{t}(g) = \mathbf{t} = \mathbf{t}(k)$.

($\Leftarrow$): Let $k \in K(\mathbf{t})$ for some type $\mathbf{t}$. If there exists $g \in G$ with $\varphi(g) = k$ and $\mathbf{t}(g) = \mathbf{t}(k)$, then $\mathbf{t}(g) \geq \mathbf{t}$ so that $g \in G(\mathbf{t})$ and thus $k \in \varphi(G(\mathbf{t}))$. Thus if every $k \in K$ has a pre-image of the same type, then for every type $\mathbf{t}$, $K(\mathbf{t}) = \varphi(G(\mathbf{t}))$, i.e. φ is balanced.

(2) ($\Rightarrow$): By Proposition 1.* $H/H[\mathbf{t}]$ is isomorphic to a pure submodule of $G/H[\mathbf{t}]$. And by Proposition 4.2 $(G/H)[\mathbf{t}] = G[\mathbf{t}]/H[\mathbf{t}]$. Now if H is cobalanced in G then $H[\mathbf{t}] = H \cap G[\mathbf{t}]$, so that $(H/H[\mathbf{t}]) \cap (G/H[\mathbf{t}])[\mathbf{t}] = (H/H[\mathbf{t}]) \cap (G[\mathbf{t}]/H[\mathbf{t}]) = (H \cap G[\mathbf{t}])/H[\mathbf{t}] = 0$. Thus by Proposition 4.21 $H/H[\mathbf{t}]$ is a quasi-summand of $G/H[\mathbf{t}]$. Now let $\mu\colon H \to Q$ and let $A = \mu(H)$ and $\mathbf{t} = \mathbf{t}(A)$. Then $\mu(H[\mathbf{t}]) = 0$ so μ induces $\bar{\mu}\colon H/H[\mathbf{t}] \to A$, and since $H/H[\mathbf{t}]$ is a quasi-summand of $G/H[\mathbf{t}]$ this extends to $\bar{\mu}' \in \mathrm{QHom}(G/H[\mathbf{t}], A)$. But then $\bar{\mu}'$ induces $\mu' \in \mathrm{QHom}(G, A)$ extending μ. Since $\mu'(G)$ is quasi-equal to A, $\mathbf{t}(\mu'(G)) = \mathbf{t}(A) = \mathbf{t}$.

($\Leftarrow$): By Proposition 4.2 $H[\mathbf{t}] \subseteq G[\mathbf{t}]$, so $H[\mathbf{t}] \subseteq H \cap G[\mathbf{t}]$. Now let A be a rank-one module with $\mathbf{t}(A) = \mathbf{t}$ and let $\mu\colon H \to A$. If μ extends to a map $\mu'\colon G \to Q$ with $\mathbf{t}(\mu'(G)) = \mathbf{t}$, then $\mu'(G[\mathbf{t}]) = 0$, so $\mu(H \cap G[\mathbf{t}]) = 0$. Thus if every $\mu \in \mathrm{Hom}(H/Q)$ extends to $\mu' \in \mathrm{Hom}(G, Q)$ with $\mathbf{t}(\mu'(G)) = \mathbf{t}(\mu(G))$, then $H \cap G[\mathbf{t}] \subseteq H[\mathbf{t}]$, so $H \cap G[\mathbf{t}] = H[\mathbf{t}]$. Since this holds for every type $\mathbf{t}$, H is a cobalanced submodule of G. $\boxed{\checkmark}$

COROLLARY 5.14. (1) *If H is a balanced pure submodule of G then $\mathbf{T}(G/H) \subseteq \mathbf{T}(G)$.*

(2) *If H is a cobalanced pure submodule of G then $\mathbf{CT}(H) \subseteq \mathbf{CT}(G)$.*

PROOF: (1) If H is a balanced pure submodule of G and $\mathbf{t} \in \mathbf{T}(G/H)$ then $\mathbf{t} = \mathbf{t}(\bar{g})$ for some $\bar{g} \in G/H$. By Proposition 5.13 $\bar{g}$ has a pre-image $g \in G$ such that $\mathbf{t}(g) = \mathbf{t}$. Thus $\mathbf{t} \in \mathbf{T}(G)$.

(2) If H is a cobalanced pure submodule of G and $\mathbf{t} \in \mathbf{CT}(H)$ then there exists $\mu \in \mathrm{Hom}(H, Q)$ with $\mathbf{t}(\mu(H)) = \mathbf{t}$. By Proposition 5.13 μ extends to $\mu' \in \mathrm{Hom}(G, Q)$ with $\mathbf{t}(\mu'(G)) = \mathbf{t}$. Thus $\mathbf{t} \in \mathbf{CT}(G)$. ☑

Using the concept of balanced surjection, we can give a generalized version of Baer's Lemma.

PROPOSITION 5.15. *If K is completely decomposable and $\varphi\colon G \to K$ is a balanced surjection, then φ splits.*

PROOF: Let $K = \bigoplus_{\mathbf{t}} K_{\mathbf{t}}$, where each $K_{\mathbf{t}}$ is $\mathbf{t}$-projective. It suffices to see that for each $\mathbf{t}$, the restriction of φ to a map $\varphi^{-1}(K_{\mathbf{t}}) \to K_{\mathbf{t}}$ is a split surjection. In fact, if for each $\mathbf{t}$, $\sigma_{\mathbf{t}}\colon K_{\mathbf{t}} \to \varphi^{-1}(K_{\mathbf{t}}) \subseteq G$ is such that $\varphi\sigma_{\mathbf{t}} = 1_{K_{\mathbf{t}}}$, then the family $\{\sigma_{\mathbf{t}}\}_{\mathbf{t}}$ combine to give a map $K = \bigoplus K_{\mathbf{t}} \to G$ splitting φ.

Now $K_{\mathbf{t}} \subseteq K(\mathbf{t}) = \varphi(G(\mathbf{t}))$ since φ is balanced. Thus since φ is surjective, $K_{\mathbf{t}} = \varphi(\, G(\mathbf{t}) \cap \varphi^{-1}(K_{\mathbf{t}}))$ and it suffices to see that the surjection $G(\mathbf{t}) \cap \varphi^{-1}(K_{\mathbf{t}}) \to K_{\mathbf{t}}$ splits. But $\varphi^{-1}(K_{\mathbf{t}}) \lhd G$ and so $G(\mathbf{t}) \cap \varphi^{-1}(K_{\mathbf{t}})$ is a pure submodule of the $\mathbf{t}$-saturated module $G(\mathbf{t})$, and hence by Proposition 4.* is $\mathbf{t}$-saturated. Since $K_{\mathbf{t}}$ is $\mathbf{t}$-bounded, it follows from Baer's Lemma (Theorem 4.13) that $G(\mathbf{t}) \cap \varphi^{-1}(K_{\mathbf{t}}) \to K_{\mathbf{t}}$ splits. ☑

PROOF OF BUTLER'S THEOREM. We now begin proving the results which will result in a proof of Butler's Theorem.

DEFINITION 5.16. If $G \lhd \bigoplus A_i$, where the A_i are rank-one modules, and if B is a pure rank-one submodule of G, we define the **support** of B to be $\mathrm{Supp}\, B = \{i \in I \mid b_i \neq 0\}$ where $b \neq 0 \in B$. (It is easy to see that this is independant of the choice of b.) We say that B is a **distinguished** rank-one submodule of G if there does not exist a rank-one submodule $B' \lhd G$ with $\mathrm{Supp}(B') \subsetneq \mathrm{Supp}(B)$. If $b \neq 0 \in B$, we also sometimes write $\mathrm{Supp}(b) = \mathrm{Supp}(B)$.

PROPOSITION 5.17. *If G is a pure submodule of a completely decomposable module $\bigoplus A_i$, then there are only finitely many distinguished pure rank-one submodules of G, and G is generated by these distinguished submodules. Furthermore, if $\{B_j\}_{j=1}^m$ are these distinguished submodules, the map $\bigoplus_1^m B_j \to G$ induced by the inclusion maps $B_j \hookrightarrow G$ is a balanced surjection, so that G is a homomorphic image of a completely decomposable module.*

PROOF: First note that if B and B' are distinguished rank-one submodules of G and $\mathrm{Supp}\, B = \mathrm{Supp}\, B'$, then $B = B'$. For if $\mathrm{Supp}\, B = \mathrm{Supp}\, B'$ and $B \neq B'$ then one can find $b \in B$, $b' \in B'$ with $b \neq b'$ but where b and b' agree in at least one coordinate. Then the pure submodule generated by $b - b'$ is non-trivial and has strictly smaller support than B, a contradiction.

To see that G is generated by the B_j, it suffices to see that for each prime ideal p, G_p is generated by the $(B_j)_p$. But now the $(B_j)_p$ are precisely the distinguished rank-one submodules of G_p. Therefore there is no loss of generality in supposing that W is a local ring. Under this assumption, there is also no loss of generality in assuming that for all i, $A_i = W$ or $A_i = Q$. Now if W is local then a rank-one submodule $B \subseteq \bigoplus A_i$ is pure if and only if either $B = pB$ or $B \not\subseteq \bigoplus pA_i$, and in either of these cases there must exist i such that B contains an element whose i^{th} coordinate is 1. Now given $g \in G$, we want to show that $g \in \sum B_j$, where the sum extends over the distinguished rank-one pure submodules of G. The idea is to use induction on the size of $\operatorname{Supp} g$. If $g \in B_j$ for some j, we are done. Otherwise, $\operatorname{Supp} g$ is not minimal and we choose B_j with $\operatorname{Supp} B_j \subsetneq \operatorname{Supp} g$. Since B_j contains an element with i^{th} coordinate 1 for some $i \in \operatorname{Supp} B_j \subseteq \operatorname{Supp} g$, B_j contains an element b whose i^{th} coordinate agrees with the ith coordinate of g. It suffices to see that $g - b$ is in the submodule generated by the B_j. But this follows by induction, since $\operatorname{Supp}(b - g) \subsetneq \operatorname{Supp} g$.

Finally, the assertion that $\bigoplus_1^m B_j \to G$ is a balanced surjection is simply the claim that for each type $\mathbf{t}$, $G(\mathbf{t})$ is generated by those B_j with $B_j \subseteq G(\mathbf{t})$ and $\mathbf{t}(B_j) \geq \mathbf{t}$. In fact, applying the proof above to $G(\mathbf{t})$ shows that $G(\mathbf{t})$ is generated by those B_j with $B_j \subseteq G(\mathbf{t})$. Furthermore $\mathbf{t}(B_j) \geq \mathbf{t}$ for all such B_j since they are pure submodules of G. $\boxed{\checkmark}$

The proof of the other direction in Butler's Theorem is dual to the above. Let G be a Butler module and $\{A_i\}_1^n$ be a finite family of rank-one submodules generating G. By a **linear functional** on G we will mean a homomorphism $\mu\colon G \to Q$. For a linear functional μ we define $\operatorname{Supp}(\mu) = \{i \mid \mu(A_i) \neq 0\}$. We say that μ is a **distinguished** linear functional if μ is non-trivial and there does not exist a non-trivial linear functional $\mu'\colon G \to Q$ with $\operatorname{Supp}(\mu') \subsetneq \operatorname{Supp}(\mu)$.

Notice that if μ and μ' are linear functionals on G and $\mu' = q\mu$ for some $q \neq 0 \in Q$, then μ and μ' have the same support. We will think of two linear functionals as equivalent if they are scalar multiples of each other, which is the same as saying that they have the same kernel.

Notice also that the property of being distinguished is not intrinsic to μ and G but depends on the family $\{A_i\}$ of rank-one modules generating G.

PROPOSITION 5.18. *If G is a Butler module then there are only finitely many distinguished linear functionals on G (up to scalar multiples) and the product map $\gamma\colon G \to \bigoplus \mu(G)$, where the μ range over the distinguished linear functionals on G, is a cobalanced pure embedding of G into the completely decomposable module $\bigoplus \mu(G)$.*

PROOF: First note that if μ is a distinguished linear functional, then $\operatorname{Supp} \mu$ uniquely determines μ up to a scalar multiple. In fact, suppose $\operatorname{Supp} \mu = \operatorname{Supp} \mu'$. Replacing μ and μ' by suitable multiples, we can arrange that μ and μ' agree on some A_j with $j \in \operatorname{Supp} \mu$ (since $\operatorname{rank} \operatorname{Hom}(A_j, Q) = 1$). Then $j \notin \operatorname{Supp}(\mu - \mu')$, so that $\operatorname{Supp}(\mu - \mu') \subsetneq \operatorname{Supp} \mu$. Since μ is distinguished, it follows that $\mu - \mu' = 0$, so that $\mu = \mu'$.

It now follows that there are only finitely many distinguished linear functionals, up to scalar multiples. Let $\mu_1, \ldots \mu_r$ be these distinguished functionals. To see that the

mapping $\gamma\colon g \mapsto (\mu_1(g), \dots, \mu_r(g))$ embeds G as a pure submodule of $\bigoplus_1^r \mu_i(G)$, it suffices to see that γ embeds G_p as a pure submodule of $\bigoplus_1^r \mu_i(G)_p$ for every prime p. We then easily see that there is no loss of generality in supposing that W is local. In this case, each A_i is isomorphic to either W or Q and $G/\mathbf{d}(G)$ is finitely generated, so G is the direct sum of $\mathbf{d}(G)$ and a free W-module. Furthermore if μ is a linear functional on G then either $\mu(G) = Q$ or $\mu(G) \approx W$. We will normalize the distinguished linear functionals so that if $\mu_i(G) \neq Q$ then $\mu_i(G) = W$. We now claim that the distinguished linear functionals $\mu_1 \dots, \mu_r$ generate $\mathrm{Hom}(G, Q)$ as a Q-space, and that the distinguished functionals with $\mu_i(G) = W$ generate $\mathrm{Hom}(G, W)$ over W. To prove the second asserion, we will prove by induction on the size of $\mathrm{Supp}\,\mu$ that for $\mu \in \mathrm{Hom}(G, W)$, μ is a linear combination with coefficients in W of μ_i with $\mu_i(G) = W$. If μ is not distinguished, then $\mathrm{Supp}\,\mu$ is not minimal and there exists a distinguished functional μ_i with $\mathrm{Supp}\,\mu_i \subseteq \mathrm{Supp}\,\mu$, and $\mu_i(G) = W$ since clearly $\mu_i(G) \neq Q$. Now since $\mu_i(G) = W$, it follows that $\mu_i(A_j) \not\subseteq p$ for some j, so $\mu_i(A_j) = W$ since W is local. Now $\mathrm{Hom}(A_j, W)$ is cyclic (since $\mathrm{rank}\, A_j = 1$) and the restriction of μ_i to A_j is a generator since $\mu_i(A_j) = W$. Thus for some $w \in W$, $w\mu_i$ and μ have the same restriction of A_j, so that $j \notin \mathrm{Supp}(\mu - w\mu_i)$. The claim then follows by induction.

The proof that the complete set of linear functionals $\mu_1, \dots, \mu_r$ generates $\mathrm{Hom}(G, Q)$ as a Q-space is analogous but easier. It then follows that for $g \neq 0 \in G$ there exists a distinguished linear functional μ_i with $\mu_i(g) \neq 0$, from which it follows that $\gamma\colon g \mapsto (\mu_1(g), \dots, \mu_r(g))$ is a monomorphism.

To prove purity, we need to show that if $g \notin pG$ then there exists a distinguished functional μ_i with $\mu_i(g) \notin p\mu_i(G) = p$. Since we have seen that the set of distinguished functionals with $\mu_i(G) = W$ generate $\mathrm{Hom}(G, W)$, it suffices to prove that there exists $\mu \in \mathrm{Hom}(G, W)$ with $\mu(g) \notin p$. But since $g \notin pG$ and in particular $g \notin \mathbf{d}(G)$, and since $G/\mathbf{d}(G)$ is free, this is clear. $\boxed{\checkmark}$

Between Proposition 5.17 and Proposition 5.18 the proof of Butler's Theorem is complete.

COROLLARY 5.19. *Let H be a Butler module and $\varphi\colon QG \to QH$ for some G. The following conditions are equivalent:*

(1) $\varphi \in \mathrm{QHom}(G, H)$
(2) $\varphi(QG[\mathbf{t}]) \subseteq QH[\mathbf{t}]$ *for every* $\mathbf{t} \in \mathbf{CT}(G)$.
(3) *For all* $\mu\colon H \to Q$, $\mathbf{t}(\mu\varphi(G)) \leq \mathbf{t}(\mu(H))$.

PROOF: (1) $\Longleftrightarrow$ (2): Since H is a pure submodule of a direct sum of rank-one modules, by parts (2) and (7) of Proposition 5.5, it suffices to suppose that $\mathrm{rank}\, H = 1$, in which case the result follows from Proposition 5.5 (5). (Note that if $\mathrm{rank}\, H = 1$ then by Lemma 2.4 $H[\mathbf{t}] = 0$ if $\mathbf{t} \geq \mathbf{t}(H)$, otherwise $H[\mathbf{t}] = H$.)

(1) $\Rightarrow$ (3): We may assume that in fact $\varphi \in \mathrm{Hom}(G, H)$. Let $\mu\colon H \to Q$ and let $\mathbf{t} = \mathbf{t}(\mu(H))$. Then by Proposition 4.4 $\varphi(G[\mathbf{t}]) \subseteq H[\mathbf{t}] \subseteq \mathrm{Ker}\,\mu$, i.e. $\mu\varphi(G[\mathbf{t}]) = 0$. Therefore by Proposition 4.4 $\mathbf{t}(\mu\varphi(G)) \leq \mathbf{t} = \mathbf{t}(\mu(H))$.

(3) $\Rightarrow$ (2): Let $g \in G[\mathbf{t}]$ for some type $\mathbf{t}$ and let A be a rank-one module with $\mathbf{t}(A) = \mathbf{t}$. Let $\mu \in \mathrm{Hom}(H, A)$. Then $\mathbf{t}(\mu(H)) \leq \mathbf{t}$ and if $\mathbf{t}(\mu\varphi(G)) \leq \mathbf{t}(\mu(H)) \leq \mathbf{t}$

then $\mu\varphi(g) = 0$ since $g \in G[\mathbf{t}]$. If this is true for every $\mu \in \mathrm{Hom}(H, A)$ it follows that $\varphi(g) \in H[\mathbf{t}]$. Thus if $\mathbf{t}(\mu\varphi(G)) \leq \mathbf{t}(\mu(H))$ for every $\mu \in \mathrm{Hom}(H, A)$ then $\varphi(QG[\mathbf{t}]) \subseteq QH[\mathbf{t}]$. ☑

COROLLARY 5.20. *Quasi-pure submodules and homomorphic images of Butler modules are Butler modules.*

PROOF: The assertion about homomorphic images is clear, since if G is a homomorphic image of a completely decomposable module C and H is a homomorphic image of G, then H is also a homomorphic image of C. Now if H is a quasi-pure submodule of G, then H is quasi-equal to $G \cap QH$, and by Proposition 5.7 H is a Butler module if and only if $G \cap QH$ is a Butler module. Thus it suffices to suppose that $H \lhd G$. Now if G is a Butler module then G is a pure submodule of a completely decomposable module C, and H is also a pure submodule of C, so by the Theorem 5.10 H is a Butler module. ☑

Arbitrary submodules of Butler modules need not be Butler modules. For instance, divisible modules are Butler modules (since they are completely decomposable) and any G is a submodule of the divisible module QG. But there exist finite rank torsion free modules with are not Butler modules, for instance the Pontryagin module (Example 1.47).

COROLLARY 5.21. *Let G be a Butler module and $H \subseteq QG$. Then H is a quasi-pure submodule of G if and only if H is a Butler module and for all $h \in H$, $\mathbf{t}_H(h) = \mathbf{t}_G(h)$.*

PROOF: Let $H' = H \cap QG$. Then $H' \lhd G$ so by Corollary 5.20 H' is a Butler module. And H is quasi-pure in G if and only if H is quasi-equal to H'.

($\Rightarrow$): If H is quasi-equal to H' then H is a Butler module by Proposition 5.7. Furthermore if $h \in H$ then by Proposition 3.7 the type of h in H is the same as its type in H', and thus the same as its type in G.

($\Leftarrow$): If every element $h \in H$ has the same type in G as in H then since $H' \lhd G$, by Proposition 2.* it has the same type in H' as in H. Thus by Proposition 5.9 H is quasi-equal to H', hence is a quasi-pure submodule of G. ☑

TYPESET AND COTYPESET.

COROLLARY 5.22. *If G is a Butler module then $\mathbf{T}(G)$ and $\mathbf{CT}(G)$ are finite and generate the same lattice.*

PROOF: Suppose that G is generated by pure rank-one submodules $A_1, \ldots, A_n$ and let $\mathbf{t}_i = \mathbf{t}(A_i) \in \mathbf{T}(G)$. If $\mathbf{t} \in \mathbf{CT}(G)$ then there exists a surjection $\varphi\colon G \to B$ where B is a rank-one module with $\mathbf{t}(B) = \mathbf{t}$. Then $B = \varphi(G) = \sum \varphi(A_i)$ and either $\varphi(A_i) \approx A_i$ or $\varphi(A_i) = 0$, so that $\mathbf{t} = \mathbf{sup}\{\mathbf{t}_i \mid \varphi(A_i) \neq 0\}$. Thus $\mathbf{CT}(G)$ is contained in the lattice generated by $\mathbf{T}(G)$ and in particular is finite.

Conversely, G is isomorphic to a pure submodule of $\bigoplus B_j$ where $\operatorname{rank} B_j = 1$ for all i and we may assume WLOG that $\pi_j(G) = B_j$, where π_j is the projection onto B_j. Thus $\mathbf{t}(B_j) \in \mathbf{CT}(G)$. Now if $\mathbf{t} \in \mathbf{T}(G)$ then $\mathbf{t} = \mathbf{t}(g)$ for some $g \in G$. Since $G \lhd \bigoplus B_j$, by Proposition 2.21 we can compute $\mathbf{t}(g)$ in $\bigoplus B_j$. Then by Proposition 2.* $\mathbf{t}(g) = \mathbf{inf}\{\mathbf{t}(B_j) \mid \pi_j(g) \neq 0\}$. Thus $\mathbf{T}(G)$ is finite and contained in the lattice generated by $\mathbf{CT}(G)$. ☑

COROLLARY 5.23. *Let G be a Butler module and H a quasi-pure submodule of G. Then $\mathbf{T}(H)$, $\mathbf{T}(G/H)$, and $\mathbf{CT}(H)$, and $\mathbf{CT}(G/H)$ are contained in the lattice generated by $\mathbf{T}(G)$.*

PROOF: By the previous Corollary, it suffices to prove that $\mathbf{T}(H)$ and $\mathbf{CT}(G/H)$ are contained in the lattice generated by $\mathbf{T}(G)$ or by $\mathbf{CT}(G)$. But $\mathbf{T}(H) \subseteq \mathbf{T}(G)$ and $\mathbf{CT}(G/H) \subseteq \mathbf{CT}(G)$ even without assuming that G is a Butler module. $\boxed{\checkmark}$

COROLLARY 5.24. *If G is a Butler module and $\mathbf{T}(G)$ contains a unique maximal type $\mathbf{t}$, then G can be written as $G_{\mathbf{t}} \oplus G'$ where $G_{\mathbf{t}}$ is $\mathbf{t}$-projective and $\mathbf{t} \notin \mathbf{T}(G')$.*

PROOF: If $\mathbf{t}$ is the unique maximal type in $\mathbf{T}(G)$ then by Corollary 5.23 $\mathbf{t} = \mathbf{OT}(G)$ and by by Proposition 4.22 $G(\mathbf{t})$ is $\mathbf{t}$-projective and $G = G(\mathbf{t}) \oplus G'$ for some G'. Since $G' \lhd G$, by Proposition 4.2 $G'(\mathbf{t}) = G' \cap G(\mathbf{t}) = 0$. Thus $\mathbf{t} \notin \mathbf{T}(G')$. $\boxed{\checkmark}$

COROLLARY 5.25. *If G is a Butler module and $\mathbf{T}(G)$ is totally ordered, then G is completely decomposable. In particular, a homogeneous Butler module is $\mathbf{t}$-projective for some $\mathbf{t}$.*

PROOF: By induction on the size of $\mathbf{T}(G)$, which is finite by Proposition 5.22. If $\mathbf{T}(G) = \{\mathbf{t}\}$ then by Proposition 5.22 $\mathbf{CT}(G) = \{\mathbf{t}\}$, so that $\mathbf{IT}(G) = \mathbf{OT}(G) = \mathbf{t}$ and G is both $\mathbf{t}$-saturated and $\mathbf{t}$-bounded. Therefore by Proposition 4.14 G is $\mathbf{t}$-projective.

In the general case, if $\mathbf{T}(G)$ is totally ordered then it contains a unique maximal type $\mathbf{t}$. Then by Proposition 5.24 $G = G_{\mathbf{t}} \oplus G'$ where $G_{\mathbf{t}}$ is $\mathbf{t}$-projective and $\mathbf{t} \notin \mathbf{T}(G')$. Thus $\mathbf{T}(G') \subsetneq \mathbf{T}(G)$ (since $G' \lhd G$), so by induction G' is completely decomposable. Thus G is completely decomposable. $\boxed{\checkmark}$

EXAMPLE 5.26. In Example 1.49 we started with two rank-one modules A and B such that $\mathbf{t}(A)$ and $\mathbf{t}(B)$ are incomparable, and constructed an indecomposable module H as the submodule of $QA \oplus QB$ generated by $A \oplus B$ together with $(a+b)/w$ for a certain $a \in A$ and $b \in B$ and $w \neq 0 \in W$. This construction will not produce an indecomposable module if $\mathbf{t}(A)$ and $\mathbf{t}(B)$ are not incomparable. In fact, since G is then quasi-equal to the completely decomposable module $A \oplus B$ it is a Butler module, and $\mathbf{T}(G) = \{\mathbf{t}(A), \mathbf{t}(B), \mathbf{t}(A) \wedge \mathbf{t}(B)\}$. if then $\mathbf{t}(A) \leq \mathbf{t}(B)$ or $\mathbf{t}(B) \leq \mathbf{t}(A)$ then by Corollary 5.25 G must be completely decomposable and thus not indecomposable.

More specifically, consider the example where W is a principal ideal domain and p and q are distinct primes and $w \in W$, $w \notin p \cup q$. Let A and B be the submodules of $Q \oplus Q$ defined by $A = (p^{-\infty}, 0)$ and $B = (0, p^{-\infty}q^{-\infty})$. Let G be the submodule of $Q \oplus Q$ generated by $A \oplus B$ together with $(1,1)/w$.

Now let A' be the submodule of G generated by $p^{-\infty}(1,1)$ together with w^{-1}. (By the existence of partial fraction expansions (Lemma 1.28) $A' = p^{-1}(1,1)/w$.) Then $A' \cap B = 0$ and since $A \subseteq p^{-\infty}(1,1) \oplus B$, $A \subseteq A' \oplus B \subseteq G$. Since $(1,1)/w \in A' \oplus B$ it follows that $G = A' \oplus B$, confirming that G is completely decomposable. $\boxed{\checkmark}$

We can now generalize Lemma 5.2 to a general existence theorem for Butler modules.

PROPOSITION 5.27. *Let $\mathbf{T}$ be a finite set of types such that for each $\mathbf{t} \in \mathbf{T}$, $\mathbf{t}$ is not the least upper bound of the set of types in $\mathbf{T}$ strictly less than $\mathbf{t}$. Let U be a finite dimensional Q-space and for each $\mathbf{t} \in \mathbf{T}$ let $U_{\mathbf{t}}$ be a subspace of U such that if $\mathbf{s} \leq \mathbf{t}$ then $U_{\mathbf{s}} \supseteq U_{\mathbf{t}}$. Then there exists a Butler module G such that $QG = U$ and for each $\mathbf{t} \in \mathbf{T}$, $QG(\mathbf{t}) = U_{\mathbf{t}}$. Furthermore G is unique up to quasi-equality.* **FIX THIS!**

PROOF: It follows from Proposition 5.9 that if such a Butler module G exists then it is unique up to quasi-equality. Now let $\mathbf{t}_0 = \mathbf{inf}\, T$ and let C_0 be an essential $\mathbf{t}_0$-projective submodule of U. And for each $\mathbf{t} \in \mathbf{T}$ let $C_{\mathbf{t}}$ be an essential $\mathbf{t}$-projective submodule of $U_{\mathbf{t}}$. By Proposition 4.15 each $C_{\mathbf{t}}$ is well defined up to quasi-equality. Now let G be the module generated by C_0 and all the $C_{\mathbf{t}}$. Then G is generated by a finite set of rank-one modules since this is true of each $C_{\mathbf{t}}$, so by Proposition 5.7 G is a Butler module. Clearly $QG = U$. We claim that for all $\mathbf{t} \in T$, $C_{\mathbf{t}}$ is a quasi-pure submodule of G and $QG(\mathbf{t}) = QC_{\mathbf{t}}$.

**** Thus WLOG we may assume that $C_{\mathbf{t}} \lhd G$. Certainly it is clear that $C_{\mathbf{t}} \subseteq G(\mathbf{t})$. Conversely, consider the quotient map $\gamma\colon G \to G/C_{\mathbf{t}}$. By Proposition 4.2 $\gamma(G(\mathbf{t})) \subseteq (G/C_{\mathbf{t}})(\mathbf{t})$. Thus to show that $G(\mathbf{t}) \subseteq C_{\mathbf{t}}$ it suffices to show that $(G/C_{\mathbf{t}})(\mathbf{t}) = 0$. Now by hypothesis $QC_{\mathbf{s}} \subseteq QC_{\mathbf{t}}$ whenever $\mathbf{s} \geq \mathbf{t}$. Thus $(G/C_{\mathbf{t}})$ is a homomorphic image of $\bigoplus_{\mathbf{s} \not\geq \mathbf{t}} C_{\mathbf{s}}$, and so by Proposition 5.22 $\mathbf{T}(G/C_{\mathbf{t}})$ is contained in the lattice generated by $\{\mathbf{s} \in \mathbf{T} \mid \mathbf{s} \not\geq \mathbf{t}\}$. ****

CRITICAL TYPES. According to Corollary 5.9, if G is a Butler module then we know G up to quasi-equality once we know the finite family of subspaces $QG(\mathbf{t})$ of QG, for $\mathbf{t} \in \mathbf{T}(G)$. And in fact a closer examination of the proof of Corollary 5.9 shows that we don't even need to know all the $QG(\mathbf{t})$, but only certain "critical" ones (see Proposition 5.* below).

DEFINITION 5.28. A **critical type** for a Butler module G is a type $\mathbf{t} \in \mathbf{T}(G)$ such that $G^*(\mathbf{t})$ is not an essential submodule of $G(\mathbf{t})$. We write $\mathbf{T}'(G)$ to denote the set of critical types of G.

LEMMA 5.29. *If G is a Butler module and $\mathbf{t} = \mathbf{IT}(G)$ then $G^*(\mathbf{t})_* = G[\mathbf{t}]$.*

PROOF: By Proposition 4.27 $G^*(\mathbf{t})_* \subseteq G[\mathbf{t}]$ (since $G[\mathbf{t}] \lhd G$). On the other hand, since G is a Butler module it is generated by a finite set of rank-one pure submodules. If $\mathbf{t} = \mathbf{IT}(G)$ then all these rank-one submodules have type at least $\mathbf{t}$, and the ones with types greater than $\mathbf{t}$ belong to $G^*(\mathbf{t})_*$. Thus $G/G^*(\mathbf{t})_*$ is generated by a finite set of rank-one modules with type $\mathbf{t}$ and thus is a homomorphic image of the direct sum of these rank-one modules, hence is $\mathbf{t}$-projective by Proposition 4.16, and in particular is $\mathbf{t}$-bounded. It then follows from Proposition 4.10 that $G[\mathbf{t}] \subseteq G^*(\mathbf{t})_*$. ☑

PROPOSITION 5.30. *Let G be a Butler module.*

1. *For every type $\mathbf{t}$, $G^*(\mathbf{t})$ is a quasi-pure submodule of G and there exists a $\mathbf{t}$-projective pure submodule $G_{\mathbf{t}}$ of G such that $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$.*
2. *$\mathbf{t}$ is a critical type of G if and only if $G_{\mathbf{t}} \neq 0$.*
3. *If $\mathbf{t} \in \mathbf{T}(G)$ and $\mathbf{t} < \mathbf{inf}\{\mathbf{s} \in \mathbf{T}(G) \mid \mathbf{s} > \mathbf{t}\}$, then $\mathbf{t}$ is critical. In particular, all maximal elements of $\mathbf{T}(G)$ are critical.*

PROOF: (1) $G^*(\mathbf{t})$ is a homomorphic image of $\bigoplus\{G(\mathbf{s}) \mid \mathbf{s} \in \mathbf{T}(G) \text{ and } \mathbf{s} > \mathbf{t}\}$, where the direct sum is finite since $\mathbf{T}(G)$ is finite. Since $\mathbf{IT}(G(\mathbf{s})) \geq \mathbf{s} > \mathbf{t}$, $\mathbf{IT}(G^*(\mathbf{t})) \geq \mathbf{t}$. Now each $G(\mathbf{s})$ is a Butler module by Proposition 5.20 since $G(\mathbf{s}) \lhd G$. Therefore $G^*(\mathbf{t})$ is a Butler module by Proposition 5.20. Thus by Proposition 5.21 $G^*(\mathbf{t})$ is quasi-pure in G if and only if for all $g \in G^*(\mathbf{t})$, the type $\mathbf{t}'$ of g in $G^*(\mathbf{t})$ is $\mathbf{t}_G(g)$. Now if $\mathbf{t}_G(g) = \mathbf{s} > \mathbf{t}$ then $g \in G(\mathbf{s}) \lhd G$ and $G(\mathbf{s}) \lhd G^*(\mathbf{t})$. Thus by Proposition 2.21 both $\mathbf{t}'$ and $\mathbf{t}_G(g)$ are the same as the type of g in $G(\mathbf{s})$ so $\mathbf{t}' = \mathbf{t}_G(g)$. Thus we need only consider $g \in G^*(\mathbf{t})$ with $\mathbf{t}_G(g) = \mathbf{t}$. But then $\mathbf{t}' \geq \mathbf{IT}(G^*(\mathbf{t})) \geq \mathbf{t}$ and $\mathbf{t}' \leq \mathbf{t}_G(g) = \mathbf{t}$ by Proposition 2.* since $G^*(\mathbf{t}) \subseteq G$. Thus $\mathbf{t}' = \mathbf{t}_G(g)$ in this case as well. It follows that $G^*(\mathbf{t})$ is a quasi-pure submodule of G.

Now since $G^*(\mathbf{t})_* \subseteq G(\mathbf{t})$ there is no loss of generality in supposing that $G = G(\mathbf{t})$, i.e. that $\mathbf{t} = \mathbf{IT}(G)$. Then by Proposition 4.23 that $G = G_{\mathbf{t}} \oplus G[\mathbf{t}]$ for some $\mathbf{t}$-projective module $G_{\mathbf{t}}$. Since $G[\mathbf{t}] = G^*(\mathbf{t})_*$ by Lemma 5.29, the result follows.

(2) By definition, $\mathbf{t} \in \mathbf{T}'(G)$ if and only if $G^*(\mathbf{t})_* \neq G(\mathbf{t})$, i.e. if and only if $G_{\mathbf{t}} \neq 0$.

(3) Suppose that $\mathbf{t} < \mathbf{inf}\{\mathbf{s} \in \mathbf{T}(G) \mid \mathbf{s} > \mathbf{t}\}$ and let $g \in G$ with $\mathbf{t}(g) = \mathbf{t}$. We claim that $g \notin G^*(\mathbf{t})_*$, which will prove that $G(\mathbf{t}) \neq G^*(\mathbf{t})_*$, so that $\mathbf{t} \in \mathbf{T}'(G)$. In fact, if $g \in G^*(\mathbf{t})_*$ then $(\exists w \neq 0 \in W)\, wg = \sum_I g_i$ with $\mathbf{t}(g_i) > \mathbf{t}$. Then by Proposition 2.* $\mathbf{t}(g) = \mathbf{t}(wg) \geq \mathbf{inf}\{\mathbf{t}(g_i) \mid i \in I\}$, contradicting the hypothesis on $\mathbf{t}$. ☑

PROPOSITION 5.31. (1) *If G and H are Butler modules and $G \sim H$, then $\mathbf{T}'(G) = \mathbf{T}'(H)$.*

(2) *If $G = \bigoplus_1^n A_i$ where the A_i are rank-one modules, then $\mathbf{T}'(G) = \{\mathbf{t}(A_i) \mid i = 1, \ldots, n\}$.*

PROOF: (2) If $G = \bigoplus_1^n A_i$ then by Proposition 4.* $G(\mathbf{t}) = \bigoplus_{\mathbf{t}(A_i) \geq \mathbf{t}} A_i$ and $G^*(\mathbf{t}) = \bigoplus_{\mathbf{t}(A_i) > \mathbf{t}} A_i$. Thus $G^*(\mathbf{t})$ is an essential submodule of $G(\mathbf{t})$ if and only if $G^*(\mathbf{t}) = G(\mathbf{t})$, and thus if and only if there exists A_i with $\mathbf{t}(Ai) \geq \mathbf{t}$ but $\mathbf{t}(Ai) \not> \mathbf{t}$, i.e. if and only if $\mathbf{t} = \mathbf{t}(A_i)$ for some i. Thus $\mathbf{T}'(G) = \{\mathbf{t}(A_1), \ldots, \mathbf{t}(A_n)\}$.

(1) If $G \sim H$ then from Proposition 4.2 it follows that $G(\mathbf{t}) \sim H(\mathbf{t})$ and $G^*(\mathbf{t}) \sim H^*(\mathbf{t})$. Thus if $\mathbf{t} \in \mathbf{T}'(G)$, so that $\operatorname{rank} G^*(\mathbf{t}) < \operatorname{rank} G(\mathbf{t})$, then $\operatorname{rank} H^*(\mathbf{t}) < \operatorname{rank} H(\mathbf{t})$, so $\mathbf{t} \in \mathbf{T}'(H)$. ☑

THEOREM 5.32. *If for each $\mathbf{t} \in \mathbf{T}'(G)$ a $\mathbf{t}$-projective submodule $G_{\mathbf{t}}$ of G is chosen as in Proposition 5.30 then G is quasi-equal to $\sum_{\mathbf{T}'(G)} G_{\mathbf{t}}$. Furthermore the map $\bigoplus_{\mathbf{T}'(G)} G_{\mathbf{t}} \to G$ induced by the inclusions $G_{\mathbf{t}} \subseteq G$ is a balanced quasi-surjection.*

PROOF: We will prove by downward induction on $\mathbf{t} \in \mathbf{T}(G)$ that $\sum\{G_{\mathbf{s}} \mid \mathbf{s} \in \mathbf{T}'(G) \text{ and } \mathbf{s} \geq \mathbf{t}\}$ for is quasi-equal to $G(\mathbf{t})$.

If $\mathbf{t}$ is maximal in $\mathbf{T}(G)$ then by Proposition 5.30 $\mathbf{t} \in \mathbf{T}'(G)$. Furthermore there are no $g \neq 0 \in G$ with $\mathbf{t}(g) > \mathbf{t}$ so $G^*(\mathbf{t}) = 0$, and so by Proposition 5.30 $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_* = G_{\mathbf{t}}$, establishing the claim in this case.

Now suppose that for all $\mathbf{t}' \in \mathbf{T}(G)$ with $\mathbf{t}' > \mathbf{t}$ it has been proved that $G(\mathbf{t}')$ is quasi-equal to $\sum_{\mathbf{s} \geq \mathbf{t}'} G_{\mathbf{s}}$. It follows that $G^*(\mathbf{t}) = \sum_{\mathbf{t}' > \mathbf{t}} G(\mathbf{t}')$ is quasi-equal to $\sum_{\mathbf{s} > \mathbf{t}} G_{\mathbf{s}}$ (using the fact that by Corollary 5.22 there are only finitely many $\mathbf{t}' \in \mathbf{T}(G)$.) Since

by Proposition 5.30 $G^*(\mathbf{t})$ is a quasi-pure submodule of G, it follows that $G^*(\mathbf{t})_*$ is quasi-equal to $\sum_{\mathbf{s}>\mathbf{t}} G_{\mathbf{s}}$. Thus $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$ is quasi-equal to $\sum_{\mathbf{s}\geq\mathbf{t}} G_{\mathbf{s}}$.

Since $\mathbf{T}(G)$ is finite, we finally conclude that for $\mathbf{t} = \mathbf{IT}(G)$, $G = G(\mathbf{t})$ is quasi-equal to the submodule of G generated by all $G_{\mathbf{t}}$ for $\mathbf{t} \in \mathbf{T}'(G)$.

Now let $\varphi\colon \bigoplus G_{\mathbf{t}} \to G$ be the map induced by the inclusions $G_{\mathbf{t}} \subseteq G$. Since for any type $\mathbf{t}$, $(\bigoplus G_{\mathbf{t}'})(\mathbf{t}) = \bigoplus_{\mathbf{t}'\geq\mathbf{t}} G_{\mathbf{t}'}$, the assertion that φ is a balanced quasi-surjection is simply the claim that for each $\mathbf{t}$, $G(\mathbf{t})$ is quasi-equal to the submodule generated by all $G_{\mathbf{t}'}$ for $\mathbf{t}' \geq \mathbf{t}$. But this has been shown already. ☑

PROPOSITION 5.33. *If G is a Butler module and $\mathbf{t}$ a type, then $G[\mathbf{t}]$ is equal to the pure closure of the submodule generated by the set of $G_{\mathbf{s}}$ such that $\mathbf{s} \in \mathbf{T}'(G)$ and $\mathbf{s} \not\leq \mathbf{t}$.*

PROOF: Let $G\langle\mathbf{t}\rangle$ be the pure submodule generated by the set of $G_{\mathbf{s}}$ such that $\mathbf{s} \in \mathbf{T}'(G)$ and $\mathbf{s} \not\leq \mathbf{t}$. By Proposition 4.4 $\mathbf{s} \not\leq \mathbf{t} \Rightarrow G(\mathbf{s}) \subseteq G[\mathbf{t}]$, so a fortiori $G_{\mathbf{s}} \subseteq G[\mathbf{t}]$. Thus $G\langle\mathbf{t}\rangle \subseteq G[\mathbf{t}]$. To see that $G[\mathbf{t}] \subseteq G\langle\mathbf{t}\rangle$ it suffices by Proposition 4.10 to see that $G/G\langle\mathbf{t}\rangle$ is $\mathbf{t}$-bounded. But by Proposition 5.32 G is quasi-equal to the submodule generated by all $G_{\mathbf{s}}$, and so $G/G\langle\mathbf{t}\rangle$ is quasi-generated by the images of these submodules, and clearly we need use only with $\mathbf{s} \leq \mathbf{t}$, since all other $G_{\mathbf{s}}$ are contained in $G\langle\mathbf{t}\rangle$. Since each such $G_{\mathbf{s}}$ is $\mathbf{s}$-projective and thus $\mathbf{t}$ bounded, $G/G\langle\mathbf{t}\rangle$ is $\mathbf{t}$-bounded by Proposition 4.10. Thus $G[\mathbf{t}] \subseteq G\langle\mathbf{t}\rangle$ and so $G[\mathbf{t}] = G\langle\mathbf{t}\rangle$. ☑

COROLLARY 5.34. *If $\mathbf{t}$ is minimal in $\mathbf{T}'(G)$ then $(G(\mathbf{t}) + G[\mathbf{t}])_* = G$.*

PROOF: If $\mathbf{s} \neq \mathbf{t} \in \mathbf{T}'(G)$ then since $\mathbf{t}$ is minimal in $\mathbf{T}'(G)$, $\mathbf{s} \not\leq \mathbf{t}$, so by Proposition 5.33 $G_{\mathbf{s}} \subseteq G[\mathbf{t}]$. Since $G_{\mathbf{t}} \subseteq G(\mathbf{t})$, $G(\mathbf{t}) + G[\mathbf{t}]$ thus contains $G_{\mathbf{s}}$ for all $\mathbf{s} \in \mathbf{T}'(G)$. Thus by Proposition 5.32 $G(\mathbf{t}) + G[\mathbf{t}]$ is quasi-equal to G. ☑

Recall that if $\mathbf{t}$ is a type then $\mathbf{t}\text{-}\operatorname{rank} G = \operatorname{rank} G/G[\mathbf{t}]$.

COROLLARY 5.35. *Let G be a Butler module and $\mathbf{t}$ a type.*

1. *G has no non-trivial $\mathbf{t}$-projective quasi-summand if and only if the set all $G_{\mathbf{s}}$ for $\mathbf{s} \in \mathbf{T}'(G)$ and $\mathbf{s} \neq \mathbf{t}$ generates an essential submodule of G.*
2. *If $\mathbf{t}$ is minimal in $\mathbf{T}'(G)$ then $\mathbf{t}\text{-}\operatorname{rank} G$ is the rank of a maximal $\mathbf{t}$-projective quasi-summand of G.*

PROOF: (2) By Proposition 4.23 the rank of a maximal $\mathbf{t}$-projective quasi-summand of G equals $\operatorname{rank}(G(\mathbf{t}) + G[\mathbf{t}])/G[\mathbf{t}]$. But if $\mathbf{t}$ is minimal in $\mathbf{T}'(G)$ then by Proposition 4.29 $G(\mathbf{t}) + G[\mathbf{t}]$ is quasi-equal to G. Thus the rank of a maximal $\mathbf{t}$-projective quasi-summand of G equals $\operatorname{rank} G/G[\mathbf{t}] = \mathbf{t}\text{-}\operatorname{rank} G$.

(1) ($\Leftarrow$): If G is quasi-equal to $C \oplus K$, where C is $\mathbf{t}$-projective, then for all $\mathbf{s} \neq \mathbf{t}$, $G_{\mathbf{s}}$ can be chosen so that $G_{\mathbf{s}} \subseteq K$ (WHY?). Thus if $C \neq 0$ then the submodule generated by all $G_{\mathbf{s}}$ is not essential in G.

($\Rightarrow$): By Proposition 4.23 G has no quasi-summand of type $\mathbf{t}$ if and only if $G(\mathbf{t}) \subseteq G[\mathbf{t}]$. But if H is the pure closure of the submodule generated by all $G_{\mathbf{s}}$ with $\mathbf{s} \neq \mathbf{t}$ then by Proposition 5.33 $G[\mathbf{t}] \subseteq H$. It follows that if G has no quasi-summand of type $\mathbf{t}$ then $G(\mathbf{t}) \subseteq H$, in which case $G_{\mathbf{t}} \subseteq H$ and so by the above H is quasi-equal to G, and so $H = G$ since $H \lhd G$. ☑

COCRITICAL TYPES. The dual of Proposition 5.30 is equally valid, although less used.

DEFINITION 5.36. A **cocritical type** for a Butler module G is a type $\mathbf{t} \in \mathbf{CT}(G)$ such that $G^*[\mathbf{t}] \neq G[\mathbf{t}]$. We write $\mathbf{CT}'(G)$ for the set of cocritical types of G.

PROPOSITION 5.37. *Let G be a Butler module.*

(1) *If $\mathbf{t} = \mathbf{OT}(G)$ then $G(\mathbf{t})$ is a $\mathbf{t}$-projective direct summand of G, and $G(\mathbf{t}) = G^*[\mathbf{t}]$.*
(2) *If $\mathbf{t} \in \mathbf{CT}(G)$ and $\mathbf{t} > \mathbf{sup}\{\mathbf{s} \in \mathbf{CT}(G) \mid \mathbf{s} < \mathbf{t}\}$, then $\mathbf{t}$ is cocritical. In particular, if $\mathbf{t}$ is minimal in $\mathbf{CT}(G)$ then $\mathbf{t}$ is cocritical.*

PROOF: (1) If $\mathbf{t} = \mathbf{OT}(G)$ then G is $\mathbf{t}$-bounded, and $G(\mathbf{t})$ is by definition $\mathbf{t}$-saturated. Thus by Proposition 4.21 $G(\mathbf{t})$ is a direct summand of the $\mathbf{t}$-bounded module G, say $G = G(\mathbf{t}) \oplus H$. By Proposition 4.27 $G(\mathbf{t}) \subseteq G^*[\mathbf{t}]$. Now by Theorem 5.10 G is a pure submodule of a completely decomposable module $\bigoplus A_i$. And if $g \in G$ and $g \notin G(\mathbf{t})$ then by Proposition 2.32 $\mathbf{t}(g) < \mathbf{OT}(G) = \mathbf{t}$, and hence by Proposition 2.21 g must have a non-trivial coordinate in some A_i with $\mathbf{t}(A_i) < \mathbf{t}$. If μ_i is the restriction to G of the projection $\bigoplus A_i \to A_i$, then $\mu_i(g) \neq 0$, so that $g \notin G[\mathbf{t}(A_i)]$ and so $g \notin G^*[\mathbf{t}]$. Thus $G^*[\mathbf{t}] \subseteq G(\mathbf{t})$.

(2) Let $\mu\colon G \to A$ with $A = \mu(G)$ and $\mathbf{t}(A) = \mathbf{t}$. Then $\mu(G[\mathbf{t}]) = 0$. Suppose $G^*[\mathbf{t}] = G[\mathbf{t}]$. Then for $g \notin G[\mathbf{t}]$, $g \notin G^*[\mathbf{t}]$ so that there exists $\mathbf{s} < \mathbf{t}$ with $g \notin G[\mathbf{s}]$, and thus there exists $A' \subset A$ with $\mathbf{t}(A') = \mathbf{s} < \mathbf{t}(A)$ and $\mu'\colon G \to A'$ with $\mu'(g) \neq 0$. Then $A = \mu(G)$ is generated by such A', and in fact by finitely many of them (since G is generated by finitely many of its rank-one submodules). Thus $\mathbf{t} = \mathbf{t}(A)$ is the least upper bound of such $\mathbf{t}(A')$. This contradicts the assumption on $\mathbf{t}$, so $G^*[\mathbf{t}] \neq G[\mathbf{t}]$ and $\mathbf{t}$ is cocritical. ☑

THEOREM 5.38. *For each $\mathbf{t} \in \mathbf{CT}'(G)$, $G^*[\mathbf{t}]/G[\mathbf{t}]$ is a direct summand of $G/G[\mathbf{t}]$. Let $\beta_{\mathbf{t}}\colon G \to G^*[\mathbf{t}]/G[\mathbf{t}]$ be the composition of the quotient mapping $G \to G/G[\mathbf{t}]$ with a projection of $G/G[\mathbf{t}]$ onto $G^*[\mathbf{t}]/G[\mathbf{t}]$. Let*

$$\beta\colon G \to \bigoplus_{\mathbf{CT}'(G)} G^*[\mathbf{t}]/G[\mathbf{t}]$$

be given by the product of the maps $\beta_{\mathbf{t}}$. Then β is a monomorphism mapping G onto a cobalanced pure submodule of $\bigoplus G^*[\mathbf{t}]/G[\mathbf{t}]$.

PROOF: Let $\mathbf{t} \in \mathbf{CT}'(G)$. Then $\mathbf{OT}(G/G[\mathbf{t}]) \leq \mathbf{t}$ since $G/G[\mathbf{t}]$ is $\mathbf{t}$-bounded and on the other hand $\mathbf{t} \in \mathbf{CT}(G/G[\mathbf{t}])$ since $\mathbf{t} \in \mathbf{CT}(G)$. Thus $\mathbf{t} = \mathbf{OT}(G/G[\mathbf{t}])$ and it follows from Proposition 4.2 that $G^*[\mathbf{t}]/G[\mathbf{t}] = (G/G[\mathbf{t}])^*[\mathbf{t}]$. Thus by Proposition 5.37 $G^*[\mathbf{t}]/G[\mathbf{t}]$ is a summand of $G/G[\mathbf{t}]$.

Now for each $\mathbf{t} \in \mathbf{CT}'(G)$ let $C_{\mathbf{t}} = G^*[\mathbf{t}]/G[\mathbf{t}]$ and let $C = \bigoplus_{\mathbf{CT}'(G)} C_{\mathbf{t}}$. For each $\mathbf{t} \in \mathbf{CT}(G)$, let

$$\tilde{\beta}_{\mathbf{t}}\colon G/G[\mathbf{t}] \to \bigoplus_{\mathbf{s} \leq \mathbf{t}} C_{\mathbf{s}}$$

be the map induced by $\beta_{\mathbf{s}}$ for $\mathbf{s} \leq \mathbf{t}$. We will show by induction on $\mathbf{t}$ that $\tilde{\beta}$ is monic with pure image. If $\mathbf{t}$ is minimal in $\mathbf{CT}(G)$, then $\tilde{\beta}_{\mathbf{t}} = \beta_{\mathbf{t}}$ and $G^*[\mathbf{t}] = G$ and $C_{\mathbf{t}} = G^*[\mathbf{t}]/G[\mathbf{t}] = G/G[\mathbf{t}]$. Thus $\tilde{\beta}_{\mathbf{t}}$ is the identity, hence is a pure monomorphism. Now suppose that it has been shown that $\tilde{\beta}_{\mathbf{s}}$ is a pure monomorphism for all $\mathbf{s} \in \mathbf{CT}(G)$ with $\mathbf{s} < \mathbf{t}$. Let p be a prime and let $g \in G$ with $g + G[\mathbf{t}] \notin p(G/G[\mathbf{t}])$. If $g \in G[\mathbf{s}]$ for all $\mathbf{s}$ with $\mathbf{s} < \mathbf{t}$ then $g \in G^*[\mathbf{t}]$ and so $\beta_{\mathbf{t}}(g) = g + G[\mathbf{t}] \notin pC_{\mathbf{t}}$ and so $\tilde{\beta}(g) \notin \bigoplus_{\mathbf{s} \leq \mathbf{t}} pC_{\mathbf{s}}$. On the other hand, if $g \notin G[\mathbf{s}]$ for some $\mathbf{s} < \mathbf{t}$, choose $\mathbf{s}$ to be minimal with this property. Then $g \in G^*[\mathbf{s}]$ and so $\beta_{\mathbf{s}}(g) = g + G[\mathbf{s}] \in G^*[\mathbf{s}]/G[\mathbf{s}]$. Now **** In both cases it follows that $\beta(g) \notin pC$.

The fact that all the induced maps $G/G[\mathbf{t}] \to C/C[\mathbf{t}]$ are monic shows that β is cobalanced, provided that it is a pure monomorphism. But this in turn follows by applying the above for $\mathbf{t} = \mathbf{CT}(G)$, since in that case $G/G[\mathbf{t}] = G$ and $C/C[\mathbf{t}] = C$. ☑

PROPOSITION 5.39. *If G is a Butler module then $\mathbf{T}(G)$ and $\mathbf{CT}(G)$ are contained in the lattice generated by either $\mathbf{T}'(G)$ or $\mathbf{CT}'(G)$.*

PROOF: By Proposition 5.32 G is a balanced homomorphic image of $C = \bigoplus G_{\mathbf{t}}$, where $\mathbf{t}$ ranges over $\mathbf{T}'(G)$ and $G_{\mathbf{t}}$ is $\mathbf{t}$-projective. Then by Proposition 2.* $\mathbf{T}(C)$ is contained in the lattice generated by $\mathbf{T}'(G)$. But by Corollary 5.23 and Corollary 5.14 $\mathbf{T}'(G) \subseteq \mathbf{T}(G) \subseteq \mathbf{T}(C)$. This shows that $\mathbf{T}(G)$ and $\mathbf{T}'(G)$ generated the same lattice.

Analogously since G is a balanced pure submodule of $D = \bigoplus \bar{G}_{\mathbf{t}}$, where $\mathbf{t}$ ranges over $\mathbf{CT}'(G)$ and $\bar{G}_{\mathbf{t}}$ is $\mathbf{t}$-projective, it follows that $\mathbf{CT}'(G) \subseteq \mathbf{CT}(G) \subseteq \mathbf{CT}(D)$ and that $\mathbf{CT}(G)$ and $\mathbf{CT}'(G)$ generate the same lattice. But by Proposition 5.23 $\mathbf{T}(G)$ is contained in the lattice generated by $\mathbf{CT}(G)$ and vice-versa. This establishes the Proposition. ☑

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THEOREM 5.40. *Let G be a finite rank torsion free W-module. The following conditions are equivalent:*

(1) *G is a Butler module.*
(2) *$\mathbf{T}(G)$ is finite and for all $\mathbf{t} \in \mathbf{T}(G)$, $G^*(\mathbf{t})$ is a quasi-pure submodule of G and there exists a pure $\mathbf{t}$-projective submodule $G_{\mathbf{t}} \subseteq G$ such that $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$.*
(3) *$\mathbf{CT}(G)$ is finite and for all $\mathbf{t} \in \mathbf{CT}(G)$, the obvious embedding $\delta\colon G/G^*[\mathbf{t}] \to \bigoplus\{G/G[\mathbf{s}] \mid \mathbf{s} \in \mathbf{CT}(G),\ \mathbf{s} < \mathbf{t}\}$ has a quasi-pure image, and $G^*[\mathbf{t}]/G[\mathbf{t}]$ is a [maximal] $\mathbf{t}$-projective direct summand of $G/G[\mathbf{t}]$.*

PROOF: (1) $\Rightarrow$ (2): If G is a Butler module, then G is a pure submodule of a completely decomposable module C and by Proposition 2.33 $\mathbf{T}(G) \subseteq \mathbf{T}(C)$, so $\mathbf{T}(G)$ is finite. And by Proposition 5.30 for each $\mathbf{t}$, $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$ for some $\mathbf{t}$-projective $G_{\mathbf{t}}$. Furthermore $G^*(\mathbf{t})$ is a quasi-pure submodule of G by Proposition 5.30.

(2) $\Rightarrow$ (1): By induction on the size of $\mathbf{T}(G)$. If $\mathbf{T}(G) = \{\mathbf{t}\}$ then $G^*(\mathbf{t}) = 0$ and by so hypothesis $G = G_{\mathbf{t}}$ where $G_{\mathbf{t}}$ is $\mathbf{t}$-projective and hence a Butler module.

Now note that if $\mathbf{t} \in \mathbf{T}(G)$ then $\mathbf{T}(G(\mathbf{t})) = \{\mathbf{s} \in \mathbf{T}(G) \mid \mathbf{s} \geq \mathbf{t}\}$ and that if $\mathbf{s} \geq \mathbf{t}$ then by Proposition 4.* $G(\mathbf{t})(\mathbf{s}) = G(\mathbf{s} \wedge \mathbf{t}) = G(\mathbf{s})$. It follows that if G

satisfies the stated hypothesis then so does $G(\mathbf{t})$. Furthermore since $G(\mathbf{t}) \lhd G$, $\operatorname{rank} G(\mathbf{t}) < \operatorname{rank} G$ unless $G(\mathbf{t}) = G$, i.e. unless $\mathbf{t} = \mathbf{IT}(G)$. Thus by induction on rank $G(\mathbf{t})$ is a Butler module except possibly for $\mathbf{t} = \mathbf{IT}(G)$. But by assumption if $\mathbf{t} = \mathbf{IT}(G)$ then $G = G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$, where $G_{\mathbf{t}}$ is $\mathbf{t}$-projective (since $\mathbf{IT}(G) \in \mathbf{T}(G)$ by Proposition 2.25). Thus if suffices to prove that for $\mathbf{t} = \mathbf{IT}(G)$, $G^*(\mathbf{t})_*$ is a Butler module. But by hypothesis $G^*(\mathbf{t})_*$ is quasi-equal to $G^*(\mathbf{t})$ and $G^*(\mathbf{t})$ is generated by $G(\mathbf{s})$ for a finite number of $\mathbf{s} > \mathbf{t}$, and hence is a homomorphic image of $\bigoplus_{\mathbf{s}} G(\mathbf{s})$. As we have seen, each $G(\mathbf{s})$ is a Butler module by induction on rank, so G is a Butler module since homomorphic images of Butler modules are Butler modules by Corollary 5.20.

(1) $\Rightarrow$ (3): If G is a Butler module, then G is a homomorphic image of a completely decomposable module C and $\mathbf{CT}(G) \subseteq \mathbf{CT}(C)$, therefore $\mathbf{CT}(G)$ is finite. For the other two claims, we may assume WLOG that $G[\mathbf{t}] = 0$. It then follows from Proposition 5.37 that $G^*[\mathbf{t}] = G(\mathbf{t})$ is $\mathbf{t}$-projective and a summand of G. Clearly it is maximal as such, since any $\mathbf{t}$-projective submodule of G must be contained in $G(\mathbf{t})$.

Now for $\mathbf{s} \in \mathbf{CT}(G)$ and $\mathbf{s} < \mathbf{t}$ write $C_{\mathbf{s}} = G/G[\mathbf{s}]$. In considering the embedding $G/G^*[\mathbf{t}] \to \bigoplus C_{\mathbf{s}}$, we may assume WLOG that $G^*[\mathbf{t}] = 0$. Thus $\mathbf{t} \notin \mathbf{CT}'(G)$ and since $G[\mathbf{t}] = 0$, G is $\mathbf{t}$-bounded. Thus $\mathbf{s} < \mathbf{t}$ for every $\mathbf{s} \in \mathbf{CT}'(G)$. Now by Proposition 5.* G can be embedded as a cobalanced pure submodule of a completely decomposable module $\bigoplus A_i$. Furthermore since **** we may assume that for all the rank-one modules A_i, $\mathbf{t}(A_i) < \mathbf{t}$. Now let p be a prime and $g \in G$ with $g \notin pG$. Then since $G \lhd \bigoplus A_i$, there is at least one i such that the coordinate of g in A_i is not in pA_i. Thus if $\mu_i \colon G \to A_i$ is the restriction to G of the projection onto A_i then $\mu_i(g) \notin pA_i$. But if $\mathbf{s} = \mathbf{t}(A_i)$ then μ_i induces a map $G/G[\mathbf{s}] \to A_i$. It follows that the image of g in $G/G[\mathbf{s}]$ is not in $p(G/G[\mathbf{s}]) = pC_{\mathbf{s}}$. Thus $\delta(g) \notin p(\bigoplus C_{\mathbf{s}})$. This shows that $\delta(G) \lhd \bigoplus C_{\mathbf{s}}$.

(3) $\Rightarrow$ (1): Note that if $\mathbf{t} \in \mathbf{CT}(G)$ then $\mathbf{CT}(G/G[\mathbf{t}]) = \{\mathbf{s} \in \mathbf{CT}(G) \mid \mathbf{s} \leq \mathbf{t}\}$ and that if $\mathbf{s} \leq \mathbf{t}$ then by Proposition 4.* $(G/G[\mathbf{t}])([\mathbf{s}]) = G[\mathbf{s}]/G[\mathbf{t}]$ and thus $(G/G[\mathbf{t}])^*[\mathbf{s}] = G^*[\mathbf{s}]/G[\mathbf{t}]$. It follows that if G satisfies the stated conditions then so does $G/G[\mathbf{t}]$ and it follows that $G/G[\mathbf{t}]$ is a Butler module by induction on rank except possibly in the case $G[\mathbf{t}] = 0$, i.e. where $\mathbf{t} = \mathbf{OT}(G)$.

Now by assumption, if $\mathbf{t} = \mathbf{OT}(G)$ then $G = G/G[\mathbf{t}] \approx G^*[\mathbf{t}] \oplus (G/G^*[\mathbf{t}])$, where $G^*[\mathbf{t}] = G^*[\mathbf{t}]/G[\mathbf{t}]$ is $\mathbf{t}$-projective (since by Proposition 2.29 $\mathbf{OT}(G) \in \mathbf{CT}(G)$). It thus suffices to prove that if $\mathbf{t} = \mathbf{OT}(G)$ then $G/G^*[\mathbf{t}]$ is a Butler module. But by hypothesis $G/G^*[\mathbf{t}]$ is isomorphic to a quasi-pure submodule of $\bigoplus_{\mathbf{s}<\mathbf{t}} G/G[\mathbf{s}]$, and as we have seen each $G/G[\mathbf{s}]$ is a Butler module by induction. Thus the result follows from the fact that quasi-pure submodules of Butler modules are Butler modules (Proposition 5.20). ☑

It is customary in presenting the following result to mention that it corrects a misprint in [Lady]. As a measure of how confusing the result is, your author feels compelled to confess that he actually stated it correctly in the original manuscript of and then erroneously "corrected" it while reading the galley proofs.

PROPOSITION 5.41. *Let G be a Butler module and $\mathbf{t}$ a type. Then $G(\mathbf{t})$ is the intersection of all $G[\mathbf{s}]$ with $\mathbf{s} \in \mathbf{CT}(G)$ and $\mathbf{s} \not\geq \mathbf{t}$.*

PROOF: Let $G\langle\mathbf{t}\rangle = \bigcap_{\mathbf{s}\not\geq\mathbf{t}} G[\mathbf{s}]$. By Proposition 4.4 if $\mathbf{s} \not\geq \mathbf{t}$ then $G(\mathbf{t}) \subseteq G[\mathbf{s}]$. Thus $G(\mathbf{t}) \subseteq G\langle\mathbf{t}\rangle$. Conversely, to see that $G\langle\mathbf{t}\rangle \subseteq G(\mathbf{t})$ it suffices to see that

$G\langle\mathbf{t}\rangle$ is $\mathbf{t}$-saturated. Now by Proposition 5.* G is a pure cobalanced submodule of a completely decomposable module C and so if we define $C\langle\mathbf{t}\rangle$ analogously to $G\langle\mathbf{t}\rangle$ then the fact that G is cobalanced in C implies that $G\langle\mathbf{t}\rangle = G \cap C\langle\mathbf{t}\rangle$ and by Proposition 4.2 the fact that G is pure in C implies that $G(\mathbf{t}) = G \cap C(\mathbf{t})$. Thus it suffices to see that $C(\mathbf{t}) = C\langle\mathbf{t}\rangle$. Now if $C = \bigoplus A_i$ where the A_i are rank-one modules, then by Lemma 4.28 $C(\mathbf{t}) = \bigoplus\{A_i \mid \mathbf{t}(A_i) \geq \mathbf{t}\} = \bigoplus A_i(\mathbf{t})$. And for $\mathbf{s} \not\geq \mathbf{t}$, $C[\mathbf{s}] = \bigoplus A_i[\mathbf{s}] = \bigoplus\{A_i \mid \mathbf{t}(A_i) \not\leq \mathbf{s}\}$. Now $A_i \not\subseteq A[\mathbf{s}] = 0$ for $\mathbf{s} = \mathbf{t}(A_i)$ so if $\mathbf{t}(A_i) \not\geq \mathbf{t}$ then $A \not\subseteq C\langle\mathbf{t}\rangle$. Conversely, if $\mathbf{t}(A_i) \geq \mathbf{t}$ and $\mathbf{s} \not\geq \mathbf{t}$ then $\mathbf{s} \not\geq \mathbf{t}(A_i)$ so $A_i[\mathbf{s}] = A_i$, and since this is true for all $\mathbf{s}$ with $\mathbf{s} \not\geq \mathbf{t}$ it follows that $A_i \subseteq C\langle\mathbf{t}\rangle$. Thus $A_i \subseteq C\langle\mathbf{t}\rangle \iff \mathbf{t}(A_i) \geq \mathbf{t}$, and so $C(\mathbf{t}) = C\langle\mathbf{t}\rangle$. $\boxed{\checkmark}$

LEMMA 5.42. *Let G be a Butler module and $\mathbf{t}$ a type, let $G_{\mathbf{t}}$ be such that $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$ (c.f. Proposition 5.*). Let $\bar{G}_{\mathbf{t}} = G^*[\mathbf{t}]/G[\mathbf{t}]$ and $\bar{G}'_{\mathbf{t}} = (G(\mathbf{t}) + G[\mathbf{t}])_*/G[\mathbf{t}]$.*

(1) $G^*(\mathbf{t})_* = G(\mathbf{t})[\mathbf{t}]$.
(2) $G_{\mathbf{t}} = G'_{\mathbf{t}} \oplus (G_{\mathbf{t}} \cap G[\mathbf{t}])$ *for some* $G'_{\mathbf{t}}$.
(3) $G^*(\mathbf{t})_* = G(\mathbf{t}) \cap G[\mathbf{t}]$ *if and only if* $G_{\mathbf{t}} \cap G[\mathbf{t}] = 0$. *In particular this is true if* $\mathbf{t} \notin \mathbf{T}'(G)$.
(4) $\bar{G}'_{\mathbf{t}}$ *is a summand of* $\bar{G}_{\mathbf{t}}$.
(5) $G'_{\mathbf{t}}$ *and* $\bar{G}'_{\mathbf{t}}$ *are* $\mathbf{t}$*-projective modules.*
(6) $G'_{\mathbf{t}} \sim \bar{G}'_{\mathbf{t}}$.

PROOF: (1) We may as well assume that $G = G(\mathbf{t})$. Then by Proposition 5.* $G = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$ for some $G_{\mathbf{t}}$ and so $G^*(\mathbf{t})_* \subseteq G[\mathbf{t}] = G_{\mathbf{t}}[\mathbf{t}] \oplus G^*(\mathbf{t})_*[\mathbf{t}] = G^*(\mathbf{t})_*[\mathbf{t}]$. Thus $G(\mathbf{t})[\mathbf{t}] = G[\mathbf{t}] = G^*(\mathbf{t})_*$.

(2) This is just a refinement of Proposition 4.23. By Proposition 4.27 $G^*(\mathbf{t})_* \subseteq G[\mathbf{t}]$, so that $G(\mathbf{t}) + G[\mathbf{t}] = G_{\mathbf{t}} + G^*(\mathbf{t})_* + G[\mathbf{t}] = G_{\mathbf{t}} + G[\mathbf{t}]$. Since $G_{\mathbf{t}}$ is $\mathbf{t}$-projective by Proposition 4.17 there exists a summand $G'_{\mathbf{t}}$ of $G_{\mathbf{t}}$ such that $G(\mathbf{t}) + G[\mathbf{t}] = G'_{\mathbf{t}} \oplus G[\mathbf{t}]$. Then $G_{\mathbf{t}} = G'_{\mathbf{t}} \oplus (G_{\mathbf{t}} \cap G[\mathbf{t}])$.

(3) Since $G^*(\mathbf{t})_* \subseteq G[\mathbf{t}]$, $G(\mathbf{t}) \cap G[\mathbf{t}] = (G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*) \cap G[\mathbf{t}] = (G_{\mathbf{t}} \cap G[\mathbf{t}]) \oplus G^*(\mathbf{t})_*$. Thus $G(\mathbf{t}) \cap G[\mathbf{t}] = G^*(\mathbf{t})_*$ if and only if $G_{\mathbf{t}} \cap G[\mathbf{t}] = 0$. In particular this is true if $\mathbf{t} \notin \mathbf{T}'(G)$, since in that case $G_{\mathbf{t}} = 0$.

(4) By Proposition 4.23 $(G(\mathbf{t}) + G[\mathbf{t}])_* \lhd G^*[\mathbf{t}]$. Therefore $\bar{G}'_{\mathbf{t}} \lhd \bar{G}_{\mathbf{t}}$. Since $\bar{G}_{\mathbf{t}}$ is $\mathbf{t}$-projective by Theorem 5.40, by Proposition 4.16 $\bar{G}'_{\mathbf{t}}$ is a summand of $\bar{G}_{\mathbf{t}}$.

(5) By Theorem 5.40 $G_{\mathbf{t}}$ and $\bar{G}_{\mathbf{t}}$ are $\mathbf{t}$-projective modules. Since $G'_{\mathbf{t}}$ is a summand of $G_{\mathbf{t}}$, and $\bar{G}'_{\mathbf{t}}$ a summand of $\bar{G}_{\mathbf{t}}$, these are also $\mathbf{t}$-projective.

(6) This was proved in Proposition 4.23. $\boxed{\checkmark}$

PROPOSITION 5.43. *Let G be a Butler module, $\mathbf{t}$ a type, and let $G_{\mathbf{t}}$ be such that $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$ and let $\bar{G}_{\mathbf{t}} = G^*[\mathbf{t}]/G[\mathbf{t}]$.*

(1) *There is a quasi-exact sequence*

$$0 \to (G(\mathbf{t}) \cap G[\mathbf{t}])/G^*(\mathbf{t})_* \to G(\mathbf{t})/G^*(\mathbf{t})_* \xrightarrow{\zeta} G^*[\mathbf{t}]/G[\mathbf{t}] \to G^*[\mathbf{t}]/(G(\mathbf{t}) + G[\mathbf{t}])_* \to 0$$

which is quasi-split in the sense that the image of each map is a quasi-summand of the module that contains it.

(2) $G(\mathbf{t}) \cap G[\mathbf{t}] = G^*(\mathbf{t})_*$ *if and only if* $G_\mathbf{t}$ *is a quasi-summand of* G*, and in this case it is a maximal quasi-summand of* G*.*

(3) $G(\mathbf{t}) + G[\mathbf{t}] = G_\mathbf{t} \oplus G[\mathbf{t}]$ *if and only if* $G_\mathbf{t}$ *is a quasi-summand of* G*.*

(4) $G(\mathbf{t}) + G[\mathbf{t}]$ *is quasi-equal to* $G^*[\mathbf{t}]$ *if and only if* $\bar{G}_\mathbf{t}$ *is isomorphic to a quasi-summand of* G*.*

PROOF: (1) Let $G'_\mathbf{t}$ be as in Lemma 5.42. Since $G(\mathbf{t})+G[\mathbf{t}] = G'_\mathbf{t} \oplus G[\mathbf{t}]$ and $G'_\mathbf{t} \subseteq G_\mathbf{t}$, the projection of $G(\mathbf{t}) + G[\mathbf{t}]$ onto $G'_\mathbf{t}$ restricts to a projection θ' of $G_\mathbf{t}$ onto $G'_\mathbf{t}$ with kernel $G_\mathbf{t} \cap G[\mathbf{t}]$. On the other hand, $G'_\mathbf{t} \approx (G(\mathbf{t}) + G[\mathbf{t}])/G[\mathbf{t}] \subseteq G^*[\mathbf{t}]/G[\mathbf{t}]$, so from θ' we get $\zeta' \colon G_\mathbf{t} \to G^*[\mathbf{t}]/G[\mathbf{t}]$ with $\operatorname{Ker}\zeta' = \operatorname{Ker}\theta' = G_\mathbf{t} \cap G[\mathbf{t}]$ and $\zeta'(G_\mathbf{t}) = (G(\mathbf{t}) + G[\mathbf{t}])/G[\mathbf{t}]$. Since $G_\mathbf{t} \approx G(\mathbf{t})/G^*(\mathbf{t})_*$, this induces a map $\zeta \colon G(\mathbf{t})/G^*(\mathbf{t})_* \to G^*[\mathbf{t}]/G[\mathbf{t}]$ with $\operatorname{Ker}\zeta = (G(\mathbf{t}) \cap G[\mathbf{t}])/G^*(\mathbf{t})_*$. Now the image of ζ is $(G(\mathbf{t}) + G[\mathbf{t}])/G[\mathbf{t}]$ and by Proposition 4.23 this is a quasi-pure submodule of G, hence is quasi-equal to $(G(\mathbf{t}) + G[\mathbf{t}])_*/G[\mathbf{t}]$. Thus the sequence as shown is quasi-exact.

(2) By Lemma 5.42 $G(\mathbf{t}) \cap G[\mathbf{t}] = G^*(\mathbf{t})_*$ if and only if $G'_\mathbf{t} = G_\mathbf{t}$. Now $G'_\mathbf{t} \subseteq G_\mathbf{t}$ and by Proposition 4.23 $G'_\mathbf{t}$ is a maximal quasi-summand of G. Thus $G'_\mathbf{t} = G_\mathbf{t}$ if and only if $G_\mathbf{t}$ is a quasi-summand of G, and so $G(\mathbf{t}) \cap G[\mathbf{t}] = G^*(\mathbf{t})_*$ if and only if $G_\mathbf{t}$ is a quasi-summand of G, and in this case it is a maximal quasi-summand of G.

(3) This follows since $G(\mathbf{t}) + G[\mathbf{t}] = G'_\mathbf{t} \oplus G[\mathbf{t}]$ and $G'_\mathbf{t}$ is a summand of $G_\mathbf{t}$ and by (2) $G'_\mathbf{t} = G_\mathbf{t}$ if and only if $G_\mathbf{t}$ is a quasi-summand of G.

(4) Let $\zeta' \colon G_\mathbf{t} \to G^*[\mathbf{t}]/G[\mathbf{t}] = \bar{G}_\mathbf{t}$ be as in (1). Since $G_\mathbf{t} = G'_\mathbf{t} \oplus \operatorname{Ker}\zeta$, $\zeta'(G'_\mathbf{t}) = \zeta'(G_\mathbf{t}) = (G(\mathbf{t}) + G[\mathbf{t}])/G[\mathbf{t}]$, which as noted in (3) is quasi-equal to $(G(\mathbf{t}) + G[\mathbf{t}])_*/G[\mathbf{t}] = \bar{G}'_\mathbf{t}$. On the other hand, by Lemma 5.42 $\bar{G}'_\mathbf{t} \approx G'_\mathbf{t}$ which is a maximal quasi-summand of G. It then follows that $\bar{G}_\mathbf{t} = \bar{G}'_\mathbf{t}$ if and only if $\bar{G}_\mathbf{t}$ is the image of a quasi-summand of G. Thus $G(\mathbf{t}) + G[\mathbf{t}]$ is quasi-equal to $G^*[\mathbf{t}]$ if and only if $\bar{G}_\mathbf{t}$ is isomorphic to a quasi-summand of G. ☑

HOM AND TENSOR PRODUCT.

PROPOSITION 5.44. *If* G *is a* $\mathbf{s}$*-projective and* H $\mathbf{t}$*-projective then* $G \otimes H$ *is* $\mathbf{st}$*-projective.*

PROOF: If G is $\mathbf{s}$-projective and H $\mathbf{t}$-projective then $\mathbf{IT}(G) = \mathbf{OT}(G) = \mathbf{s}$ and $\mathbf{IT}(H) = \mathbf{OT}(H) = \mathbf{t}$. By Proposition 2.35 $\mathbf{IT}(G \otimes H) = \mathbf{OT}(G \otimes H) = \mathbf{st}$. Thus $G \otimes H$ is both $\mathbf{st}$-saturated and $\mathbf{st}$-bounded, hence is $\mathbf{st}$-projective by Proposition 4.*. ☑

PROPOSITION 5.45. (1) *If* G *and* H *are Butler modules, then so are* $G \otimes H$ *and* $\operatorname{Hom}(G, H)$*.*

(2) *If* H *is a Butler module such that* $\mathbf{T}(H)$ *consists solely of idempotent types, then for any finite rank torsion free module* G*,* $\operatorname{Hom}(G, H)$ *is a Butler module whose typeset contains only idempotent types.*

(3) *If* G *is a Butler module such that* $\mathbf{T}(G)$ *contains only idempotent types, then for any* H*,* $\operatorname{Hom}(G, H)$ *is is isomorphic to a quasi-pure submodule of* $\bigoplus_{\mathbf{T}'(G)} H(\mathbf{t})^{k_\mathbf{t}}$*, where* $k_\mathbf{t} = \operatorname{rank} G_\mathbf{t}$*.*

PROOF: (1) If G is a homomorphic image of $\bigoplus A_i$, and H a homomorphic image of $\bigoplus B_j$, where the A_i and B_j are rank-one modules, then $G \otimes H$ is a homomorphic image of $\bigoplus A_i \otimes B_j$, and each $A_i \otimes B_j$ is a rank-one module. Therefore $G \otimes H$ is a Butler module. Furthermore if H is a pure submodule of $\bigoplus C_k$, where the C_k are rank-one modules, then $\mathrm{Hom}(G,H) \lhd \bigoplus \mathrm{Hom}(G,C_k) \lhd \bigoplus_{i,k} \mathrm{Hom}(A_i,C_k)$. Since each $\mathrm{Hom}(C_k,A_i)$ is a rank-one module, $\mathrm{Hom}(G,H)$ is a Butler module by Theorem 5.10.

(2) If $\mathbf{T}(H)$ contains only idempotent types then by Proposition 5.22 the same is true of $\mathbf{CT}(H)$. Therefore the rank-one modules C_k in the previous paragraph all have idempotent types. Thus by Proposition 4.39 each $\mathrm{Hom}(G,C_k)$ is $\mathbf{t}_k$-projective, where $\mathbf{t}_k = \mathbf{t}(C_k)$. Thus $\mathrm{Hom}(G,H)$ is isomorphic to a pure submodule of the completely decomposable module $\bigoplus \mathrm{Hom}(G,C_k)$ and therefore is a Butler module by Theorem 5.10.

(3) If A is a rank-one module and $\mathbf{t} = \mathbf{t}(A)$ is idempotent, then by Proposition 4.36 $\mathrm{Hom}(A,H) \approx H(\mathbf{t})$. Thus if $\mathbf{t} \in \mathbf{T}'(G)$ and $G_{\mathbf{t}}$ is a $\mathbf{t}$-projective module such that $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$, then $\mathrm{Hom}(G_{\mathbf{t}},H) \approx H(\mathbf{t})^{k_{\mathbf{t}}}$, where $k_{\mathbf{t}} = \mathrm{rank}\, G_{\mathbf{t}}$. But by Proposition 5.32 G is a quasi-homomorphic image of $\bigoplus_{\mathbf{T}'(G)} G_{\mathbf{t}}$, so that $\mathrm{Hom}(G,H)$ is quasi-isomorphic to a pure submodule of $\bigoplus_{\mathbf{T}'(G)} \mathrm{Hom}(G_{\mathbf{t}},H) \approx \bigoplus H(\mathbf{t})^{k_{\mathbf{t}}}$. ☑

PROPOSITION 5.46. *Let $\mathbf{t}$ be an idempotent type. Let G and H be Butler modules such that G has no non-trivial $\mathbf{t}$-projective quasi-summand. For each $\mathbf{s} \in \mathbf{T}'(G) \cup \mathbf{T}'(H)$, choose $G_{\mathbf{s}}$ and $H_{\mathbf{s}}$ such that $G(\mathbf{s}) = G_{\mathbf{s}} \oplus G^*(\mathbf{s})$ and $H(\mathbf{s}) = H_{\mathbf{s}} \oplus H^*(\mathbf{s})$ (c.f. Proposition 5.30).*

1. *Let G' and H' be the submodules of G and H generated by $G_{\mathbf{s}}$ and $H_{\mathbf{s}}$ for all $\mathbf{s} \neq \mathbf{t}$. Then $G \otimes H$ is quasi-equal to $(G' \otimes H_{\mathbf{t}}) + (G \otimes H')$.*
2. *Every element of $\mathbf{T}'(G \otimes H)$ has the form $\mathbf{st}$ for $\mathbf{s} \in \mathbf{T}'(G)$, $\mathbf{t} \in \mathbf{T}'(H)$.*
3. *If $\mathbf{t}$ is minimal in $\mathbf{T}'(H)$ then $\mathbf{t} \notin \mathbf{T}'(G \otimes H)$.*

PROOF: (1) By Proposition 5.30 H is quasi-equal to $H_{\mathbf{t}} + H'$, and so $G \otimes H$ is quasi-equal to $(G \otimes H_{\mathbf{t}}) + (G \otimes H')$. Thus it suffices to see that $G \otimes H_{\mathbf{t}}$ is quasi-equal to $G' \otimes H_{\mathbf{t}}$. By Proposition 5.30 G' is an essential submodule of G and $G' + G_{\mathbf{t}}$ is quasi-equal to G, so we may assume WLOG that $G = G' + G_{\mathbf{t}}$. Thus G/G' is a torsion module which is a homomorphic image of $G_{\mathbf{t}}$, so there is a short exact sequence $0 \to K \to G_{\mathbf{t}} \to G/G' \to 0$ where $\mathrm{rank}\, K = \mathrm{rank}\, G$. Now $G_{\mathbf{t}}$ and $H_{\mathbf{t}}$ are $\mathbf{t}$-projective and since $\mathbf{t}$ is idempotent it follows from Proposition 5.44 that $G_{\mathbf{t}} \otimes H_{\mathbf{t}}$ is $\mathbf{t}$-projective. Then $K \otimes H_{\mathbf{t}}$ is $\mathbf{t}$-saturated and also $\mathbf{t}$-bounded since it is a submodule of $G_{\mathbf{t}} \otimes H_{\mathbf{t}}$. Thus $K \otimes H_{\mathbf{t}}$ is $\mathbf{t}$-projective and quasi-equal to $G_{\mathbf{t}} \otimes H_{\mathbf{t}}$ by Proposition 3.9. Therefore $(G/G') \otimes H_{\mathbf{t}}$ has finite length, so $G' \otimes G_{\mathbf{t}}$ is quasi-equal to $G \otimes H_{\mathbf{t}}$. This completes the proof that $G \otimes H$ is quasi-equal to $(G \otimes H_{\mathbf{t}}) + (G \otimes H')$.

(2) ****

(3) By (1) G is quasi-equal to the submodule generated by $G_{\mathbf{s}} \otimes H_{\mathbf{t}}$ and $G_{\mathbf{t}} \otimes H_{\mathbf{s}'}$ and $G_{\mathbf{s}} \otimes H_{\mathbf{s}'}$ for all $\mathbf{s} \neq \mathbf{t} \in \mathbf{T}'(G)$ and $\mathbf{s}' \neq \mathbf{t} \in \mathbf{T}'(H)$. Thus if $\mathbf{t}$ is minimal in $\mathbf{T}'(G) \cup \mathbf{T}'(H)$ it follows that $G \otimes H$ is generated by rank-one modules all having types $\mathbf{ss}' \not\leq \mathbf{t}$. From this one sees that $G(\mathbf{t}) \subseteq G^*(\mathbf{t})_*$, so that $\mathbf{t} \notin \mathbf{T}'(G \otimes H)$. ☑

Consider a Butler module G such that $\mathbf{T}'(G)$ contains only idempotent types. Form the sequence $\mathbf{T}'(G)$, $\mathbf{T}'(G \otimes G)$, $\mathbf{T}'(G \otimes G \otimes G)$, ... Recall from Proposition 2.* that

for idempotent types $\mathbf{s}$ and $\mathbf{t}$, $\mathbf{st} = \mathbf{s} \vee \mathbf{t}$. It thus follows from (2) of Proposition 5.46 that $\mathbf{T}'(G^{\otimes n}) \subseteq \mathbf{T}'(G)$ for all n. On the other hand, by (3) of Proposition 5.46 as long as no rank-one quasi-summands occur there is an upward pressure on the critical typesets $\mathbf{T}'(G^{\otimes n})$, in the sense that the minimal elements of $\mathbf{T}'(G^{\otimes n})$ cannot belong to the critical typeset of $G^{\otimes n+1}$. If this were continue indefinitely, we would eventually derive the absurdity that for large enough n, $\mathbf{T}'(G^{\otimes n}) = \varnothing$. Since this cannot happen, it follows that for some n $G^{\otimes n}$ must have a rank-one quasi-summand.

This illustrates the idea that when one tensors Butler modules whose typesets are idempotent there is a tendancy for rank-one quasi-summands to appear, and the more one tensors the stronger this tendancy becomes. One can contrast this with what happens with Butler modules whose typesets contains only locally trivial types. We will see in Chapter 9 that if G and H are Butler modules of this sort then it is extremely difficult for $G \otimes H$ to have a rank-one quasi-summand.

ALMOST COMPLETELY DECOMPOSABLE MODULES. Consider again Example 4.25. In this example we constructed two indecomposable modules G and H, with completely decomposable submodules $C = A_1 \oplus \cdots \oplus A_n \approx D = B_1 \oplus \cdots \oplus B_n$ such that $G/C \approx W/p$ and $H/D \approx W/q$. We then saw that $G \oplus H = K \oplus L$ where L is completely decomposable. Thus $G \oplus H$, a direct sum of 2 indecomposable modules, can also be written as a direct sum of $n+1$ indecomposable modules (since L is the direct sum of n rank-one modules).

Heuristically, we can explain Example 4.25, at least after a fashion, as follows: Since G/C is cyclic, the module G consists of the completely decomposable module C glued together by a single element in $p^{-1}(a_1 + \cdots + a_n)$. Likewise H consists of D glued together by a single element belonging to $q^{-1}(b_1 + \cdots + b_n)$. But since $G/C \oplus H/D \approx W/p \oplus W/q \approx W/pq$, a cyclic module, we can find a single element $g \in G \oplus H$ such that the image of g is a generator of $G \oplus H \ / \ C \oplus D$. If we can then find a pure submodule K of $G \oplus H$ of rank n containing g, then K contains all the glue for $G \oplus H$ and consequently $G \oplus H$ must be the direct sum of K and a direct sum of rank-one modules (since there is no glue left to stick the remaining rank-one quasi-summands together).

The key point here seems to be that modules which are obtained from completely decomposable modules by using a finite number of pieces of "glue" are likely to be easy to manipulate in terms of obtaining various direct sum decompositions. And that if G is such a module, and C the underlying completely decomposable module, then the quotient G/C is a good measure of how far G is from being completely decomposable and how G will behave in direct sum decompositions.

To formalize this, we introduce the concept of almost completely decomposable modules.

We say that G is **almost completely decomposable** if G is quasi-isomorphic to a completely decomposable module. Since completely decomposable modules are Butler modules, by Proposition 5.7 so are almost completely decomposable modules.

PROPOSITION 5.47. *Let G be almost completely decomposable.*

(1) $\mathbf{T}'(G)$ *consists of those types* $\mathbf{t}$ *such that* G *has a quasi-summand of type* $\mathbf{t}$.
(2) $\mathbf{T}'(G) = \mathbf{CT}'(G)$.
(3) $\mathbf{T}(G)$ *consists of all types of the form* $\mathbf{inf}\,\mathbf{S}$, *for* $\mathbf{S} \subseteq \mathbf{T}'(G)$.
(4) $\mathbf{CT}(G)$ *consists of all types of the form* $\mathbf{sup}\,\mathbf{S}$ *for* $\mathbf{S} \subseteq \mathbf{T}'(G)$.
(5) *If* $\mathbf{s}, \mathbf{t} \in \mathbf{T}'(G)$ *and* $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$ *and* $\mathbf{t} \leq \mathbf{s}$ *then* $G_{\mathbf{t}} \cap G[\mathbf{s}] = 0$.

PROOF: Since G is quasi-equal to a completely decomposable module, by Proposition 3.8 and Proposition 5.30 it suffices to prove the theorem for completely decomposable modules. But in this case (1) was proved in Proposition 5.30, (2) was proved in Proposition 5.31, and (3) and (4) in Corollary 2.24. And (5) is clear for completely decomposable modules since if $G = \bigoplus G_{\mathbf{t}}$ then $G[\mathbf{s}] = \bigoplus_{\mathbf{s}' \not\leq \mathbf{s}} G_{\mathbf{s}'}$ so if $\mathbf{t} \leq \mathbf{s}$ then $G_{\mathbf{t}} \cap G[\mathbf{s}] = 0$. $\boxed{\checkmark}$

PROPOSITION 5.48. *Let G be a Butler module Then G is completely decomposable if and only if for each* $\mathbf{t} \in \mathbf{CT}(G)$, $G[\mathbf{t}] = \sum\{G(\mathbf{s}) \mid \mathbf{s} \not\leq \mathbf{t}\}$.

PROOF: ($\Rightarrow$): Easy.

($\Leftarrow$): For each $\mathbf{t} \in \mathbf{T}'(G)$ let $G_{\mathbf{t}} \lhd G$ be such that $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$. We will prove that $G = \bigoplus G_{\mathbf{t}}$. Suppose first, by way of contradiction, that for some $\mathbf{t} \in \mathbf{T}'(G)$, $G_{\mathbf{t}} \cap \sum\{G_{\mathbf{s}} \mid \mathbf{s} \neq \mathbf{t}\} \neq 0$. ????

Let $\mathbf{t} = \mathbf{IT}(G)$. Then by Proposition 4.40 $G = G_{\mathbf{t}} \oplus G[\mathbf{t}]$ for some $\mathbf{t}$-projective module $G_{\mathbf{t}}$. It then suffices to prove that $G[\mathbf{t}]$ is completely decomposable. By assumption, $G[\mathbf{t}] = \sum\{G(\mathbf{s}) \mid \mathbf{s} \not\leq \mathbf{t}\} = \sum\{G(\mathbf{s}) \mid \mathbf{s} \in \mathbf{T}(G) \mid \mathbf{s} \neq \mathbf{t}\}$, since $\mathbf{t} = \mathbf{IT}(G)$. But this simply says that $G = G^*(\mathbf{t})$ and so $G^*(\mathbf{t})_* = G^*(\mathbf{t})$. ????

By Proposition 4.29 G is completely decomposable if and only if for all $\mathbf{t} \in \mathbf{CT}(G)$, $G^*[\mathbf{t}] = G(\mathbf{t}) + G[\mathbf{t}]$. Now if $G[\mathbf{t}] = \sum\{G(\mathbf{s}) \mid \mathbf{s} \not\leq \mathbf{t}\}$ then $G(\mathbf{t}) + G[\mathbf{t}] = \sum\{G(\mathbf{s}) \mid \mathbf{s} \not< \mathbf{t}\}$. Now by Proposition 4.27 $G(\mathbf{t}) + G[\mathbf{t}] \subseteq G^*[\mathbf{t}]$. ****

PROPOSITION 5.49. *Let G be a Butler module and for every* $\mathbf{t} \in \mathbf{T}'(G)$ *let* $G_{\mathbf{t}} \lhd G(\mathbf{t})$ *be such that* $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$. *The following conditions are equivalent:*

(1) *G is almost completely decomposable.*
(2) *For every type* $\mathbf{t}$, $G_{\mathbf{t}}$ *is a quasi-summand of* G.
(3) *The sum* $\sum_{\mathbf{T}'(G)} G_{\mathbf{t}}$ *is direct.*
(4) *For every type* $\mathbf{t}$, $G^*[\mathbf{t}]/G[\mathbf{t}]$ *is isomorphic to a quasi-summand of* G.
(5) *For every type* $\mathbf{t}$, $G^*(\mathbf{t})_* = G(\mathbf{t}) \cap G[\mathbf{t}]$.
(6) *For every* $\mathbf{t}$, $G^*[\mathbf{t}] = (G(\mathbf{t}) + G[\mathbf{t}])_*$.
(7) *For every* $\mathbf{t} \in \mathbf{CT}(G)$, $G[\mathbf{t}]$ *is quasi-equal to* $\sum\{G(\mathbf{s}) \mid \mathbf{s} \not\leq \mathbf{t}\}$.
(8) *For every* $\mathbf{t} \in \mathbf{T}'(G)$, $G(\mathbf{t}) + G[\mathbf{t}] = G_{\mathbf{t}} \oplus G[\mathbf{t}]$.

PROOF: (1) $\Rightarrow$ (2): This just comes down to seeing that (2) is true for completely decomposable modules. If G is quasi-equal to a completely decomposable module $C = \bigoplus C_{\mathbf{t}}$, then by Proposition 4.2 $G(\mathbf{t})$ is quasi-equal to $C(\mathbf{t})$ and $G^*(\mathbf{t})_*$ is quasi-equal to $C^*(\mathbf{t})_*$. Furthermore $C(\mathbf{t}) = C_{\mathbf{t}} \oplus C^*(\mathbf{t})$, so that $G(\mathbf{t})$ is quasi-equal to $C_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$. It follows that we may choose $G_{\mathbf{t}} = (C_{\mathbf{t}})_*$. Since $C_{\mathbf{t}}$ is a summand of C and G is

quasi-equal to C, it follows that $G_{\mathbf{t}}$ is a quasi-summand of G. And in fact this will be true no matter what choice of $G_{\mathbf{t}}$ we make, as can be seen, for instance, from the equivalence of (2) and (3).

(2) $\Rightarrow$ (3): Suppose that each $G_{\mathbf{t}}$ is a quasi-summand of G. Note that no quasi-summand of $G_{\mathbf{t}}$ is quasi-isomorphic to any quasi-summand of $G_{\mathbf{t}'}$ for $\mathbf{t}' \neq \mathbf{t}$. Thus by Theorem 3.24 $\sum G_{\mathbf{t}}$ is direct, and generates a quasi-summand of G.

(3) $\Rightarrow$ (1): By Proposition 5.32 G is quasi-equal to $\sum_{\mathbf{T}'(G)} G_{\mathbf{t}}$. Thus if $\sum_{\mathbf{T}'(G)} G_{\mathbf{t}} = \bigoplus_{\mathbf{T}'(G)} G_{\mathbf{t}}$ then G is quasi-equal to a completely decomposable module, hence is almost completely decomposable.

(1) $\Longleftrightarrow$ (6): Proved in Proposition 4.29.

(4) $\Longleftrightarrow$ (6): Proved in Proposition 5.43.

(2) $\Longleftrightarrow$ (5): Proved in Proposition 5.43.

(1) $\Rightarrow$ (7): By analogous reasoning, it suffices to prove that $C[\mathbf{t}] = \sum_{\mathbf{s} \not\leq \mathbf{t}} C(\mathbf{s})$, and this is clear from the formulas above, since $\bigoplus_{\mathbf{s} \not\leq \mathbf{t}} C_{\mathbf{s}} = \sum_{\mathbf{s} \not\leq \mathbf{t}} C(\mathbf{s})$.

(7) $\Rightarrow$ (1): ****

(2) $\Longleftrightarrow$ (8): By Proposition 5.43. $\boxed{\checkmark}$

PROPOSITION 5.50. *Let G be an almost completely decomposable module and $\mathbf{s}$, $\mathbf{t}$ types. Then*

(1) $G[\mathbf{s}][\mathbf{t}] = G[\mathbf{s}] \cap G[\mathbf{t}]$.
(2) $G(\mathbf{s})[\mathbf{t}] = G(\mathbf{s}) \cap G[\mathbf{t}]$.

PROOF: (1) It suffices to prove this for completely decomposable modules, since if G is quasi-equal to a completely decomposable module C then $G[\mathbf{s}]$, $G[\mathbf{t}]$, $G[\mathbf{s}][\mathbf{t}]$, and $G[\mathbf{s}] \cap G[\mathbf{t}]$ are all pure submodules of G and are quasi-equal to $C[\mathbf{s}]$, $C[\mathbf{t}]$, $C[\mathbf{s}][\mathbf{t}]$, and $C[\mathbf{s}] \cap C[\mathbf{t}]$. But if $G = \bigoplus A_i$, where the A_i are rank-one modules, then $C[\mathbf{s}] = \bigoplus\{A_i \mid \mathbf{t}(A_i) \not\leq \mathbf{s}\}$, and $C[\mathbf{s}][\mathbf{t}] = \bigoplus\{A_i \mid \mathbf{t}(A_i) \not\leq \mathbf{s} \text{ and } \mathbf{t}(A_i) \not\leq \mathbf{t}\} = C[\mathbf{s}] \cap C[\mathbf{t}]$.

(2) Analogous. $\boxed{\checkmark}$

REGULATING SUBMODULES. Since an almost completely decomposable module G is quasi-isomorphic to a completely decomposable module, by Proposition 3.* there exist completely decomposable submodules $C \subseteq G$ such that G/C has finite length. We want to single out certain of these completely decomposable submodules, called *regulating submodules*, which have special importance. In order to describe their properties, we first need to define the module-theoretic analogue of the *index* of a subgroup in a group.

NOTATION 5.51. If H is a submodule of G such that G/H has finite length, we define an ideal $[G : H]$ such that the following properties hold:

(1) If $G = H$ then $[G : H] = W$.
(2) If G/H is simple, then $[G : H] = p$ where p is the unique prime ideal such that $G/H \approx W/p$.
(3) If $H \subseteq K \subseteq G$, then $[G : H] = [G : K][K : H]$.

It follows from the Jordan-Hölder Theorem that these three properties recursively determine $[G : H]$ for all $H \subseteq G$ such that G/H has finite length, and that $[W : \mathfrak{a}] = \mathfrak{a}$ for any non-trivial ideal $\mathfrak{a}$.

In other words, if G/H has finite length consider a composition series $H = G_0 \subsetneq G_1 \subsetneq \ldots \subsetneq G_n = G$. Then each G_i/G_{i-1} is a simple module, so $G_i/G_{i-1} \approx W/p_i$ for some prime (maximal) ideal p_i. Then $[G\colon H] = p_1 \cdots p_n$.

(Unfortunately, the notation $[G\colon H]$ is used in some books on commutative ring theory in a slightly different sense. This is one case where we have decided to be consistent with group theoretic language instead of with ring theory. Since our use of this notation does not extend beyond the present chapter, your author hopes that any confusion will be minimal and brief.)

LEMMA 5.52. (1) *Let $C \subseteq G$ be such that G/C has finite length and let $I = [G\colon C]$. Then $IG \subseteq C$.*

(2) *If $C \subseteq G$ and $D \subseteq H$ are such that G/C and H/D have finite length, then $[G \oplus H\colon C \oplus D] = [G\colon C][H\colon D]$.*

(3) *If $H \subsetneq K \subsetneq G$ then $[G\colon H] \subsetneq [G\colon K]$ and $[G\colon H] \subsetneq [K\colon H]$.*

PROOF: (1) By an easy induction on length G/C one sees that $I(G/C) = 0$ so that $IG \subseteq C$.

(2) $C \oplus D \subseteq C \oplus H \subseteq G \oplus H$. Thus $[G \oplus H\colon C \oplus D] = [G \oplus H\colon C \oplus H][C \oplus H\colon C \oplus D]$. But it is easy to see that $[G \oplus H\colon C \oplus H] = [G\colon C]$ and $[C \oplus H\colon C \oplus D] = [H\colon D]$.

(3) In fact, by the defining property of the index function, $[G\colon H] = [G\colon K][K\colon H]$ and $[K\colon H] \neq W$, $[G\colon K] \neq W$. Thus $[G\colon H]$ is a proper subideal of both $[G\colon K]$ and $[K\colon H]$. $\boxed{\checkmark}$

Note that (3) of Lemma 5.52 says that the index function $[G\colon C]$ is, as it were, covariant in the second variable C. Your author tends to find this sometimes confusing.

DEFINITION 5.53. Let G be almost completely decomposable and for each $\mathbf{t} \in \mathbf{T}'(G)$ let $G_{\mathbf{t}}$ be a pure completely decomposable submodule of G such that $G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$ (see Proposition 5.40). Recall from Proposition 5.49 that G is quasi-equal to $\bigoplus G_{\mathbf{t}}$. We say that $\bigoplus G_{\mathbf{t}}$ is a **regulating submodule** of G.

Regulating submodules are not unique, except in certain special cases. In fact, it can be shown that in general not even the quotient $G/\bigoplus G_{\mathbf{t}}$ is uniquely determined independently of the choice of $G_{\mathbf{t}}$. However we now see that the index $[G\colon C]$ is the same for all regulating submodules C of G.

LEMMA 5.54. *Let G be an almost completely decomposable module and let $C = \bigoplus C_{\mathbf{t}}$ be a submodule of G quasi-equal to G, where each $C_{\mathbf{t}}$ is $\mathbf{t}$-projective. Let $\mathbf{t}$ be minimal in $\mathbf{T}'(G) = \mathbf{C}\mathbf{T}'(G)$.*

(1) *C is a regulating submodule of G if and only $G(\mathbf{t}) + G[\mathbf{t}] = C_{\mathbf{t}} \oplus G[\mathbf{t}]$ and $C[\mathbf{t}]$ is a regulating submodule of $G[\mathbf{t}]$.*

(2) *If C is a regulating submodule of G then $[G : C] = [G\colon G(\mathbf{t}) + G[\mathbf{t}]]\,[G[\mathbf{t}]\colon C[\mathbf{t}]]$.*

(3) *$G(\mathbf{t}) + G[\mathbf{t}] = G^*[\mathbf{t}] = G$.*

PROOF: (1) C is a regulating submodule of G if and only if for each $\mathbf{s} \in \mathbf{T}'(G)$, $G(\mathbf{s}) = C_{\mathbf{s}} \oplus G^*(\mathbf{s})_*$. By Proposition 5.49 this will be true if and only if $G(\mathbf{s}) + G[\mathbf{s}] = C_{\mathbf{s}} \oplus G[\mathbf{s}]$. Furthermore since $\mathbf{t}$ is minimal IN $\mathbf{t}'(g)$, $C[\mathbf{t}] = \bigoplus_{\mathbf{s}\neq\mathbf{t}} C_{\mathbf{t}}$ and by Proposition 4.4 $G(\mathbf{s}) \subseteq G[\mathbf{t}]$ for $\mathbf{s} \neq \mathbf{t}$, so that $G[\mathbf{t}](\mathbf{s}) = G(\mathbf{s})$ and $G[\mathbf{t}]^*(\mathbf{s}) = G^*(\mathbf{s})_*$. It follows that $G(\mathbf{s}) = C_{\mathbf{s}} \oplus G^*(\mathbf{s})$ for all $\mathbf{s} \neq \mathbf{t} \in \mathbf{T}'(G)$ if and only if $C[\mathbf{t}]$ is a regulating submodule of $G[\mathbf{t}]$. Thus C is a regulating submodule of G if and only if $C[\mathbf{t}]$ is a regulating submodule of $G[\mathbf{t}]$ and $G(\mathbf{t}) + G[\mathbf{t}] = C_{\mathbf{t}} \oplus G[\mathbf{t}]$.

(2) Since $C \subseteq G(\mathbf{t})+G[\mathbf{t}] \subseteq G$, $[G\colon C] = [G\colon G(\mathbf{t})+G[\mathbf{t}]]\,[G(\mathbf{t})+G[\mathbf{t}]\colon C]$. Furthermore if C is a regulating submodule of G then $G(\mathbf{t}) + G[\mathbf{t}] = C_{\mathbf{t}} \oplus G[\mathbf{t}]$ and $C = C_{\mathbf{t}} \oplus C[\mathbf{t}]$, so that $[G(\mathbf{t}) + G[\mathbf{t}]\colon C] = [G[\mathbf{t}]\colon C[\mathbf{t}]]$. Thus $[G\colon C] = [G\colon G(\mathbf{t}) + G[\mathbf{t}]]\,[G[\mathbf{t}]\colon C[\mathbf{t}]]$.

(3) By Proposition 5.49, $(G(\mathbf{t}) + G[\mathbf{t}])_* = G^*[\mathbf{t}]$. Now by Proposition 5.47, $\mathbf{CT}(G)$ is contained in the lattice generated by $\mathbf{T}'(G)$. Thus if $\mathbf{t}$ is minimal in $\mathbf{T}'(G)$ and $\varphi \in \mathrm{Hom}(G, Q)$ then $\mathbf{t}(\varphi(G)) \not< \mathbf{t}$ since $\mathbf{t}(\varphi(G)) \in \mathbf{CT}(G)$. Thus $G[\mathbf{s}] = G$ for all $\mathbf{s} < \mathbf{t}$, so $G^*[\mathbf{t}] = G$. $\boxed{\checkmark}$

PROPOSITION 5.55. *Let G be an almost completely decomposable module, and let $C = \bigoplus G_{\mathbf{t}}$ be a regulating submodule of G.*

(1) *If C' is a completely decomposable submodule of G which is quasi-equal to G, then $[G : C'] \subseteq [G\colon C]$.*
(2) *If C' is a completely decomposable submodule of G which is quasi-equal to G then C' is a regulating submodule of G if and only if $[G : C'] = [G\colon C]$.*

PROOF: (1) Let C' be a completely decomposable submodule of G which is quasi-equal to G, and write $C' = \bigoplus_{\mathbf{T}'(G)} C'_{\mathbf{t}}$, where $C'_{\mathbf{t}}$ is $\mathbf{t}$-projective. We use induction on the size of $\mathbf{T}'(G)$, the result being clear if G is homogeneous since in that case G is quasi-isomorphic to a $\mathbf{t}$-projective module for some $\mathbf{t}$ and so by Proposition 4.16 G itself is $\mathbf{t}$-projective and thus $G = G_{\mathbf{t}}$ so $G = C$ and $[G\colon C] = W$.

If G is not homogeneous, choose $\mathbf{t}$ minimal in $\mathbf{T}'(G) = \mathbf{CT}'(G)$. Then by Lemma 5.54 $(G(\mathbf{t}) + G[\mathbf{t}])_* = G^*[\mathbf{t}] = G$. Then by Lemma 5.54 $C[\mathbf{t}]$ is a regulating submodule of $G[\mathbf{t}]$. Now $C'_{\mathbf{t}} \oplus G^*(\mathbf{t})_* \subseteq G(\mathbf{t})$ and so $C'_{\mathbf{t}} \oplus G[\mathbf{t}] \subseteq G(\mathbf{t}) + G[\mathbf{t}]$. Thus, recalling that the index operation is covariant in the second variable,

$$[G\colon C'] = [G\colon C'_{\mathbf{t}} \oplus G[\mathbf{t}]]\,[C'_{\mathbf{t}} \oplus G[\mathbf{t}]\colon C'] \subseteq [G\colon G(\mathbf{t}) + G[\mathbf{t}]]\,[G[\mathbf{t}]\colon C'[\mathbf{t}]],$$

since $C' = C'_{\mathbf{t}} \oplus C'[\mathbf{t}]$. By induction $[G[\mathbf{t}]\colon C'[\mathbf{t}]] \subseteq [G[\mathbf{t}]\colon C[\mathbf{t}]]$ so that

$$\begin{aligned}[G\colon C'] \subseteq [G\colon G(\mathbf{t}) + G[\mathbf{t}]]\,[G[\mathbf{t}]\colon C[\mathbf{t}]] &= [G\colon G(\mathbf{t}) + G[\mathbf{t}]]\,[G_{\mathbf{t}} \oplus G[\mathbf{t}]\colon G_{\mathbf{t}} \oplus C[\mathbf{t}]] \\ &= [G\colon G(\mathbf{t}) + G[\mathbf{t}]]\,[G(\mathbf{t}) + G[\mathbf{t}]\colon C] = [G\colon C],\end{aligned}$$

since $G(\mathbf{t}) + G[\mathbf{t}] = G_{\mathbf{t}} \oplus G[\mathbf{t}]$ by Proposition 5.49 and $G_{\mathbf{t}} \oplus C[\mathbf{t}] = G_{\mathbf{t}} \oplus \bigoplus_{\mathbf{s}\neq\mathbf{t}} G_{\mathbf{s}} = \bigoplus G_{\mathbf{t}} = C$.

(2) It follows from (1) that if C and C' are regulating submodules of G then $[G\colon C'] \subseteq [G\colon C]$ and $[G\colon C] \subseteq [G\colon C']$ so that $C = C'$.

Conversely, suppose that C' is a completely decomposable submodule module of G quasi-equal to G and that $[G\colon C'] = [G\colon C]$, where C is a regulating submodule. Then

the inclusions in the displayed formulas above must be equalities. It that follows for $\mathbf{t}$ minimal in $\mathbf{T}'(G)$, $[G\colon C'_{\mathbf{t}} \oplus G[\mathbf{t}]] = [G\colon G(\mathbf{t}) + G[\mathbf{t}]]$, so that $G(\mathbf{t}) + G[\mathbf{t}] = C'_{\mathbf{t}} \oplus G[\mathbf{t}]$. Likewise it must be true that $[G[\mathbf{t}]\colon C'[\mathbf{t}]] = [G[\mathbf{t}]\colon C[\mathbf{t}]]$. By induction on the size of $\mathbf{T}'(G)$ it then follows that $C'[\mathbf{t}]$ is a regulating submodule of $G[\mathbf{t}]$, and thus by Lemma 5.54 C' is a regulating submodule of C. $\boxed{\checkmark}$

DEFINITION 5.56. If G is an almost completely decomposable module and C a regulating submodule, then we call $[G\colon C]$ the **regulating index** of G and write $i(G) = [G\colon C]$.

Proposition 5.55 says that among the ideals of the form $[G\colon C]$, where C ranges over the completely decomposable submodules quasi-equal to G, there is a unique largest one, namely $i(G)$, and that a completely decomposable submodule C of G which is quasi-equal to G is a regulating submodule if and only if $[G\colon C] = i(G)$. Thus $i(G)$ is in some sense a measure of how far G is from being completely decomposable. The larger the idea $i(G)$ is, the closer G is to being completely decomposable.

Those who do not like the usage of the group-theoretic word "index" in this context might prefer to call $i(G)$ the regulating *defect*. Or perhaps regulating *excess* might be more appropriate.

Lemma 5.54 asserts that if $\mathbf{t}$ is minimal in $\mathbf{T}'(G)$ then $i(G) = i(G[\mathbf{t}])[G\colon G(\mathbf{t}) + G[\mathbf{t}]]$. This gives a recursive procedure for computing $i(G)$ without the need to actually find a regulating submodule of G. In applying this it is useful to remember that by Proposition 5.50 and Proposition 4.4 for any type $\mathbf{s}$, $G[\mathbf{t}][\mathbf{s}] = G[\mathbf{t}] \cap G[\mathbf{s}]$ and $G[\mathbf{t}](\mathbf{s}) = G[\mathbf{t}] \cap G(\mathbf{s})$.

EXAMPLE 5.57. Let A_1, A_2, A_3, and B be non-trivial submodules of Q containing 1 such that $\mathbf{t}(A_1)$, $\mathbf{t}(A_2)$, $\mathbf{t}(A_3)$ are mutually incomparable and $\mathbf{t}(B) < \mathbf{t}(A_1)$, $\mathbf{t}(A_2)$ and $\mathbf{t}(B) \not\leq \mathbf{t}(A_3)$ (and necessarily $\mathbf{t}(A_i) \not\leq \mathbf{t}(B)$ for any i, otherwise $\mathbf{t}(A_i) \leq \mathbf{t}(B) < \mathbf{t}(A_1)$). Suppose also that there exists a prime p such that $1 \notin pA_1, pA_2, pA_3, pB$. Let p' be an element of p with $p' \notin p^2$. Let $G = (A_1 \oplus A_2 \oplus A_3 \oplus B) + p^{-2}(1, 1, -p', -p')$ Then $i(G) = p^2$.

PROOF: Let $C = A_1 \oplus A_2 \oplus A_3 \oplus B$, so that $G = C + p^{-2}(1, 1, -p', -p')$. It is easy to se that C is a regulating submodule of G (although not the only one). However instead of using this fact, we can use Lemma 5.54, as indicated above. If $\mathbf{s} = \mathbf{t}(B)$ and $\mathbf{t}_i = \mathbf{t}(A_i)$, then the minimal elements in $\mathbf{T}'(G)$ are $\mathbf{s}$ and $\mathbf{t}_3$. Now since $p^{-1}p' \subseteq p^{-1}p = W \subseteq A_3$, B, then $p^{-1}(1, 1, p', p') \subseteq p^{-1}(1, 1, 0, 0) + C$. It follows that EXPLAIN!

$$G[\mathbf{t}_1] = (0 \oplus A_2 \oplus A_3 \oplus 0).$$

Likewise

$$G(\mathbf{s}) = C(\mathbf{s})_* = (A_1 \oplus A_2 \oplus 0 \oplus B) + p^{-1}(1, 1, 0, 0)$$

and

$$G[\mathbf{s}] = (A_1 \oplus A_2 \oplus A_3 \oplus 0) + p^{-1}(1, 1, 0, 0).$$

Thus $G(\mathbf{s})+G[\mathbf{s}] = C+p^{-1}(1,1,0,0)$, and $pG \subseteq (G(\mathbf{s})+G[\mathbf{s}])$ so that $[G\colon G(\mathbf{s})+G[\mathbf{s}]] = p$. Now $\mathbf{T}'(G[\mathbf{s}]) = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ and all these are minimal critical types for $G[\mathbf{s}]$. Now since $G(\mathbf{t}_1) = A_1 \oplus 0 \oplus 0 \oplus 0 \ \lhd\ G[\mathbf{s}] \ \lhd\ G$, then $G[\mathbf{s}](\mathbf{t}_1) = G(\mathbf{t}_1) = A_1 \oplus 0 \oplus 0 \oplus 0$. And $G[\mathbf{s}][\mathbf{t}_1] = (0 \oplus A_2 \oplus A_3 \oplus 0)$. Thus $G[\mathbf{s}]\,/(G[\mathbf{s}](\mathbf{t}_1) + G[\mathbf{s}][\mathbf{t}_1])$ is generated by the image of $p^{-1}(1,1,0,0)$, so that $[G[\mathbf{s}]\colon (G[\mathbf{s}](\mathbf{t}_1) + G[\mathbf{s}][\mathbf{t}_1])] = p$. Therefore

$$\begin{aligned} i(G) = i(G[\mathbf{s}])[G\colon G(\mathbf{s}) + G[\mathbf{s}]] = pi(G[\mathbf{s}]) &= pi(G[\mathbf{s}][\mathbf{t}_1])[G[\mathbf{s}]\colon G[\mathbf{s}](\mathbf{t}_1) + G[\mathbf{s}][\mathbf{t}_1]] \\ &= p^2 i(G[\mathbf{s}][\mathbf{t}_1]) = p^2 \end{aligned}$$

since $G[\mathbf{s}][\mathbf{t}_1] = 0 \oplus A_2 \oplus A_2 \oplus 0$, a completely decomposable module. ☑

PROPOSITION 5.58. (1) *If G and H are almost completely decomposable modules then $i(G \oplus H) = i(G)i(H)$.*

(2) *An almost completely decomposable module G is completely decomposable if and only if $i(G) = W$.*

(3) *If G and H are almost completely decomposable modules and $G \oplus L \approx H \oplus L$ for some finite rank torsion free module L, then $i(G) = i(H)$.*

PROOF: (1) If C and D are regulating submodules of G and H then it is immediately apparent that $C \oplus D$ is a regulating submodule of $G \oplus H$. Thus $i(G \oplus H) = [G \oplus H\colon C \oplus D] = [G\colon C][H\colon D] = i(G)i(H)$.

(2) G is completely decomposable if and only if G is a regulating submodule for itself, i.e. if and only if $i(G) = [G\colon G] = W$.

(3) We use induction on the size of $\mathbf{T}'(G)$. If $\mathbf{T}'(G) = \{\mathbf{t}\}$, then G is $\mathbf{t}$-projective and so if $G \oplus L \approx H \oplus L$ for some L then $H \sim G$ by Theorem 3.24, and so by Proposition 4.16 H must also be $\mathbf{t}$-projective. Thus $i(G) = i(H) = W$. If $\mathbf{T}'(G)$ has more than one type, let $\mathbf{t}$ be minimal in $\mathbf{T}'(G)$. By Lemma 5.54 $i(G) = i(G[\mathbf{t}])[G\colon G(\mathbf{t}) + G[\mathbf{t}]]$. Furthermore by Lemma 5.54 $G = (G(\mathbf{t}) + G[\mathbf{t}])_*$, and likewise $H = (H(\mathbf{t}) + H[\mathbf{t}])_*$. Now let $\mathbf{F}$ be the additive functor given by $\mathbf{F}(X) = (X(\mathbf{t}) + X[\mathbf{t}])_*/(X(\mathbf{t}) + X[\mathbf{t}])$. By Proposition 4.23 $\mathbf{F}(L)$, $\mathbf{F}(G)$, and $\mathbf{F}(H)$ all have finite length. Since $G \oplus L \approx H \oplus L$, $\mathbf{F}(G) \oplus \mathbf{F}(L) \approx \mathbf{F}(H) \oplus \mathbf{F}(L)$, and so $\mathbf{F}(G) \approx \mathbf{F}(H)$ by the Krull-Schmidt Theorem for finite length modules, so that $[G\colon G(\mathbf{t}) + G[\mathbf{t}]] = [(G(\mathbf{t}) + G[\mathbf{t}])_*\colon G(\mathbf{t}) + G[\mathbf{t}]] = [(H(\mathbf{t}) + H[\mathbf{t}])_*\colon H(\mathbf{t}) + H[\mathbf{t}]] = [H\colon H(\mathbf{t}) + H[\mathbf{t}]]$. And by induction $i(G[\mathbf{t}]) = i(H[\mathbf{t}])$. Thus $i(G) = i(H)$. ☑

We can now make explicit the assertion in the remark preceding Proposition 5.47 that in Example 4.25 the submodule K of $G \oplus H$ "uses up all the glue" that holds G and H together, so that if $G \oplus H = K \oplus L$ then L must be completely decomposable. In fact, we can now see that this simply means that $i(K) = i(G \oplus H)$ so that $i(L) = W$ and therefore L is completely decomposable.

We now get a Theorem, which anticipates the ideas of Chapter 10. We first need the following lemma, which is vaguely analogous to (but much weaker than) Nakayama's Lemma.

LEMMA 5.59. *If G is a reduced finite rank torsion free module and I is a non-trivial ideal, then there exists $0 \neq J \subseteq I$ such that $JM \subsetneq M$ for every non-trivial submodule M of G.*

PROOF: If $IM = M$ and $IN = N$ then $I(M+N) = M+N$. Thus there exists a unique largest submodule M of G such that $IM = M$. Now if $IM = M$ then $IM_* = M_*$, so it follows that $M \lhd G$. If $M \neq 0$ then $M \not\subseteq p^\infty G$ for some prime p since G is reduced, so M is not p-divisible. Thus $M \supsetneq pM = pIM$ and $pI \subseteq I$. Thus there exist non-trivial ideals $I' \subseteq I$ such that $I'M \neq M$. Now if $M' \subseteq G$ is such that $I'M' = M'$ then $M' \supseteq IM' \supseteq I'M' = M'$ so that $M' \subseteq M$ by the maximality of M, and again if we choose M' maximal with this property then $M' \lhd G$ and $M' \subsetneq M$. Since G has the descending chain condition on pure submodules, eventually we find an ideal $J \subseteq I$ such that $JM = 0$ only for $M = 0$. $\boxed{\checkmark}$

PROPOSITION 5.60. *If G, H, and L are finite rank torsion free modules such that $G \oplus L \approx H \oplus L$, and let I be a non-trivial ideal of W. Then there exists a monomorphism $\varphi\colon G \to H$ such that $H/\varphi(G)$ has finite length and $[H\colon \varphi(G)]$ is comaximal with I.*

PROOF: It suffices to prove the Proposition for $G/\mathbf{d}(G)$ and $H/\mathbf{d}(H)$ since ****. Thus we may suppose WLOG that G is reduced. Thus it follows from Lemma 5.59 that, replacing I by a still smaller non-trivial ideal, there is no loss of generality in suppose that $IM \subsetneq M$ for every non-trivial submodule M of G.

Now let $R = \operatorname{End} G$ and define a function $\mathbf{H}$ from finite rank torsion free modules to right R-modules by $\mathbf{H}(X) = \operatorname{Hom}(G,X)/I\operatorname{Hom}(G,X)$. Then $\mathbf{H}(X)$ has finite length (as a W-module and a fortiori as an R-module) and so from the isomorphism $\mathbf{H}(G) \oplus \mathbf{H}(L) \approx \mathbf{H}(H) \oplus \mathbf{H}(L)$ we deduce that $\mathbf{H}(G) \approx \mathbf{H}(L)$ by the Krull-Schmidt Theorem for finite length modules. In particular, $\mathbf{H}(G)$ is a cyclic R-module.

Now define a functor $\mathbf{F}$ from finite rank torsion free modules to finite length W-modules by letting $\mathbf{F}(X)$ be the image of the composition $G \otimes \operatorname{Hom}(G,X) \to X \to X/IX$, where the first map is simply given by $g \otimes \varphi \mapsto \varphi(g)$. Since every $g \in G$ has the form $\varphi(g)$ where $\varphi = 1 \in \operatorname{Hom}(G,G)$, it follows that $\mathbf{F}(G) = G/IG$. As with the functor $\mathbf{H}$ in the previous paragraph we conclude from the classical Krull-Schmidt Theorem that $\mathbf{F}(H) \approx \mathbf{F}(G) = G/IG$. But now $H/IH \oplus L/IL \approx G/IG \oplus L/IL$, and since this is an isomorphism of finite length modules we conclude that $H/IH \approx G/IG$. Thus $\mathbf{F}(H) \approx \mathbf{F}(G) = G/IG \approx H/IH$, and since $\mathbf{F}(H) \subseteq H/IH$ and these are finite length modules we conclude that $\mathbf{F}(H) = H/IH$.

Now let $\gamma \in \operatorname{Hom}(G,H)$ be a generator for the right R-module $\mathbf{H}(H)$. Then $\mathbf{F}(H)$ is generated by elements of the form $\gamma\varphi(g) + IH \in H/IH$, so that $H/IH = \mathbf{F}(H) = (\gamma(H) + IH)/IH$. This shows that γ induces a surjection $\bar\gamma\colon G/IG \to H/IH$. Since G/IG and H/IH are isomorphic finite length modules it follows that $\bar\gamma$ is an isomorphism. Thus if $K = \operatorname{Ker}\gamma$ then $K \subseteq IG$, but $K \lhd G$ so $K \subseteq IK \subseteq K$. But by assumption, $IK = K$ implies $K = 0$. Thus γ is monic.

Now $0 = \operatorname{Ker}\bar\gamma = (\gamma^{-1}(IH) + IG)/IG$ and it follows that $I(H/\gamma(G)) = (\gamma(G) + IH)/\gamma(G) = H/\gamma(G)$, so that if $I \subseteq p$ for some prime p then $p \notin \operatorname{Ass}(H/\gamma(G))$ and thus W/p does not appear in a composition series for $H/\gamma(G)$, so that $[H\colon \gamma(G)] \not\subseteq p$. Thus no prime ideal contains both I and $[H\colon \gamma(G)]$, so $[H\colon \gamma(G)] + I = W$, as required. $\boxed{\checkmark}$

We will postpone looking at all the fascinating consequences of Proposition 5.60 for finite rank torsion free modules in general until Chapter 10. For almost completely decomposable modules, however, we have a very specific result, as follows:

THEOREM 5.61. *Suppose that W is a principal ideal domain and let G and H be almost completely decomposable W-modules. The following conditions are equivalent:*

(1) *There exists a finite rank torsion free module L such that $G \oplus L \approx H \oplus L$.*
(2) *$G \oplus C \approx H \oplus C$, where C is a regulating submodule of G.*
(3) *$i(G) = i(H)$ and G contains a submodule G' such that G' is quasi-equal to G, $G' \approx H$ and the ideals $[G : G']$ and $i(G)$ are comaximal.*

PROOF: (1) $\Rightarrow$ (3): Suppose that $G \oplus L \approx H \oplus L$ for some L. Then by Proposition 5.58 $i(G) = i(H)$. Furthermore by Proposition 5.60 there exists $\gamma\colon G \to H$ such that $\gamma(G) \approx G$ and $\gamma(G)$ is quasi-equal to H and $[H\colon \gamma(G)] + i(G) = W$.

(2) $\Rightarrow$ (1): Clear.

(3) $\Rightarrow$ (2): Suppose that $i(G) = i(H)$ and there there exists $\varphi \in \mathrm{Hom}(H, G)$ such that φ is a quasi-isomorphism and $i(G) + [G\colon \varphi(H)] = W$. Choose $u \in [G\colon \varphi(H)]$, $v \in i(G)$, such that $u + v = 1$. Let C be a regulating submodule of G. Then $[G\colon C] = i(G)$. Thus by Lemma 5.52 $vG \subseteq i(G)G \subseteq C$ and $uG \subseteq \varphi(H)$. Let $\alpha\colon G \to H \oplus C$ be defined by $\alpha(g) = (\varphi^{-1}(ug), vg)$, and let $\beta\colon H \oplus C \to G$ by $\beta(h, c) = \varphi(h) + c$. Then $\beta\alpha(g) = \varphi(\varphi^{-1}(ug)) + vg = (u + v)g = g$ so that α is a split monomorphism. Thus $H \oplus C \approx G \oplus D$ for some D. Since $G \sim H$, by Theorem 3.24 $D \sim C$. Furthermore by Proposition 5.58 $i(G)i(D) = i(H)i(C)$ and since $i(G) = i(H)$ it follows that $i(D) = i(C) = W$, since C is completely decomposable. Thus by Proposition 5.58 D is completely decomposable. Since W is a principal ideal domain and C and D are quasi-isomorphic completely decomposable modules, $D \approx C$ by Proposition 4.28. Thus $G \oplus C \approx H \oplus C$. ☑

EXAMPLE 5.62. Example 1.49 is one good example for Theorem 5.62. We will use the same idea, but make it a little more complicated. Let A_1, A_2, A_3 be rank-one modules with mutually incomparable types, such that $A_i \cap A_j$ is the same for all pairs $i \neq j$ and such that there exists a prime p such that none of the A_i is p-divisible. We may assume WLOG that $1 \in A_i$ and $1 \notin pA_i$ for all i. Let u, $v \notin p$. Let

$$\begin{aligned} C &= A_1 \oplus A_2 \oplus A_3 \\ G &= (A_1 \oplus A_2 \oplus A_3) + p^{-1}(1, 1, 1) \\ H &= (A_1 \oplus A_2 \oplus A_3) + p^{-1}(1, u, v) \subseteq Q \oplus Q \oplus Q. \end{aligned}$$

Then C is a regulating submodule of both G and H, $i(G) = i(H) = p$, and $G \oplus C \approx H \oplus C'$, for some completely decomposable module C' quasi-isomorphic to C.

PROOF: Let $\mathbf{t}_i = \mathbf{t}(A_i)$. By the assumption on the A_i, $\mathbf{t}_i \wedge \mathbf{t}_j = \mathbf{IT}(G)$ for all pairs $i \neq j$. Then $\mathbf{T}(G) = \mathbf{T}(H) = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{IT}(G)\}$, $\mathbf{T}'(G) = \mathbf{T}'(H) = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$. For $i = 1, 2, 3$, $G(\mathbf{t}_i) = A_i \subseteq A_1 \oplus A_2 \oplus A_3$ and $G^*(\mathbf{t}_i) = 0$, so that $G_{\mathbf{t}_i} = A_i$. Thus $C = A_1 \oplus A_2 \oplus A_3 = G_{\mathbf{t}_1} \oplus G_{\mathbf{t}_2} \oplus G_{\mathbf{t}_3}$, so C is a regulating submodule of G and

$i(G) = i(H) = [G\colon C] = [H\colon D] = p$. Now let G' be the submodule of G generated by $uvA_1 \oplus vA_2 \oplus uA_3$ together with $uvp^{-1}(1,1,1)$ and let $\varphi\colon QH \to QG$ be defined by $\varphi(x,y,z) = (uvx,\, vy,\, uz)$. Then $\varphi(A_1) = uvA_1$, $\varphi(A_2) = vA_2$, $\varphi(A_3) = uA_3$, and $\varphi(p^{-1}(1,u,v)) = uvp^{-1}(1,1,1)$ so that in fact $\varphi(H) = G'$ and $G' \approx H$. Furthermore $G/G' \approx W/(uv) \oplus W/(u) \oplus W/(v)$ so that $[G\colon G']$ is the principal ideal generated by u^2v^2, which by assumption is relatively prime to p. Therefore by (the proof of) Theorem 5.61 $G \oplus C \approx H \oplus C'$ for some $C' \sim C$.

In fact, this can be seen more explicitly from Proposition 5.15, and one can even see that we can choose $C' = C$. Since p is maximal and $u,\, v \notin p$ there exist $r,\, s \in W$ and $p_1,\, p_2 \in p$ with $ru + p_1 = sv + p_2 = 1$. Let $\tilde{G}$ be the pure closure of the submodule of $H \oplus A_2 \oplus A_3$ generated by $(1,0,0,0,0)$, $(0,u,0,p_1,0)$, and $(0,0,v,0,p_2)$. Then $G \approx \tilde{G}$ by the mapping

$$\begin{aligned}\psi\colon (1,0,0) &\mapsto (1,0,0,0,0)\\ (0,1,0) &\mapsto (0,u,0,p_1,0)\\ (0,0,1) &\mapsto (0,0,v,0,p_2).\end{aligned}$$

Note that $\psi(p^{-1}(1,1,1)) = p^{-1}(1,u,v,p_1,p_2) \subseteq Q\psi(G) \cap (H \oplus W \oplus W) \subseteq \tilde{G}$. Furthermore, with $C \subseteq H$ as in the previous paragraph, $\tilde{G} + (C \oplus A_2 \oplus A_3) = H \oplus A_2 \oplus A_3$ since

$$\begin{aligned}p^{-1}(1,u,v,0,0) &\subseteq p^{-1}[(1,0,0,0,0) + (0,u,0,p_1,0) + (0,0,v,0,p_2)] + p^{-1}(0,0,0,p_1,p_2)\\ &\subseteq H \oplus A_2 \oplus A_3\end{aligned}$$

because $p_ip^{-1} \subseteq pp^{-1} = W \subseteq A_2,\, A_3$. Now we claim that $\tilde{G}$ is a balanced submodule of $H \oplus A_2 \oplus A_3$. Define $\gamma\colon H \oplus A_2 \oplus A_3 \to A_2 \oplus A_3$ by $\gamma(x,y,z,y',z') = (p_1y - uy',\, p_2z - vz')$. (Note that $\gamma(p^{-1}(1,u,v,0,0)) \in p^{-1}(p_1u, p_2v) \in A_2 \oplus A_3$.) Since $\gamma(0,p_1,0,-r,0) = (p_1 + ru, 0) = (1,0)$ and $\gamma(0,0,1,0,-s) = (0, p_2 + sv) = (0,1)$ it follows that $A_2 = \gamma((H \oplus A_2 \oplus A_3)(\mathbf{t}_2))$ and $A_3 = \gamma((H \oplus A_2 \oplus A_3)(\mathbf{t}_3))$, so that γ is a balanced surjection. Clearly $\tilde{G} \subseteq \operatorname{Ker}\gamma$ and since the image of γ has rank 2 and $\tilde{G} \lhd H \oplus A_2 \oplus A_3$, $\tilde{G} = \operatorname{Ker}\gamma$. By Proposition 5.15 γ is split, so that

$$H \oplus A_2 \oplus A_3 \approx \tilde{G} \oplus A_2 \oplus A_3 \approx G \oplus A_2 \oplus A_3.$$

Thus a fortiori $H \oplus C \approx G \oplus C$. $\boxed{\checkmark}$

A module G which is known to be completely decomposable is determined up to quasi-isomorphism once we know that rank of $G(\mathbf{t})$ for each $\mathbf{t} \in \mathbf{T}(G)$. Thus completely decomposable modules are a model in the search for classes of modules which can be determined by simple invariants.

Unfortunately there are very few classes of modules which can be completely described by a finite family of integers. However Butler modules are a natural generalization

of completely decomposable modules in that a Butler module is determined up to quasi-isomorphism by a finite family of subspace of QG*, namely the subspaces* $QG(\mathbf{t})$ *for* $\mathbf{t} \in \mathbf{T}(G)$*.*

The category whose objects consist of finite dimensional Q*-vector spaces together with a finite family of subspaces (and whose morphisms consist of those linear transformations which respect those subspaces) seems (and is) far simpler than the category of finite rank torsion free modules under quasi-homomorphisms. However understanding it is by no means child's play. One advantage of this category – the category of representations of quivers – is that a lot of algebraists have studied it recently.*