The Logarithm Function

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A logarithm function is characterized by the equation $L(xy) = L(x) + L(y)$. If $L(x)$ is any differentiable function defined on the positive reals and satisfying this equation, then its derivative is given by $L'(x) = C/x$, where $C = L'(1)$. Furthermore, $L(x) = \log_B x$, where $B = e^C$.

The main purpose of these notes is to give a modern definition of the logarithm function, which is the definition found in several contemporary calculus books, and to motivate the definition by examining the properties of the logarithm function as classically defined. This will involve going through the fairly standard calculation of the derivative of the logarithm, and thinking rather carefully about what is really going on in this calculation. (A somewhat condensed version of this treatment is given in the latest (i.e. 7^{th}) edition of Salas & Hille, the book I've been teaching from for the past 20 years.)

1. The Classical Definition

I think that many students find it difficult to become comfortable with the logarithm function. I know that I did. And I have an idea as to why this is so.

The definitions of some functions tell you directly how to compute the value of that function at any given point. An example of such a definition would be $f(x) = \dfrac{x^4 - 9x^3 + 27}{x^5 + 8x^3 - 7x^2 + 9}$. These functions are easy for the mind to accept.

Even the trig functions fall in this category, although the method for computing them described by the usual definitions involves constructing a geometric figure, and thus is not very helpful in practice. (To determine $\sin\theta$, for instance, one can construct a right triangle having θ as one of the acute angles and a hypotenuse of length 1. The value of $\sin\theta$ can then be measured as the length of the side opposite the angle θ.)

The definitions of other functions, though, don't tell you how to compute the value. Instead, they tell you how to recognize whether a particular answer is the correct one or not.

An example of such a function is $f(x) = \sqrt[3]{x}$. One can check that $\sqrt[3]{125}$ is 5 by verifying that $5^3 = 125$. But when required to compute, say, $\sqrt[3]{216}$, one winds up resorting to trial and error to discover the answer 6. If one is asked for $\sqrt[3]{50}$, then trial and error does not work, since the answer is not an integer (and is in fact not even a rational number). If one doesn't have a calculator or set of tables handy, then one will be able to come up with at best only a very crude approximation (unless one knows a sophisticated trick such a Newton's Method, which can produce a fairly good approximation). The answer is clearly between 3 and 4, and a fairly quick calculation shows that 3.5 is too small and 3.7 is slightly too big, since $3.7^3 = 50.653$.

It's not really the practical difficulty that matters, though. In practice, a function such as $f(x) = \dfrac{x^4 - 9x^3 + 27}{x^5 + 8x^3 - 7x^2 + 9}$ is also not very easy to compute by hand. And one usually has a calculator handy in any case.

But conceptually, I believe that a function that's described by a specific algorithm is easier for the mind to accept than one that's essentially described as the solution of an equation. (For instance, the definition of the cube root function basically says that $\sqrt[3]{a}$ is defined to be the solution to the equation $x^3 = a$.)

Classically, $\log_B a$ is defined to be the solution to the equation $B^x = a$. For instance, one verifies that $\log_2 32 = 5$ by noting that $2^5 = 32$.

If one wants to calculate by hand $\log_2 12$, then one is in real trouble. One sees quickly that the answer is between 3 and 4, since $2^3 = 8$ and $2^4 = 16$. It seems reasonable to try 3.5. This involves seeing whether $2^{3.5}$ is larger or smaller than 12.

But what is $2^{3.5}$?

Since $3.5 = 7/2$,

$$2^{3.5} = 2^{\frac{7}{2}} = \sqrt{2^7} = \sqrt{128}.$$

Now $\sqrt{128}$ is not something one can compute instantaneously. However since $11^2 = 121$, one can say that $2^{3.5} = \sqrt{128}$ is slightly bigger than 11, and certainly smaller than 12, so that $\log_2 12$ is definitely larger than 3.5.

One might try 3.6. This would involve comparing $2^{3.6}$ with 12. To do this, note that $3.6 = 36/10 = 18/5$, so that $2^{3.6} = \sqrt[5]{2^{18}}$. At this point, any reasonable person is going to go find a good calculator, but if one resorts to a little cleverness one can see that

$$12^5 = (2^2)^5 3^5 = 2^{10}\, 3^5 = 2^{10}\, 243$$

which is smaller than

$$2^{18} = 2^{10}\, 256.$$

Since $12^5 < 2^{18}$, we see that $12 < \sqrt[5]{2^{18}} = 2^{3.6}$, so that the exponent x for which $2^x = 12$ must be smaller than 3.6. In other words, $\log_2 12 < 3.6$.

Now at this point, almost any mathematician will object and say that all the above is irrelevant and has almost nothing to do with calculus. It's the **concept** of the logarithm that's important. One doesn't have to know how to actually calculate logarithms in order to understand how the logarithm function behaves for purposes of calculus.

This is quite correct. But I think that accepting this point of view is a big step in sophistication for many students, and one should not simply gloss over it. If the student finds the logarithm concept somehow very alien, then I think that the going carefully though the preceding calculations may help.

The important thing, in any case, is to remember that the classical definition of the logarithm amounts to saying that

$$\boxed{\ell = \log_B a \qquad \Longleftrightarrow \qquad B^\ell = a.}$$

It can be seen that the word "logarithm" is almost equivalent to the word "exponent." It's just that one is looking at the exponent from a different point of view.

If one understands exponents, then one essentially understand logarithms.

For instance, it is well known that if one adds exponents, then one multiplies the corresponding values created by those exponents. In other words,

$$\begin{aligned} \text{If} \quad a_1 = B^{\ell_1} \quad &\text{and} \quad a_2 = B^{\ell_2} \\ \text{then} \quad a_1 a_2 = B^{\ell_1} B^{\ell_2} &= B^{\ell_1 + \ell_2}. \end{aligned}$$

Since the exponents here are the logarithms

$$(\ \ell_1 = \log_B a_1, \qquad \ell_2 = \log_B a_2, \qquad \ell_1 + \ell_2 = \log_B (a_1 a_2) \quad)$$

this says that

$$\boxed{\log_B (a_1 a_2) = \log_B a_1 + \log_B a_2.}$$

Similarly one sees that

$$\boxed{\log_B (a_1 / a_2) = \log_B a_1 - \log_B a_2.}$$

Also note that since $B^0 = 1$, it follows that

$$\boxed{\log_B 1 = 0.}$$

2. The Derivative of the Logarithm

By definition,

$$\frac{d}{dx} \log_B x = \lim_{h \to 0} \frac{\log_B (x+h) - \log_B x}{h}.$$

In trying to evaluate this, the most important thing is *not to get hung up on thinking about what the logarithm really is.* Instead, just use the fact that the logarithm of a product is the sum of the logarithms of the two factors, plus a few other tricks.

$$\begin{aligned}
\lim_{h\to 0} \frac{\log_B(x+h) - \log_B x}{h} &= \lim_{h\to 0} \frac{\log_B(x\left(1+\frac{h}{x}\right)) - \log_B x}{h} \\
&= \lim_{h\to 0} \frac{\log_B x + \log_B\left(1+\frac{h}{x}\right) - \log_B x}{h} \\
&= \lim_{h\to 0} \frac{\log_B(1+\frac{h}{x}) - 0}{h} \\
&= \lim_{\frac{h}{x}\to 0} \frac{\log_B(1+\frac{h}{x}) - \log_B 1}{h} \\
&= \lim_{h/x\to 0} \frac{\log_B(1+\frac{h}{x}) - \log_B 1}{x\,h/x} \\
&= \frac{1}{x} \lim_{h/x\to 0} \frac{\log_B(1+\frac{h}{x}) - \log_B 1}{h/x}
\end{aligned}$$

We have used here the fact that $\log_B 1 = 0$ and that as we let h approach 0 in order to take the limit, x does not change, so that saying that h approaches 0 is the same as saying that h/x approaches 0. For the same reason, the factor $\frac{1}{x}$ can be brought outside the limit sign.

We have been able to do the whole calculation above without giving any thought to what the logarithm function really means. On the other hand, the final result seems, if anything, even more complicated than what we started with.

In fact, though, we have achieved a substantial simplification. This is because

$$\lim_{h/x\to 0} \frac{\log_B(1+\frac{h}{x}) - \log_B 1}{h/x}$$

does not actually depend on x at all. The quantity h/x is simply something that is approaching 0, and if we write $k = h/x$, then the calculation so far yields

$$\frac{d}{dx}\log_B x = \frac{1}{x} \lim_{k\to 0} \frac{\log_B(1+k) - \log_B 1}{k} = \frac{1}{x}\, C,$$

where C is a constant independent of x.

Computing C is the most difficult part of the whole calculation. It turns out that $C = \log_B e$, where e is a special irrational number which is as important in calculus as that other more famous irrational number π. But it's more interesting to notice that

$$C = \lim_{k\to 0} \frac{\log_B(1+k) - \log_B 1}{k}$$

is simply the derivative of the logarithm function evaluated at $x = 1$. In other words, if we let $L(x)$ denote $\log_B x$, then the calculation above shows that

$$\boxed{L'(x) = \frac{1}{x}\, L'(1).}$$

And we derived this simply by using the algebraic rules obeyed by the logarithm.

3. Some Problems with the Classical Definition

As stated in the beginning, the real purpose of these notes is not to simply repeat the standard calculation of the derivative of the logarithm function, but to replace the classical definition of the logarithm by a more subtle one, which is in some ways superior.

There are actually a number of problems with the classical definition. Although these problems are by no means insurmountable, they do require quite a bit of effort to get around.

We have defined $\log_B x$ as the solution to an equation. To wit, we have said that

$$\ell = \log_B x \qquad \text{means} \qquad B^\ell = x.$$

There's a very important principle that should cause one to object to this definition. Namely, before one can defined a function as the solution to an equation, one needs to find a reason for believing that such a solution exists.

In other words,

How do we know that there is an exponent ℓ such that $B^\ell = x$ (for positive x)?

(We might also ask how we know that there's only one. But this turns out to be an easier question.)

This is not a frivolous objection, when one considers the fact that one is almost never able to calculate the value for the logarithm ℓ precisely and in fact, as seen in the beginning of these notes, it's not even all that easy to see whether a particular value for ℓ is the correct choice or not, since computing B^ℓ is itself not at all easy if ℓ is not an integer.

Getting around these difficulties leads one into some rather deep mathematical waters, involving the use of something called the *Intermediate Value Theorem* for continuous functions.

But even without getting into all this, one can note that the function B^ℓ is not at all as straightforward as it might as first seem. For instance, it is not even completely obvious that, for fixed B, the function $f(\ell) = B^\ell$ is a continuous function of ℓ. Take two numbers, for instance, such as $5/8$ and $7/11$. These are fairly close to each other. In fact, $5/8 = .625$ and $7/11 \approx .636$. Now choose a fixed B, say $B = 10$. Then

$$\begin{aligned} 10^{5/8} &= \sqrt[8]{10^5} = \sqrt[8]{100,000}\,, \\ 10^{7/11} &= \sqrt[11]{10^7} = \sqrt[11]{10,000,000}. \end{aligned}$$

It doesn't seem at all obvious that these two values will be at all close to each other. (They are, however. To see why, put both fractions over a common denominator.

$$\begin{aligned} 10^{5/8} &= 10^{55/88} = \sqrt[88]{10^{55}}\,, \\ 10^{7/11} &= 10^{56/88} = \sqrt[88]{10^{56}}. \end{aligned}$$

It's not to hard to prove that these are fairly close.)

The most serious objection to the use of the function B^ℓ is the question of what one does when ℓ is irrational. If $\ell = \sqrt{3}$, then it's impossible to write ℓ as a fraction p/q with p and q both integers, and so, technically at least, B^ℓ isn't even defined. An engineer or a physicist might not have a big problem with this, and suggest just using a good approximation for ℓ. Since $\sqrt{3} \approx 1.732$, one could just say that $B^{\sqrt{3}} \approx B^{1.732} = \sqrt[1000]{B^{1732}}$.

Mathematicians, though, like to know that things can be done precisely, at least in theory. And the theory for $B^{\sqrt{3}}$ breaks down. It can be fixed, but it takes a bit of work to do so.

4. A More Modern Approach

In my opinion, there is one idea that more than anything else distinguishes the mathematics of the Twentieth Century from the more classical theory. And that is the following:

> It doesn't matter a whole lot what things **are**.
> What really matters is how they **behave.**

This idea, which when formalized becomes the *Axiomatic Method* in mathematics, is something that students intuitively take for granted. For instance, the derivative, when first encountered in Calc I, seems like a very intimidating concept. However one quickly derives a bunch of rules for calculating derivatives and students realize that it doesn't even matter if they don't understand what a derivative is; as long as one knows the rules, one can do all the problems. (In order to use the derivative in applications, though, one has to understand what it really is.)

In the classical approach to mathematics, one would define a concept by some often fairly complicated formula, and then do a lot of hard calculations in order to establish some nice rules that the concept would obey. Then once one had these nice rules, everything would flow fairly smoothly.

In the modern approach, **one starts with the nice rules,** and then proves all the formulas and theorems which are consequences of these nice rules. Then, after all this nice theory has been established, one goes back and shows how the original concept can be computed in a concrete way (and thus establishes its existence).

Basically, the classical approach and the modern approach both need to obtain the same body of results. The only difference is a matter of where one starts.

Those who have learned to think in terms of the modern approach talk about it being *cleaner* or *more elegant*. On the other hand, the disadvantage is that one has to be willing to prove a lot of theorems about a concept before one knows in concrete terms what that concept is.

The other disadvantage is that the more modern approach tends to be highly verbal, where there are a lot more paragraphs than equations, and at first this sometimes seems to turn mathematics into a mere word game. This presents a real problem for the many students who decide to major in the hard sciences precisely because their language skills are not very good.

Using the modern approach, we don't worry in the beginning about how the logarithm function is concretely defined. Instead, one simply considers a differentiable function $L(x)$ defined for all strictly positive real numbers x and satisfying the axiom

$$\boxed{L(xy) = L(x) + L(y).}$$

As already shown above, this axiom is all we need to derive the formula

$$\boxed{L'(x) = C/x \qquad \text{where} \qquad C = L'(1).}$$

But it turns out that we could also have started out with this second formula.

Theorem. Let $L(x)$ be a differentiable function defined on the (strictly) positive real numbers. Then

$$L(xy) = L(x) + L(y) \text{ for all } x \text{ and } y$$

if and only if

$$L(1) = 0 \text{ and } L'(x) = C/x \text{ for some constant } C.$$

Furthermore, in this case $C = L'(1)$ and for also $L(x^r) = rL(x)$ for all r.

PROOF: If $L(xy) = L(x) + L(y)$, then in particular $L(1) = L(1 \cdot 1) = L(1) + L(1)$. Subtracting $L(1)$ from both sides, we see that $L(1) = 0$. Also, we get $L(x) = L(y \cdot x/y) = L(y) + L(x/y)$. Subtracting $L(y)$ from both sides of this equation yields $L(x/y) = L(x) - L(y)$. Furthermore, the calculation on p. 4 can be adapted to show that $L'(x) = L'(1)/x$. Namely (where this time we skip a few steps),

$$\begin{aligned}
L'(x) &= \lim_{h\to 0} \frac{L(x+h) - L(x)}{h} \\
&= \lim_{h\to 0} \frac{L(x) + L\left(1 + \dfrac{h}{x}\right) - L(x)}{h} \\
&= \lim_{\frac{h}{x}\to 0} \frac{L(1 + \frac{h}{x}) - L(1)}{h} \\
&= \frac{1}{x} \lim_{h/x\to 0} \frac{L(1 + \frac{h}{x}) - L(1)}{h/x} \\
&= \frac{1}{x} L'(1).
\end{aligned}$$

On the other hand, start with the principles $L(1) = 0$ and $L'(x) = C/x$ for some constant C. Now let a be an arbitrary but fixed real number and compute the derivative of the function $L(x + a)$. By the chain rule,

$$\frac{d}{dx} L(xa) = \frac{C}{xa} \frac{d\,(xa)}{dx} = \frac{C}{xa}\, a = L'(x).$$

Therefore the two functions $L(xa)$ and $L'(x)$ have the same derivative, so they must differ by a constant:

$$L(xa) = L(x) + K.$$

In particular, $L(a) = L(1 \cdot a) = L(1) + K = 0 + K = K$, so that $K = L(a)$. Therefore we have $L(xa) = L(x) + L(a)$. Writing y instead of a, we have established that

$$L(xy) = L(x) + L(y).$$

Now compute the derivative of $L(x^r)$ by the Chain Rule.

$$\begin{aligned}\frac{d}{dx}L(x^r) &= \frac{C}{x^r}\frac{d}{dx}(x^r)\\ &= \frac{C\,rx^{r-1}}{x^r}\\ &= \frac{rC}{x}\\ &= rL'(x).\end{aligned}$$

From this we see that the two functions $rL(x)$ and $L(x^r)$ have the same derivative and therefore differ by a constant. But substituting $x = 1$ shows that this constant must be 0. Therefore

$$L(x^r) = rL(x). \qquad \boxed{\checkmark}$$

The above theorem is at first glance amazing, because it shows that the formula $L'(x) = L'(1)/x$ is true not only for logarithm functions, but for any function satisfying the axiom $L(xy) = L(x) + L(y)$.

It turns out, though, that the only functions satisfying this axiom are logarithms. In fact,

Theorem. If $L(x)$ satisfies the axiom $L(xy) = L(x) + L(y)$ and if B is a number such that $L(B) = 1$, then

$$\boxed{\ell = L(x) \text{ if and only if } B^\ell = x.}$$

(Thus $L(x)$ is the logarithm of x with respect to the base B in the traditional sense.)

PROOF: On the one hand, if $x = B^\ell$ then $L(x) = L(B^\ell) = \ell L(B) = \ell \cdot 1 = \ell$.

Conversely, if $\ell = L(x)$, then the above shows that $L(x) = L(B^\ell)$. But since $L'(x) = L'(1)/x$ and we are considering only positive x, then $L'(x) > 0$ for all x if $L'(1) > 0$, and therefore L is a strictly increasing function; likewise L is strictly decreasing if $L'(1) < 0$. In either case, $L(x) = L(B^\ell)$ is only possible if $x = B^\ell$, as claimed.

(Note: $L'(1) = 0$ is not possible. Otherwise $L'(x) = L'(1)/x = 0$ for all x, so that $L(x)$ is a constant function. Thus $L(B) = L(1) = 0$, contrary to the assumption that $L(B) = 1$.) $\boxed{\checkmark}$

NOTE: If one is being ultra-careful, one should ask, *How do we know that there exists a number B such that $L(B) = 1$?* This will in fact be the case if and only if $L'(1) \neq 0$. As pointed out in the above proof, if $L'(1) = 0$ then L must be a constant function, and in fact $L(x) = 0$ for all x. On the other hand, if $L'(x) > 0$ or $L'(x) < 0$, then one can see from the Mean Value Theorem, for instance, that there must exist numbers a such that $L(a) \neq 0$. Now solve for r such that $rL(a) = 1$ and set $B = a^r$. Then $L(B) = L(a^r) = rL(a) = 1$. (There's actually still a small glitch here if one is being totally rigorous. But this is pretty convincing, and a more rigorous proof is indeed possible.)

5. Construction of the Natural Logarithm Function

Definition. The **natural logarithm** is defined to be the differentiable function $\ln x$ defined for all $x > 0$ and satisfying the following two axioms:

(1) $\ln(xy) = \ln x + \ln y$ for all $x,\ y > 0$.

(1) The derivative of $\ln x$ at $x = 1$ equals 1.

If for convenience we momentarily write $L(x) = \ln x$, then the preceding discussion shows that $L'(x) = 1/x$ for all positive x.

If the reader has the taste of a contemporary mathematician, then, after taking time to really think about everything which has been done (starting from p.7), his response may be something like, *Gee, this is really neat! We've found the derivative of the natural logarithm and all its customary properties without having to do any ugly calculations at all.*

If, on the other hand, he has a more classical taste, his (her) thought may be more like, *Ugh, this is a lot of really abstract and difficult thinking. It would be so much easier just to do whatever calculations are required!*

In any case, a valid objection can be made. Namely, everything that has been done so far was done under the **assumption** that we have a function $\ln x$ such that $\ln(xy) = \ln x + \ln y$ and that $\dfrac{d(\ln x)}{dx}(1) = 1$. But, so far at least, *we haven't actually constructed a function that has these properties!*

There are in fact two issues here:

(1) How do we know that there is any such function? (The problem of **existence**.)

(2) How do we know that there's only one function with these properties? (The problem of **uniqueness**.)

There's an important distinction to be made here.

The problem of constructing the natural logarithm function (i. e. the problem of existence and uniqueness) is not the same as the problem of how to calculate the natural logarithm in practice.

We don't need to know how computers actually calculate the natural logarithm any more than we need to know how they compute the sine and cosine functions. All that a mathematician needs to know is that there is some specific theoretical way (even if impractical) in which the value of $\ln x$ is determined.

The most obvious way of constructing the natural logarithm function is simply to use the classical definition. Namely, define $\ln x$ to be the exponent ℓ such that $e^{\ell} = x$. Here, we want e to be a number which satisfies that condition $\ln e = 1$.

Unfortunately, we can't define the natural logarithm by using e and then define e by using the natural logarithm, because that would be circular reasoning. So we have to figure out a way of deciding what e needs to be, and that involves some hard work. (Unfortunately, it's not a convenient number like 5 or -3.)

If one uses the classical definition of the logarithm at this stage, one winds up doing basically all the same work involved in developing the logarithm function in the traditional way. The only difference in the two approaches is whether one does the really hard work first or last.

6. The Bottom Line

The advantage of doing the theoretical work first, though, is that knowing the theory of the logarithm can suggest easier methods for actually constructing it.

In particular, the method used here will be based on the fact that $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$. Since the function $1/x$ is continuous for x positive, and since the Fundamental Theorem of Calculus guarantees that every continuous function has an anti-derivative, we can **define** the natural logarithm to be an anti-derivative of the function $1/x$. To make this specific, we will adjust the "constant of integration," as it were, to make $\ln 1 = 0$.

The way we actually do this is as follows. We choose a (theoretical) anti-derivative $F(x)$ for $1/x$. And then we **define** $\ln x = F(x) - F(1)$. It follows immediately that

$$\frac{d}{dx}(\ln x) = \frac{dF}{dx} = \frac{1}{x}$$

and that $\ln 1 = F(1) - F(1) = 0$.

To a mathematician, this is everything that's needed to show the existence of the natural logarithm function.

To most non-mathematicians, though, this definition seems somehow too mystical. This anti-derivative $F(x)$, although guaranteed by the Fundamental Theorem of Calculus, seems to live somewhere in Never Never Land. We have no idea what it actually looks like.

There is a way of making this definition seem more concrete. If $F(x)$ is an anti-derivative for $1/x$, then

$$\int_a^b \frac{dt}{t} = F(b) - F(a).$$

(Changing the variable of integration to t doesn't affect the answer, but avoids confusion in the next step.) In particular, we now get

$$\ln x = F(x) - F(1) = \int_1^x \frac{dt}{t}\,.$$

This definition,

$$\boxed{\ln x = \int_1^x \frac{dt}{t}\,,}$$

is the one given in several modern calculus books, for instance Salas & Hille. And while it may at first seem a bit difficult to think of a function as being defined by an integral which one doesn't know how to calculate (except by using the function $\ln x$ itself), the definition is actually quite concrete and one can visualize $\ln x$ as being the area under the appropriate part of the curve $y = \dfrac{1}{x}$. And one can compute it approximately (to as much accuracy as desired) by approximately calculating that area. (This is not the best way of calculating the logarithm function, though.)

In some ways, I think that once one gets used to it, this definition is actually more tangible than the traditional one.

7. The Exponential Function

By definition, e is the unique number such that $\ln e = 1$, i. e. $\int_1^e \frac{1}{t}\,dt = 1$. It turns out that e is irrational, but an approximate value is 2.718.

From the preceding, it can be seen that

$$\ln e^x = x \ln e = x.$$

As discussed earlier, for certain values of x, for instance $x = \sqrt{3}$ (as well as $x = \pi$ and $x = \sqrt[3]{2}$), it is not really clear how to raise a number to the x power, because x cannot be written precisely as a fraction p/q with p and q being integers. So what we can really say is that $\ln e^x = x$ whenever e^x is defined.

This actually gives us a cheap solution to the problem of deciding how to define e^x when x is irrational. Namely, we can specify, for instance, that $e^{\sqrt{3}}$ is the number y such that $\ln y = \sqrt{3}$. (This works because $\ln x$ is an increasing function for positive x, since its derivative $1/x$ is positive. Therefore there can't be two different values of y such that $\ln y = \sqrt{3}$. Furthermore, one can prove that there does exist a value of y that works by using something called the *Intermediate Value Theorem.*)

Since this new definition of e^x agrees with the usual definition whenever the usual one applies, in this way we get e^x defined as a differentiable function of x for all real numbers x. And it is also easy to show that the usual rule for exponents applies:

$$e^{x+y} = e^x e^y.$$

(To see that this is true, just take the logarithm of both sides.

$$\ln(e^{x+y}) = x + y = \ln e^x + \ln e^y = \ln(e^x e^y),$$

using the principle that $\ln a + \ln b = \ln(ab)$.)

It is easy to compute the derivative of e^x. In fact, if $y = e^x$ then $x = \ln y$ and so

$$1 = \frac{dx}{dx} = \frac{d}{dx}(\ln y) = \frac{y'}{y}$$

(using the chain rule), so that $y' = y$.

In other words, $\frac{d}{dx}(e^x) = e^x$.

It is also interesting to note the following more general principle, analogous to the way we computed the derivative of the logarithm function.

Theorem. Suppose that $E(x)$ is a non-zero differentiable function defined for all x. Then $E(x)$ satisfies the functional equation

$$E(x+y) = E(x)E(y)$$

if and only if

$$E(0) = 1 \quad \text{and} \quad E'(x) = CE(x),$$

where C is a constant. In this case, $C = E'(0)$.

PROOF: On the one hand, if $E(x+y) = E(x)E(y)$, then $E(0) = E(0+0) = E(0)E(0)$, and so either $E(0) = 0$ or one can divide by $E(0)$ to get $E(0) = 1$. But $E(0) = 0$ is not possible, otherwise for all x, $E(x) = E(0+x) = E(0)E(x) = 0$, contrary to the assumption that $E(x)$ is not the zero function.

Now one gets

$$\begin{aligned}
E'(x) &= \lim_{h\to 0} \frac{E(x+h) - E(x)}{h} \\
&= \lim_{h\to 0} \frac{E(x)E(h) - E(x)}{h} \\
&= \lim_{h\to 0} \frac{E(x)\,(E(h) - 1)}{h} \\
&= E(x)\,\lim_{h\to 0} \frac{E(h) - 1}{h} \\
&= E(x)\,\lim_{h\to 0} \frac{E(0+h) - E(0)}{h} \\
&= E(x)\,E'(0).
\end{aligned}$$

On the other hand, start by assuming it known that $E(0) = 1$ and $E'(x) = CE(x)$, where C is a constant. Then $C = C \cdot 1 = C\,E(0) = E'(0)$.
Let a be arbitrary but fixed, and differentiate $E(x+a)$ by the chain rule.

$$\begin{aligned}
\frac{d}{dx}E(x+a) &= CE(x+a)\,\frac{d}{dx}(x+a) \\
&= CE(x+a)
\end{aligned}$$

Now use the quotient rule to see that

$$\frac{d}{dx}\big(\,E(x+a)/E(x)\,\big) = \frac{CE(x)\,E(x+a) - E(x)\,CE(x+a)}{E(x)^2} = 0\,,$$

showing that $E(x+a)/E(x)$ is a constant. Substituting $x = 0$ and using the fact that $E(0) = 1$ shows that this constant equals $E(a)$. Thus $E(x+a)/E(x) = E(a)$ so that $E(x+a) = E(x)E(a)$. Since a is arbitrary, $E(x+y) = E(x)E(y)$ for all y. $\boxed{\checkmark}$

8. Complex Variables

Looking at the exponential function as characterized by the fact that it takes addition to multiplication leads in a natural and fairly simple way to Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta\,,$$

where i is the square root of -1. This is pointed out in the very interesting book *Where Mathematics Comes From* by George Lakoff and Rafael E. Núñez.

Euler's formula can be derived quite easily by looking at the Taylor series expansions for e^x, $\sin x$, and $\cos x$. But for some reason, undoubtedly personal prejudice on my part, I have always found describing e^x in terms of its Taylor series unsatisfactory on an intuitive level.

It will be useful to begin by reviewing the most basic facts of the algebra of complex numbers. The complex number system is obtained from the set of real numbers by adjoining the number $i = \sqrt{-1}$. All complex numbers have the form $a+bi$, where a and b are real numbers. One thinks of the number $a+bi$ as represented by the point (or vector) (a,b) in the Euclidean plans. Addition of complex numbers then corresponds to elementary vector addition. This fact is completely non-profound.

What is deeper is that multiplication of complex numbers can also be described geometrically if one thinks in terms of polar coordinates. The usual rules for change of coordinates show that a point with polar coordinates r and θ will have cartesian coordinates (x,y) where $x = r\cos\theta$ and $y = r\sin\theta$. Thus the complex number corresponding to polar coordinates r and θ will be

$$r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)\,.$$

And here, in the absolutely most basic conceptual framework of the complex number system, we suddenly encounter the very expression that constitutes the right-hand side of Euler's formula

But this in itself is not very deep. What is more significant is the fact that when one multiplies complex numbers what happens is that the absolute values multiply and the corresponding angles add. (Here, by the absolute value (or modulus) of a complex number $a+bi$ is meant $\sqrt{a^2+b^2}$, which is the value for r when we write (a,b) in polar coordinates. There are of course many numerical values corresponding to the angle θ (called the *argument*) for the polar coordinates of (a,b), but one choice that works is $\tan^{-1}\left(\dfrac{b}{a}\right)$ in case $a+bi$ is in the right half-plane, i. e. when $a \geq 0$.)

The proof that multiplication of complex numbers corresponds to addition of the angles and multipication of the moduli follows from the addition formulas for the sine and cosine:

$$\begin{aligned} r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\ &= r_1r_2(\,\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\,)\,. \end{aligned}$$

Thus algebra for complex numbers works just fine as long as one is adding, subtracting, and multiplying. Division is also not a problem. But when it comes to exponentiation, things are not so clear. Expressions such as i^i, or even 2^i, don't have any obvious interpretation. And one can't hope to prove Euler's formula until one define what $e^{i\theta}$ means.

Now having just looked at the exponential function as being characterized by the fact that it takes addition to multiplication, the fact that multiplication of complex numbers corresponds to addition of the corresponding angles should certainly attract our interest. If we set $F(\theta) = \cos\theta + i\sin\theta$, then as just seen, $F(\theta_1 + \theta_2) = F(\theta_1)F(\theta_2)$, which suggests that the function F is something like an exponential function.

We also notice that

$$F'(\theta) = \frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = iF(\theta)\,.$$

Now if the function $e^{i\theta}$ is meaningful and behaves according to the familiar rules of algebra and calculus, then we must have

$$e^{i(\theta_1+\theta_2)} = e^{\theta_1 e^i \theta_2}$$

$$= \frac{d}{d\theta}\,e^{i\theta} = ie^{i\theta}\,.$$

We have seen that the function $F(\theta) = \cos\theta + i\sin\theta$ (defined when θ is a real number) has these properties. Furthermore, it is the only function which has them. Because if we look at the quotient $e^{i\theta}/F(\theta)$ and apply the quotient rule, using the properties we have established, we get

$$\frac{d}{d\,\theta}\,\frac{e^{i\theta}}{F(\theta)} = \frac{F(\theta)ie^{i\theta} - iF(\theta)e^{i\theta}}{F(\theta)^2} = 0\,.$$

So we see that $e^{i\theta}/F(\theta)$ is constant. Since it equals 1 when $\theta = 0$, it follows that $e^{i\theta} = F(\theta) = \cos\theta + i\sin\theta$.

So here we get Euler's formula almost as a matter of definition. Alternatively, we could plausibly define $e^{i\theta}$ by its Taylor series expansion. Or we could define it as the solution to the differential equation $F' = iF$ with $F(0) = 1$. Ultimately it doesn't matter, because all these characterizations can be shown to be equivalent.

At this point, we have e^z defined when z is a real number and when z is pure imaginary: $z = iy$. But if $z = x + iy$, we get

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)\,.$$

Thus e^z is meaningful for all complex numbers.

As for logarithms, clearly we should require that $\ln(\cos y + i\sin y) = \ln e^{iy} = iy$. Then if we write a complex number $x + iy$ in terms of polar coordinates, i. e. $x + iy = re^{i\theta}$, we get

$$\ln(x + iy) = \ln r + \ln(e^{i\theta}) = \ln r + i\theta\,.$$

Since r is a real number, $\ln r$ is defined, and thus $\ln z$ is defined for all complex numbers z.

There is a slight catch, though. The value θ as given above is not uniquely determined by $x + iy$. We can replace θ by $\theta + 2\pi$, $\theta + 4\pi$, $\theta - 2\pi$, etc., and $\sin\theta$ and $\cos\theta$ will not change, and thus $x + iy$ will not change. Thus $\ln z$ for a complex number z is only defined up to multiple of $2\pi i$.

This is extremely unfortunate. But that's the way it is. Those who work in complex analysis use the oxymoron "multiple-valued function" (multivalent function) to describe this situation.