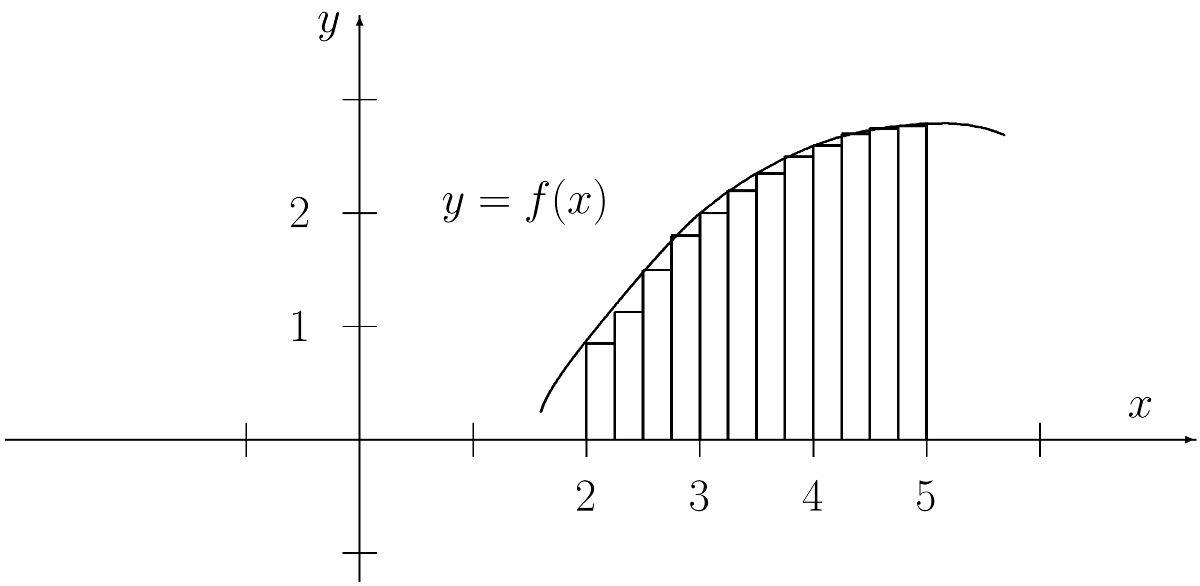
The Traditional Definition of the Integral



$$\int_2^5 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i')\,\Delta x$$

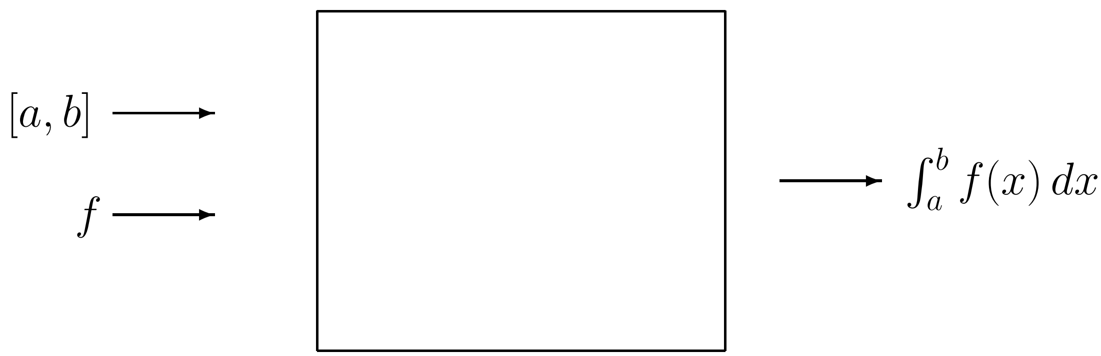
$$\text{where } \Delta x = \frac{5-2}{n}$$

and x_i' lies somewhere between

$2 + (i-1)\Delta x$ and $2 + i\,\Delta x$.

1. What Is an Integral?

An integral is a process (a black box) that takes an input consisting of a function f and a closed interval $[a, b]$ and produces an output which is a numerical value.



(For mathematicians, the most important issue here is: What sort of function can $f(x)$ be? We will assume here that we are working with functions which are continuous or break up into a finite number of continuous pieces pieced together. This is adequate generality for most applications of integration in the physical sciences and engineering.)

The integral is characterized by three important properties.

Axioms for the Integral

1. Cumulative.
$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$

2. Quantizable. If we replace the function $f(x)$ by a function by one which instead of increasing continuously increases by an incredibly large number of incredibly small quantum jumps, then we will get a very good approximation of the integral. More precisely, if we take a sequence of step functions which approaches $f(x)$ as the limit, then $\int_a^b f(x)\,dx$ will be the limit of the integrals of these step functions.

3. Constant functions. If $f(x) = C$, where C is a constant, then $\int_a^b f(x)\,dx = (b-a)C$.

Properties 1 and 3 imply that if we have two apparently different black boxes with these same properties, then they would produce the same value whenever $f(x)$ is a step function. Since every Riemann-integrable function is a limit of step functions, Property 2 then implies that the two black boxes actually produce the same value for all functions.

Many black boxes in the "Real World," satisfy the three properties which characterize the integral.

"Real World" Relationships With These Properties

1. The area under a curve.

2. The work done by a force applied over a certain distance.

3. The distance traveled by an object moving with a given velocity function over a given time interval.

4. The volume of a solid obtained by revolving the graph of a function $\mathbf{y = f(x)}$ around the x-axis.

5. The force created by a given pressure over an area. (This will correspond to a double integral.)

6. The mass of a solid corresponding to a given density function. (This will correspond to a triple integral.)

Since the three properties **uniquely** characterize the integral, we can then conclude that the above relationships are given by integrals. We don't need to re-invent the integral for each new application as is commonly done in calculus books.

$$\mathbf{2}. \quad \text{Work} = \int_a^b \text{Force}(x)\, dx$$

$$\mathbf{4}. \quad \text{Volume} = \int_a^b \pi f(x)^2\, dx$$

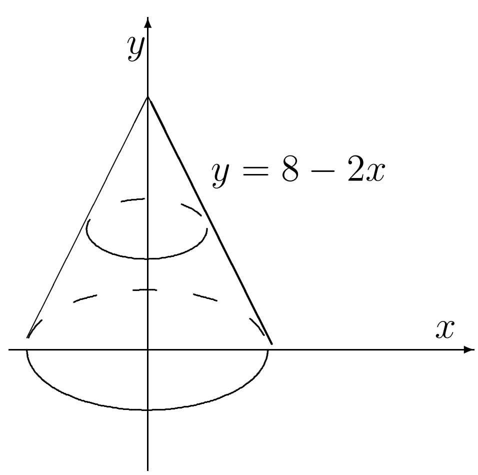
$$\mathbf{5}. \quad \text{Force} = \int_\Omega \text{Pressure}(x, y)\, dx\, dy$$

etc.

HOWEVER

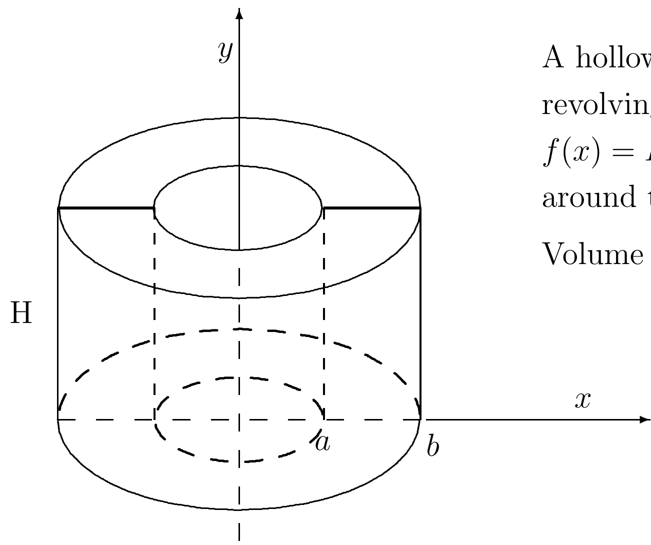
Many important applications of integration satisfy the **first** and **second** conditions that characterize the integral but do not satisfy the **third**. Namely, when the function involved is constant, the quantity to be calculated is not given simply by a multiplication by $b - a$.

For instance, let V be the **volume of the solid of revolution** obtained by revolving that portion of the graph of a function $y = f(x)$ between $x = a$ and $x = b$ (assuming $0 < a < b$) **around the y-axis.**



A cone, seen as a volume of revolution obtained by revolving the line $y = 8 - 2x$ around the y-axis.

Then when $f(x)$ is a constant H, we get a hollow cylinder and the volume is $V = \pi(b^2 - a^2)H$.



A hollow cylinder, obtained by revolving the constant function $f(x) = H$ between $x = a$ and $x = b$ around the y-axis.

$$\begin{aligned} \text{Volume} &= \pi b^2 H - \pi a^2 H \\ &= \pi H(b^2 - a^2). \end{aligned}$$

The multiplier $b^2 - a^2$ doesn't fit our paradigm for the integral.

If we try to **force** it into the previous paradigm, we might try this:

$$\text{Volume} = \int_{a^2}^{b^2} \pi f(x)\, d(x^2) \qquad (??)$$

Now as good calculus students, we do in fact know that the volume of the solid of revolution obtained from revolving a curce around the y-axis is given by an integral, namely

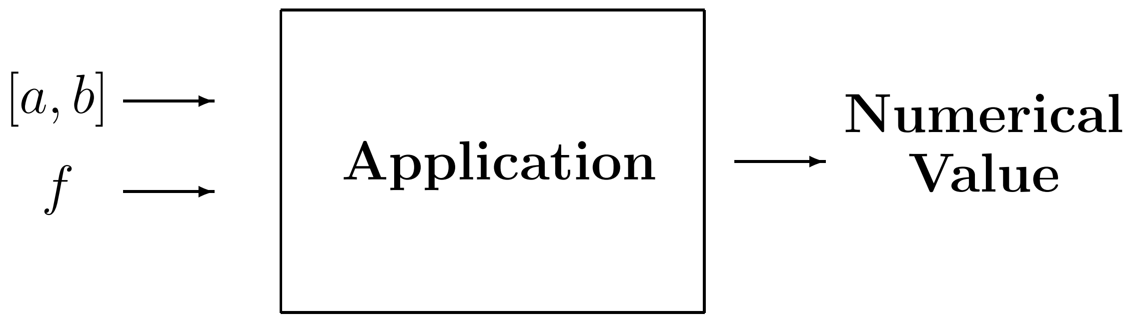
$$V = 2\pi \int_a^b x f(x)\, dx\,.$$

This is a different sort of integral than the ones we have been considering, since the integral contains not only the function $f(x)$ but also the independent variable x.

(In the language of integration theory, for this application of integration we need to use a new "measure," namely $2\pi x\, dx$ instead of dx.)

The Structure of an Application of Integration

In general, a particular application of integration will have the following structure.



All applications of integration will satisfy Property 1 of the integral.

1. Cumulative. If $a < b < c$ then the value determined by looking at the function over the interval $[a, c]$ will be the sum of the value determined over $[a, b]$ and the value determined over $[b, c]$.

Most applications will also satisfy Property 2.

2. Quantizable. If we replace the function $f(x)$ by one which instead of increasing continuously increases by an incredibly large number of incredibly small quantum jumps, then we will get a very good approximation. More precisely, if we take a sequence of step functions which approaches $f(x)$ as the limit, then the value corresponding to $f(x)$ will be the limit of the values corresponding to the step functions.

Many important applications, however, will not satisfy the previously stated Property 3. When $f(x)$ is constant, the value of the quantity in question will not simply be given by multiplication by $b - a$.

However we assume that it is known what the correct formula is when the function $f(x)$ is constant then we will be able to figure out the correct integral formula.

The Golden Rule of Thumb for Integrals

If a reasonable* formula given by an integral gives the correct answers when applied to constant functions, and if the particular application in question satisfies Property 2, i. e. is quantizable, then the formula is in fact correct.

*** The fine print.** Basically, this will be good for formulas of the form

$$\int_a^b \Phi(x, f(x))\, dx\,,$$

where Φ is a continuous function of two variables.

The only easy way to screw up is to include the derivative $f'(x)$ (or higher derivatives) in the integrand.

For instance, if $v(t)$ is velocity, then the formula

$$\text{Distance} = \int_{t_0}^{t_1} v(t)\,(1 + v'(t))\, dt \quad (??)$$

gives the correct answer when the velocity $v(t)$ is a constant, or for that matter a step function, but gives the wrong answer in all other cases.

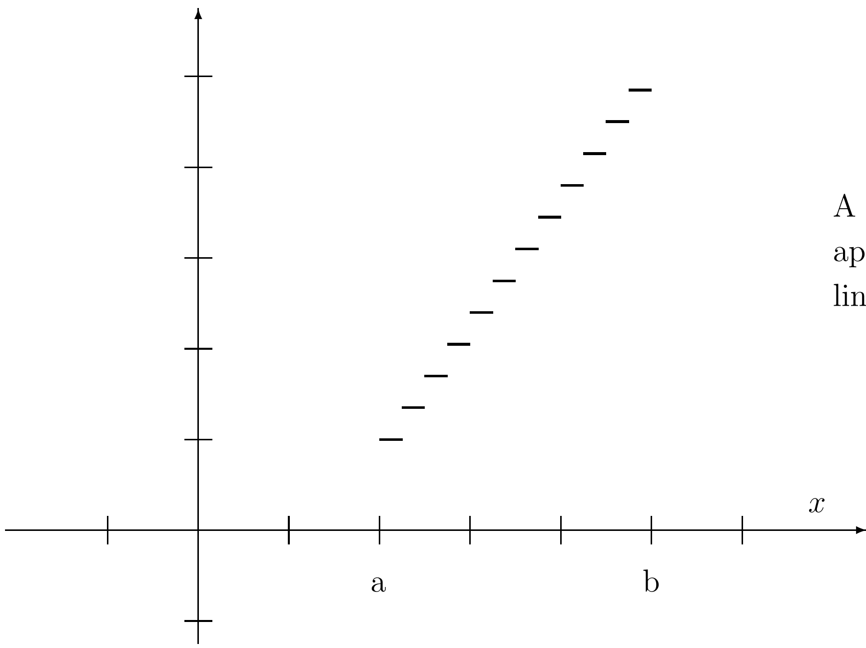
Why the Golden Rule Works

Because the integral is cumulative (i. e. additive over disjoint intervals), if a formula given by an integral gives the correct answer for constant functions, then it will also give the correct answer for step functions. But if the application in question is quantizable, then the correct value corresponding to the function $f(x)$ can be obtained by taking the limit of the values for a sequence of step functions converging to $f(x)$.

The reason for calling the principle above a "Rule of Thumb" is that there are two ways it can go wrong: First, the application in question might not in fact be quantizable. And second, the formula in question might involve $f'(x)$ or otherwise fail to meet the conditions of the footnote.

A few important applications of integration are not quantizable according to our definition and hence do not satisfy Property 2. These applications cannot be ignored.

The most obvious example is the formula for the **length** of that portion of the graph of a function $y = f(x)$ between points $x = a$ and $x = b$. **If one "quantizes" a function, i. e. approximates it by a step function, one will not get a good approximation to the length of its graph.** In fact, it is fairly clear that for step functions, the length is always $b - a$, whereas this is never true for functions whose slope is non-zero in at least some places.



A step-function approximation to a straight line.

Theorem. Suppose the quantity being computed by an application has an **increasing relationship** to the function that determines it (i. e. making the function bigger always makes the quantity bigger), and that for constant functions, over a given interval the quantity depends continuously on the (constant) value of the function. Then the relationship is quantizable.

Example. The volume obtained by revolving that portion of the graph of a positive function $y = f(x)$ between $x = a$ and $x = b$ around the y-axis. (Clearly making the function bigger will increase the volume.)

The continuity assumption in the theorem is included to rule out the possibility of bizarre, contrived counter-examples. In most cases, the theorem can be proved without it in any case.

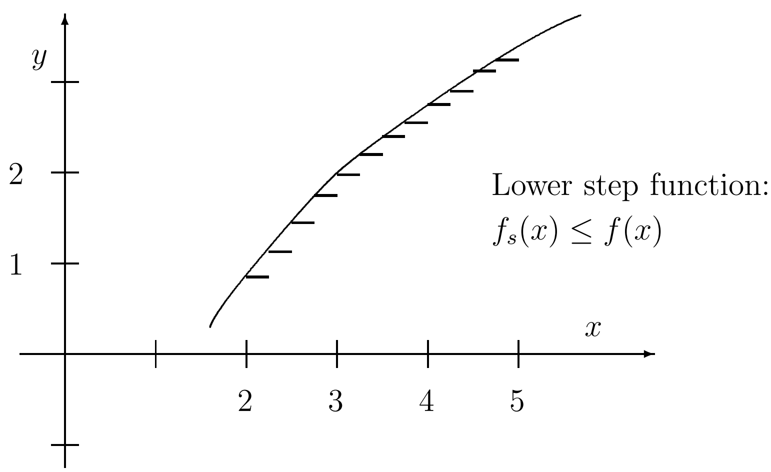
Most common applications of integration are increasing.

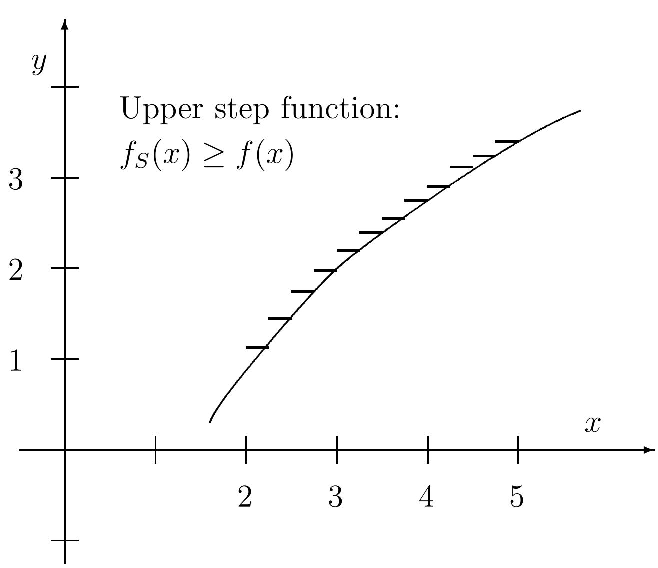
The same thing holds if the relationship between the function and quantity being computed is decreasing rather than increasing.

The idea of the proof of the theorem is that one brackets a function between a step function below it and a step function above it. Since by assumption, the application in question is increasing, the value corresponding to the function will lie between the values corresponding to these two step functions. But, as I will show in a moment, by making the steps small enough, the values corresponding to the step functions can be made arbitrarily small.

Thus the value corresponding to these step functions can be made arbitrarily close to the value for the function under consideration.

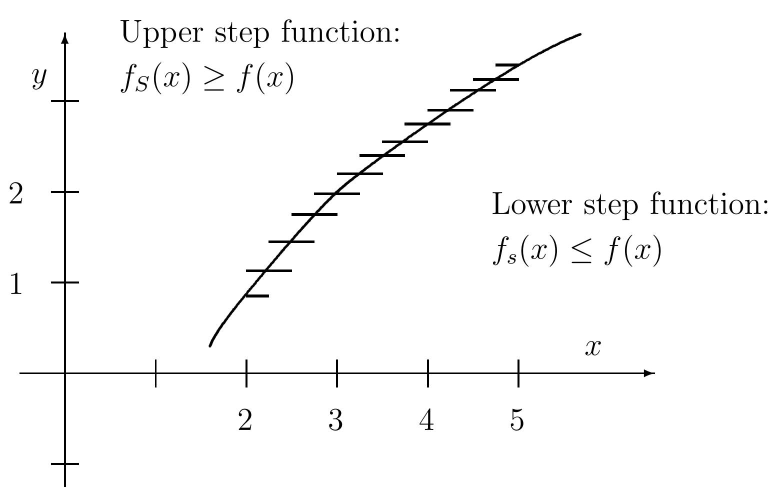
This is what we mean when we say that the application is quantizable.





Proof of my claim that the values for the application in question corresponding to the two step functions bracketing $f(x)$ will be extremely close together if the steps are made small enough:

I will prove this only in the special case that the function $f(x)$ is increasing from a to b. Then looking at these two step functions together, we can see that except for the first step, the lower one is the same as the upper one except shifted one step to the right. (This works because we have made all the steps of the same width.)



As an example, for the volume of revolution around the x-axis, using the lower step function gives an approximation (assuming that the function $y = f(x)$ is increasing, as in the picture)

$$V_s = \pi f(x_0)^2 \Delta x + \pi f(x_1)^2 \Delta x + \pi f(x_2)^2 \Delta x + \cdots + \pi f(x_{n-1})^2 \Delta x$$

and using the upper step function gives an approximation

$$V_S = \pi f(x_1)^2 \Delta x + \pi f(x_2)^2 \Delta x + \cdots + \pi f(x_{n-1})^2 \Delta x + \pi f(x_n)^2 \Delta x \,,$$

where $\Delta x = (b-a)/n$. But it is evident that

$$\begin{aligned} V_S - V_s &= \left(\pi f(x_n)^2 - \pi f(x_0)^2\right) \Delta x \\ &= \left(\pi f(b)^2 - \pi f(a)^2\right) \Delta x \,, \end{aligned}$$

and this can be made as small as we want by making Δx small enough (i. e. making n large enough). (Note that the continuity assumption in the theorem was not required for the proof in this case that $f(x)$ is an increasing function. A similar proof works if $f(x)$ is a decreasing function. And almost all continuous functions can be obtained by gluing together pieces of increasing functions with pieces of decreasing ones.)

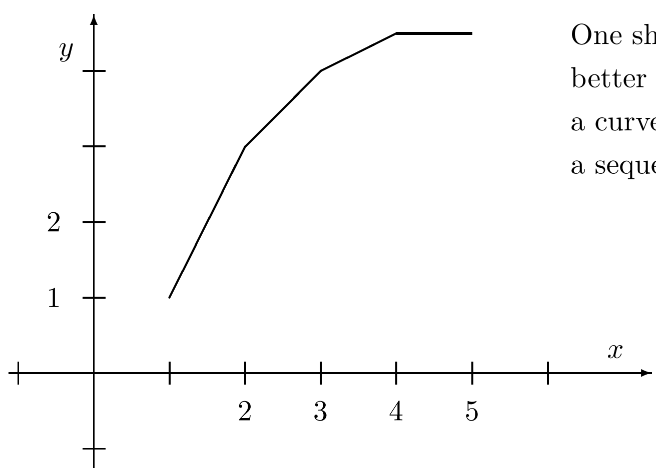
And now for something different:

An important bad example.

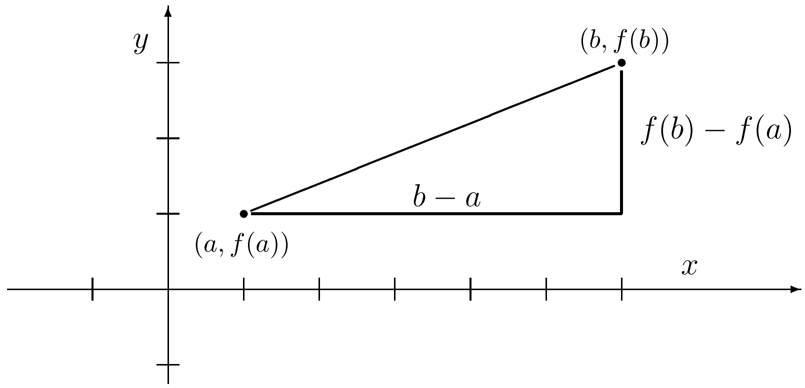
Note that the **length of the graph** of a function does not have an increasing relationship to the function. Making the function $f(x)$ larger makes its graph between $x = a$ and $x = b$ higher but not longer. If we did not already know it, this would be a hint that length does not have a quantizable relationship to the function.

However if one makes the derivative $f'(x)$ larger (in absolute value), then the curve gets steeper and does in fact become longer. This suggests the the length of the curve might be given by some integral involving $f'(x)$.

We have seen that we cannot find the length of the graph of a function $f(x)$ by approximating $f(x)$ by a step function. However it seems plausible that one could find the length by approximating the graph of the function by a set of line segments.



One should be able to get better and better approximations to the length of a curve by approximating the curve by a sequence of line segments.



If $f(x) = mx + c$ is a straight-line function, then the length of this curve from the point $(a, f(a))$ to the point $(b, f(b))$ is

$$\begin{aligned}\sqrt{(b-a)^2 + (f(b)-f(a))^2} &= \sqrt{(b-a)^2 + m^2(b-a)^2} \\ &= (b-a)\sqrt{1+m^2} \\ &= (b-a)\sqrt{1+f'(x)^2}\,.\end{aligned}$$

Thus the formula

$$\text{Length} = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

gives the correct answer for straight line segments.

Since the length of any curve can be approximated arbitrarily closely by replacing the curve by a union of very short line segments, we see that this formula gives the correct result for the length of the graph of any function.