A COURSE IN HOMOLOGICAL ALGEBRA

Chapter **: Syzygies, Projective Dimension, Regular Sequences, and Depth

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Syzygies, Torsionless Modules, and Reflexive Modules

In this section, modules are usually assumed to be finitely generated.

Definition. A Λ-module A is called an 0^{th} **syzygy** if it is isomorphic to a submodule of a projective Λ-module.
We now define the concept of an n^{th} syzygy recursively. A is an n^{th} syzygy if there exists a projective module P such that A can be embedded in P in such a way that P/A is a $(n-1)^{\text{st}}$ syzygy.
In other words, A is an n^{th} syzygy if there exists an exact sequence

$$0 \longrightarrow A \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0\,,$$

where the P_i are all projective Λ-modules.

The fascinating thing is that zero$^{\text{th}}$ syzygies and first syzygies have an intrinsic significance in terms of the duality functor $A \mapsto A^* = \operatorname{Hom}_\Lambda(A, \Lambda)$. Namely, a left Λ-module A is a first syzygy if and only if it is the dual of some right Λ-module, and is a zero$^{\text{th}}$ syzygy if and only if natural isomorphism $A \to A^{**}$ is monic.

Another interesting fact is that $A^* = 0$ if and only if $A \approx \operatorname{Ext}^1_\Lambda(B, \Lambda)$ for some right Λ-module B. We will now proceed to prove these results.

References. Bass, Trans. Amer. Math Soc. 95(1960), 466-88.
Jans, Trans. Amer. Math. Soc. 106(1963),330-40.
Jans, *Rings and Homology.*
(Auslander, "Coherent Functors," *in* the La Jolla Conference on Categorical Algebra.)

Notation. If A is a left Λ-module, we set $A^* = \operatorname{Hom}_\Lambda(A, \Lambda)$. A^* is a **right** Λ-module. We define $\delta_A \colon A \to A^{**}$ by

$$\delta_A(a)(\varphi) = \varphi(a) \qquad \text{for } \varphi \in A^*.$$

Lemma. δ is a natural transformation. Furthermore, δ_P is an isomorphism if P is finitely generated projective.

PROOF: For the second statement it suffices to prove that δ_Λ is an isomorphism. ☑

Lemma. $\delta_{A^*} : A^* \to A^{***}$ is split monic.

PROOF: We claim that $(\delta_A)^* \delta_{A^*} = 1_{A^*}$. In fact,

$$A^* \xrightarrow{\delta_{A^*}} A^{***} \qquad\qquad A^{***} \xrightarrow{\delta_A^*} A^*$$

and for $\varphi \in A^*$,

$$(\forall a \in A) \quad (\delta_A)^*(\delta_{A^*}(\varphi))(a) = \delta_{A^*}(\varphi)(\delta_A(a)) = \delta_A(a)(\varphi) = \varphi(a)$$

so that $(\delta_A)^*(\delta_{A^*}(\varphi)) = \varphi$ and so $(\delta_A)^* \delta_{A^*} = 1_{A^*}$. ☑

Theorem. Let Λ be a left and right noetherian ring (not necessarily commutative) and let A be a finitely generated left Λ-module such that $A^* = 0$. Then there exists a finitely generated right Λ-module B such that $A \approx \operatorname{Ext}^1_\Lambda(B, \Lambda)$.

PROOF: Suppose that $A^* = 0$ and consider a projective resolution

$$P_1 \xrightarrow{\partial} P_0 \to A \to 0\,.$$

Let $B = \operatorname{Coker} \partial^*$, so that

$$0 \to A^* = 0 \to P_0^* \xrightarrow{\partial^*} P_1^* \to B \to 0$$

is exact. Since $P_i^{**} \approx P_i$, this yields an exact sequence

$$0 \to B^* \to P_1 \xrightarrow{\partial} P_0 \to \operatorname{Ext}^1_\Lambda(B, \Lambda) \to 0 = \operatorname{Ext}^1_\Lambda(P_1^*, \Lambda)\,,$$

showing that $A = \operatorname{Coker} \partial \approx \operatorname{Ext}^1_\Lambda(B, \Lambda)$. ☑

Definition. We say that a Λ-module A is **torsionless** if δ_A is a monomorphism and **reflexive** if δ_A is an isomorphism.

Theorem. Let Λ be **right** noetherian and let A be a finitely generated left Λ-module. Then A is torsionless if and only if A is a 0^{th} syzygy.

PROOF: To say that δ_A is monic is to say that for every $a \in A$ there exists $\varphi \in A^*$ such that $\varphi(a) \neq 0$.

($\Leftarrow$): Easy. ($\Rightarrow$): Let $F \twoheadrightarrow A$ be a surjection with F finitely generated and free. This induces $A^* \rightarrowtail F^*$. Thus A^* is finitely generated since Λ is right noetherian. Thus there exists a finitely generated free right Λ-module G and a surjection $G \twoheadrightarrow A^*$. Thus if δ_A is monic there are monomorphisms

$$A \rightarrowtail A^{**} \rightarrowtail G^*.$$

Since G^* is free, A is a 0^{th} syzygy. ☑

Lemma. Let A be a submodule of the left Λ-module P. Define

$$A' = \{\varphi \in P^* \mid \varphi(A) = 0\}$$
$$A'' = \{p \in P \mid (\forall \varphi \in A')\ \varphi(p) = 0\}.$$

Then there is an exact sequence

$$0 \to A' \to P^* \xrightarrow{\rho} A^*.$$

Furthermore, if P/A is torsionless then $A'' = A$.

PROOF: A' is clearly the kernel of the map $P^* \xrightarrow{\rho} A^*$ induced by the inclusion $A \hookrightarrow P$ (since $\rho(\varphi)$ is the restriction of φ to A). This justifies the first assertion.

Now let $\varphi' \in (P/A)^*$. Then φ' is induced by a map $\varphi \in P^*$ such that $\varphi(A) = 0$, i.e. $\varphi \in A'$. Then for $p \in A'' \subseteq P$,

$$\delta_{P/A}(p + A)(\varphi') = \varphi'(p + A) = \varphi(p) = 0,$$

by definition of A''. Thus if P/A is torsionless, so that $\operatorname{Ker} \delta_{P/A} = 0$, then $A''/A = 0$ so that $A'' = A$. ☑

Theorem. Let Λ be both left and right noetherian and let A be a finitely generated left Λ-module. Then A is a first syzygy if and only if $A \approx C^*$ for some right Λ-module C.

PROOF: ($\Rightarrow$): Suppose that A is a first syzygy. Then there is an exact sequence

$$0 \to A \to P_1 \to P_0,$$

where P_0 and P_1 are finitely generated projective. Let $A' \subseteq P_1^*$ and $A'' \subseteq P_1$ be defined as in the previous lemma. Then

$$A'' = \{p \in P_1 \mid \delta_{P_1}(p)(A') = 0\},$$

so that there is an exact sequence

$$0 \to A'' \to P_1 \to (A')^*,$$

where the right hand map is induced by δ_{P_1}, so that there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A'' & \longrightarrow & P_1 & \longrightarrow & (A')^* \\ & & & & \delta_{P_1}\big\downarrow \approx & & \big\| \\ 0 & \longrightarrow & (P_1^*/A')^* & \longrightarrow & P_1^{**} & \xrightarrow{\rho} & (A')^*, \end{array}$$

where ρ is induced by the inclusion $A' \hookrightarrow P_1^*$, so that $\operatorname{Ker} \rho = (P_1^*/A')^*$, as indicated. Thus $A'' \approx (P_1^*/A')^*$. But P_1/A is a zero$^{\text{th}}$ syzygy since $P_1/A \subseteq P_0$ and is thus torsionless, so that by the preceding Lemma $A'' = A$. Thus $A \approx C^*$ with $C \approx P_1^*/A'$.

($\Leftarrow$): Suppose that $A = C^*$. Let P_1 be a projective which maps onto C, so that we have

$$0 \longrightarrow K \longrightarrow P_1 \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow C^* \longrightarrow P_1^* \longrightarrow K^*.$$

But K^* is a 0^{th} syzygy since $\delta_{K^*}: K^* \to K^{***}$ is monic (in fact, split monic). Thus C^* is a first syzygy. ☑

Theorem. Let Λ be left and right noetherian and let A be a finitely generated torsionless left Λ-module. Then there exists a torsionless right Λ-module B such that there are exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\delta_A} & A^{**} & \longrightarrow & \mathrm{Ext}^1_\Lambda(B,\Lambda) & \longrightarrow & 0 \\ 0 & \longrightarrow & B & \xrightarrow{\delta_B} & B^{**} & \longrightarrow & \mathrm{Ext}^1_\Lambda(A,\Lambda) & \longrightarrow & 0. \end{array}$$

PROOF: Starting with a short exact sequence $0 \to K \to P \to A \to 0$ with P projective, one gets

$$0 \to A^* \to P^* \to B \to 0, \tag{1}$$

with $B = P^*/A^*$. Since $A = P/K$, this can be written

$$0 \to K' \to P^* \to B \to 0.$$

Since P^* is projective, (1) induces

$$0 \to B^* \to P^{**} \to A^{**} \to \mathrm{Ext}^1_\Lambda(B,\Lambda) \to 0. \tag{2}$$

As above, let

$$K'' = \{p \in P \mid (\forall \varphi \in K')\ \varphi(p) = 0\}.$$

Then the image of B^* in P^{**} is $\delta_P(K'')$ and $K'' = K$ by the Lemma above, since by hypothesis $P/K = A$ is torsionless. Thus we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K'' & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \\ & & \delta_K \big\downarrow \approx & & \delta_P \big\downarrow \approx & & \big\downarrow \delta_A & & \\ 0 & \longrightarrow & B^* & \longrightarrow & P^{**} & \longrightarrow & A^{**} & & \end{array}$$

where δ_A is monic since A is a zero$^{\text{th}}$ syzygy. Thus $A \approx P^{**}/B^*$. Combining this with (2) one has

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^{**}/B^* & \longrightarrow & A^{**} & \longrightarrow & \mathrm{Ext}^1_\Lambda(B,\Lambda) & \longrightarrow & 0 \\ & & \big\downarrow \approx & & \Vert & & \Vert & & \\ 0 & \longrightarrow & A & \xrightarrow{\delta_A} & A^{**} & \longrightarrow & \mathrm{Ext}^1_\Lambda(B,\Lambda) & \longrightarrow & 0\,, \end{array} \tag{3}$$

as required.

Applying the same reasoning to (3) we get

$$0 \to B \to B^{**} \to \mathrm{Ext}^1_\Lambda(A,\,\Lambda) \to 0\,. \quad \boxed{\checkmark}$$

Corollary. Λ has the property that every finitely generated Λ-module is reflexive if and only if Λ is left and right self-injective.

PROOF: ($\Leftarrow$): If A is a finitely generated Λ-module, consider a projective resolution $P_1 \to P_0 \twoheadrightarrow A$. Now if Λ is injective as a left and a right Λ-module, then the functor $A \mapsto A^{**}$ is exact, so the rows in the following diagram are exact:

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ {\scriptstyle\delta_{P_1}}\big\downarrow\approx & & {\scriptstyle\delta_{P_0}}\big\downarrow\approx & & {\scriptstyle\delta_A}\big\downarrow & & \\ P_1^{**} & \longrightarrow & P_0^{**} & \longrightarrow & A^{**} & \longrightarrow & 0\,, \end{array}$$

showing that δ_A is an isomorphism, so A is reflexive.

($\Rightarrow$): If every finitely generated Λ-module is reflexive, then a fortiori every finitely generated A is torsionless. Thus by the Theorem there exists a Λ-module B and an exact sequence

$$0 \longrightarrow B \xrightarrow{\delta_B} B^{**} \longrightarrow \operatorname{Ext}^1_\Lambda(A,\Lambda) \longrightarrow 0\,.$$

Since by hypothesis δ_B is an isomorphism, $\operatorname{Ext}^1_\Lambda(A,\Lambda) = 0$. Since this is true for all finitely generated A, Λ is an injective module. ☑

Corollary. Let Q denote the injective envelope of Λ. Then all finitely generated torsionless modules are reflexive if and only if Q/Λ is injective (i.e. $\operatorname{inj\,dim}\Lambda \le 1$).

PROOF: ($\Rightarrow$): By the same reasoning as in the proof of the previous corollary, if all finitely generated torsionless modules are reflexive, then the theorem implies that $\operatorname{Ext}^1_\Lambda(A,\Lambda) = 0$ whenever A is torsionless, i.e. whenever A is a zero$^{\text{th}}$ syzygy. Now let M be any finitely generated Λ-module and let P be a projective module which maps onto M, giving a short exact sequence

$$0 \to K \to P \to M \to 0\,.$$

Then K is a zero$^{\text{th}}$ syzygy and thus torsionless, so $\operatorname{Ext}^1_\Lambda(K,\Lambda) = 0$. But then

$$\operatorname{Ext}^2_\Lambda(M,\Lambda) \approx \operatorname{Ext}^1_\Lambda(K,\Lambda) = 0.$$

On the other hand, from the sequence $0 \to \Lambda \to Q \to Q/\Lambda \to 0$ it follows that

$$\operatorname{Ext}^1_\Lambda(M,\, Q/\Lambda) \approx \operatorname{Ext}^2_\Lambda(M,\, \Lambda) = 0\,.$$

Since this is true for every finitely generated M, Q/Λ must be injective. ☑

Corollary. Let Λ be a commutative noetherian ring and let T be its total quotient ring. Then Λ has the property that all finitely generated duals A^* are reflexive if and only if T/Λ is an injective Λ-module.

PROOF: T is the injective envelope of Λ, since it is a maximal essential extension of Λ. ☑

Projective Dimension

Schanuel's Lemma. Let

$$0 \to K \to P \to M \to 0$$
$$0 \to K_1 \to P_1 \to M \to 0$$

be short exact sequences such that P and P_1 are projective. Then

$$K \oplus P_1 \approx K_1 \oplus P.$$

Definition. Projective Dimension

Theorem. The projective dimention of a Λ-module A is the smallest n such that $\mathrm{Ext}_\Lambda^{n+1}(A, \_\_) = 0$, or ∞ if there is no such n.

Theorem. If Λ is left noetherian, then proj. $\dim A$ is the smallest n such that $\mathrm{Ext}_\Lambda^{n+1}(A, B) = 0$ for all finitely generated B.

PROOF: It suffices to see that K is projective if and only if $\mathrm{Ext}_\Lambda^1(K, B) = 0$ for all finitely generated B. To see this, take B to be a zero$^{\text{th}}$ syzygy for K. ☑

Theorem. If R is commutative then proj. $\dim_R A = \sup_{\mathfrak{m}}$ proj. $\dim_{R_\mathfrak{m}} A_\mathfrak{m}$.

Theorem. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence. If any two of these modules have finite projective dimension, then so does the third. Furthermore, either

$$\text{proj. } \dim B < \text{proj. } \dim C = \text{proj. } \dim A + 1$$

or

$$\text{proj. } \dim B = \max\{\text{proj. } \dim A, \text{ proj. } \dim C\}.$$

Regular M-sequences and Depth

Reference. Matsumura, Commutative Algebra.

In the rest of this chapter, R will denote a commutative ring.

Definition. $\mathrm{Ass}\, M$. Regular M-sequence.

Theorem. If R is noetherian then an R-module M is trivial if and only if $\mathrm{Ass}\, M = \varnothing$. Furthermore, if x is a zero divisor on an R-module M, then there exists a prime ideal $\mathfrak{p} \in \mathrm{Ass}\, M$ with $x \in \mathfrak{p}$.

Proposition. If $\mathfrak{p}$ is a prime ideal in R, then $\operatorname{Ass} R/\mathfrak{p} = \{\mathfrak{p}\}$.

Theorem. If R is noetherian and M is finitely generated, then $\operatorname{Ass} M$ is finite.

PROOF: Since M is noetherian, there is a maximal submodule M' of M such that $\operatorname{Ass} M'$ is finite. If $M' \neq M$, one easily gets a contradiction. ☑

Lemma. For $N \subseteq M$,

$$\operatorname{Ass} M \subseteq \operatorname{Ass} N \cup \operatorname{Ass}(M/N).$$

Definition. The **support** of an R-module M consists of the set of primes $\mathfrak{p}$ such that $M_{\mathfrak{p}} \neq 0$.

Proposition. Every element of $\operatorname{Ass} M$ belongs to $\operatorname{Supp} M$. Furthermore, the primes which are minimal in $\operatorname{Supp} M$ belong to $\operatorname{Ass} M$.

Lemma. If an ideal is contained in a finite union of prime ideals, then it is contained in one of those primes.

Corollary. Let R be noetherian and M a finitely generated R-module. If $\mathfrak{a}$ is an ideal consisting of zero divisors on M, then $\mathfrak{a}$ is contained in some associated prime $\mathfrak{p}$ for M.

Definition. Regular M-sequence. $\operatorname{Depth}_{\mathfrak{a}} M$.

Theorem. Let R be a noetherian ring and M a finitely generated R-module and $\mathfrak{a}$ an ideal such that $\mathfrak{a}M \neq M$. If r is the smallest integer such that $\operatorname{Ext}^r_R(R/\mathfrak{a}, M) \neq 0$, then every regular M-sequence in $\mathfrak{a}$ has length r. Thus $r = \operatorname{depth}_{\mathfrak{a}} M$.

PROOF: Clear if $r = 0$, since if $\operatorname{Hom}_R(R/\mathfrak{a}, M) \neq 0$ then no element of $\mathfrak{a}$ is regular on M and conversely, if no regular M-sequences exist in $\mathfrak{a}$ then $\mathfrak{a}$ consists of zero divisors on M and consequently $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} M$ by the preceding Corollary, and $\operatorname{Hom}_R(R/\mathfrak{p}, M) \neq 0$ (since M contains a submodule isomorphic to $R/\mathfrak{p}$), so also $\operatorname{Hom}_R(R/\mathfrak{a}, M) \neq 0$.

Now let $r \geq 1$. Then by the preceding paragraph, we can choose $x_1 \in \mathfrak{a}$ which is regular on M. Now use induction, since clearly $x_2, \ldots, x_r$ is a maximal regular sequence of M/x_1M if and only if $x_1, \ldots, x_r$ is a maximal regular M-sequence. ☑

Lemma. Let $(R, \mathfrak{m})$ be local and suppose that $\operatorname{depth}_{\mathfrak{m}} R = 0$ (i. e. $\mathfrak{m}$ consists of zero divisors). Then any R-module M with finite projective dimension is in fact projective.

PROOF: It suffices to prove that all modules M with $\operatorname{proj.\,dim} M \leq 1$ are projective. If M is such a module, take a projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\ \varepsilon\ } M \longrightarrow 0$$

such that ε induces an isomorphism from $P_0/\mathfrak{m}P_0$ onto $M/\mathfrak{m}M$. Then $P_1 \subseteq \mathfrak{m}P_0$. Since $\mathfrak{m}$ consists of zero divisors, $\mathfrak{m} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} R$. Since $\mathfrak{m}$ is maximal, thus $\mathfrak{m} \in \operatorname{Ass} R$. Thus there exists $x \in R$ with $x\mathfrak{m} = 0$, and consequently $xP_0 = 0$. Since x can't be invertible, $x \in \mathfrak{m}$, so by Nakayama's Lemma, $P_1 = 0$, so that ε is an isomorphism and M is projective. ☑

Lemma. Let $(R, \mathfrak{m})$ be local, let M be an R-module with finite projective dimension, and let $x \in M$ be regular on M. Then $\text{proj. dim}(M/xM) = 1 + \text{proj. dim}\, M$.

PROOF: Since x is regular on M, the sequence

$$0 \longrightarrow M \xrightarrow{\ x\ } M \longrightarrow M/xM \longrightarrow 0$$

is exact. The induced long exact Ext sequence looks like

$$\ldots \to \operatorname{Ext}^k_R(M, \_\!\_) \xrightarrow{x} \operatorname{Ext}^k_R(M, \_\!\_) \to \operatorname{Ext}^{k+1}_R(M/xM, \_\!\_) \to \operatorname{Ext}^{k+1}_R(M, \_\!\_) \to \ldots$$

Now if $\operatorname{Ext}^k_R(M, \_\!\_) \neq 0$, then $\operatorname{Ext}^k_R(M, X) \neq 0$ for some finitely generated X, and in this case $\operatorname{Ext}^k_R(M, X)$ is finitely generated. Since $x \in \mathfrak{m}$, multiplication by x cannot be surjective on $\operatorname{Ext}^k_R(M, X)$ by Nakayama's Lemma. It then follows the $\operatorname{Ext}^{k+1}_R(M/xM, \_\!\_) \neq 0$.

But conversely, the long exact sequence shows that if $\operatorname{Ext}^k_R(M, \_\!\_) = 0$ (and therefore also $\operatorname{Ext}^{k+1}_R(M, \_\!\_) = 0$), then $\operatorname{Ext}^{k+1}_R(M/xM, \_\!\_) = 0$. ☑

Theorem. Let $(R, \mathfrak{m})$ be local and $x_1, \ldots, x_r$ a regular M-sequence. Then

$$\text{proj. dim}\, M/(x_1, \ldots, x_r)M = r + \text{proj. dim}\, M.$$

Theorem. Let $(R, \mathfrak{m})$ be local and let M be a finitely generated R-module with finite projective dimension. Then

$$\text{proj. dim}\, M = \operatorname{depth}_{\mathfrak{m}} R - \operatorname{depth}_{\mathfrak{m}} M.$$

PROOF: A previous lemma covers the case $\operatorname{depth}_{\mathfrak{m}} R = 0$ (i.e. $\mathfrak{m} \in \operatorname{Ass} R$).

Now use a double induction on $\operatorname{depth}_{\mathfrak{m}} R$ and $\operatorname{depth}_{\mathfrak{m}} M$ by choosing $x \in \mathfrak{m}$ regular on R and using the Lemma below. The difficult part concerns the case when $\operatorname{depth}_{\mathfrak{m}} M = 0$ and $\operatorname{depth}_{\mathfrak{m}} R > 0$.

Lemma. Let $(R, \mathfrak{m})$ be local, M finitely generated, and let $x \in \mathfrak{m}$ be regular on both R and M. Write $\bar{R} = R/xR$ and $\bar{M} = M/xM$. Then

$$\text{proj. dim}_R\, M = \text{proj. dim}_{\bar{R}}\, \bar{M}.$$

PROOF: Consider a minimal projective resolution for M,

$$0 \to P_n \to \ldots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to M \to 0.$$

(I.e. suppose that the induced maps $P_i/\mathfrak{m}P_i \to \partial_i(P_i)/\mathfrak{m}\partial_i(P_i)$ are all isomorphisms.) Then $\text{proj. dim}\, M$ is the largest n such that $P_n \neq 0$. Now since x is regular on M, $\operatorname{Tor}^R_i(R/xR,\, M) = 0$ for all $i > 0$. Therefore the induced sequence

$$0 \to P_n/xP_m \to \ldots \to P_2/xP_2 \to P_1/xP_1 \to P_0/xP_0 \to \bar{M} \to 0$$

is exact. But this is a minimal projective resolution for $\bar{M}$ over $\bar{R}$, so that $\text{proj. dim}_{\bar{R}}\, \bar{M} = n = \text{proj. dim}_R\, M$. ☑

Addendum. If $x_1, \dots, x_r$ is a regular M-sequence in $\mathfrak{a}$ and $r = \operatorname{depth}_{\mathfrak{a}} M$, then

$$\operatorname{Ext}^r_R(R/\mathfrak{a},\, M) \approx \operatorname{Hom}_R(R/\mathfrak{a},\, M/(x_1, \dots, x_r)M).$$

Proposition. Let R be noetherian, M a finitely generated R-module, N a submodule of M, and $\mathfrak{a}$ an ideal such that $\mathfrak{a}M \neq M$, $\mathfrak{a}N \neq N$, and $\mathfrak{a}(M/N) = M/N$. Then either

(*) $$\operatorname{depth}_{\mathfrak{a}} M = \min\{\operatorname{depth}_{\mathfrak{a}} N,\, \operatorname{depth}_{\mathfrak{a}} M/N\}$$

or

(**) $$\operatorname{depth}_{\mathfrak{a}} M > \operatorname{depth}_{\mathfrak{a}} M/N = \operatorname{depth}_{\mathfrak{a}} N - 1\,.$$

PROOF: If $\operatorname{depth}_{\mathfrak{a}} M$ and $\operatorname{depth}_{\mathfrak{a}} M/N$ are non-zero, choose an element $x \in \mathfrak{a}$ which is regular on both M and M/N. Since $\operatorname{Tor}_1^R(R/xR,\, M/N) = 0$,

$$0 \longrightarrow N/xN \longrightarrow M/xM \longrightarrow (M/N)/x(M/N) \longrightarrow 0$$

is exact, making an induction possible.

Now it is easily seen that if $\operatorname{depth}_{\mathfrak{a}} M = 0$ then either $\operatorname{depth}_{\mathfrak{a}} N = 0$ or $\operatorname{depth}_{\mathfrak{a}} M/N = 0$, so that $(*)$ holds in this case.

Now if $\operatorname{depth}_{\mathfrak{a}} M/N = 0$ and $\operatorname{depth}_{\mathfrak{a}} M \neq 0$, then there exists a coset $m + N \in M/N$ such that $\mathfrak{a}(m+N) = 0$. There also exists an element $x \in \mathfrak{a}$ which is regular on M. Then $xm \notin xN$ but $\mathfrak{a}xm \subseteq xN$ since $\mathfrak{a}(m+N) = 0 \in M/N$, so that $\mathfrak{a}$ consists of zero divisors on N/xN, and thus $\operatorname{depth}_{\mathfrak{a}} N = 1$ and (**) holds. ☑

Lemma. Let $(R, \mathfrak{m})$ be local and let $k = R/\mathfrak{m}$. Let M be a finitely generated non-trivial R-module. Then proj. $\dim M$ is the smallest n such that $\operatorname{Tor}^R_{n+1}(M, k) = 0$.

PROOF: By considering the n^{th} syzygy in a projective resolution for M, it suffices to see that a finitely generated R-module N is projective if $\operatorname{Tor}^R_1(N, k) = 0$. Consider an short exact sequence

$$0 \to K \to F \xrightarrow{\varepsilon} N \to 0$$

such that the induced map $F/\mathfrak{m}F \to N/\mathfrak{m}N$ is an isomorphism. If $\operatorname{Tor}^R_1(N, k) = 0$, then

$$0 \to K/\mathfrak{m}K \to F/\mathfrak{m}F \to N/\mathfrak{m}N \to 0$$

is exact, so that $K/\mathfrak{m}K = 0$. Therefore $K = 0$ by Nakayama's Lemma, so that N is free. ☑

Theorem. Let Λ be a ring which need not be either commutative nor noetherian. The following conditions are equivalent:

(1) proj. $\dim M \leq n$ for all left Λ-modules M.
(2) proj. $\dim M \leq n$ for all finitely generated left Λ-modules M.
(3) inj. $\dim M \leq n$ for all finitely generated left Λ-modules M.
(4) $\mathrm{Ext}_\Lambda(\_\,,\,\_) = 0$.

Definitions. Left global dimension.

A commutative local ring $(R, \mathfrak{m})$ is **regular** if $\mathfrak{m}$ is generated by a regular R-sequence.

A commutative ring is **regular** if $R_\mathfrak{m}$ is regular for every maximal ideal $\mathfrak{m}$.

$\mathrm{height}_M\, \mathfrak{p}$.

The **Krull dimension** of M is the supremum of $\mathrm{height}_M\, \mathfrak{p}$ for all maximal ideals $\mathfrak{p}$.

Definition. Let M be a finitely generated R-module and let $\mathfrak{a}$ be its annihilator. Then the **grade** of M is defined to be $\mathrm{depth}_\mathfrak{a} R$, in other words $\mathrm{grade}\, M$ is the length of the longest R-sequence consisting completely of elements that annihilate M.

Warning. Unfortunately, the words **depth** and **grade** have been used inconsistently in the literature. Kaplansky uses *grade* to mean what we have defined as *depth.* I guess some people feel rather strongly about this issue, or at least such was true many years ago. The definitions given are the ones that seemed to be most prevalent at the time these notes were written.

Theorem. If M is finitely generated, then $\mathrm{grade}\, M$ is the smallest integer n such that $\mathrm{Ext}^n_R(M,\, R) \neq 0$.

PROOF: If $\mathrm{grade}\, M = 0$, then $\mathfrak{a} = \mathrm{ann}\, M$ consists of zero-divisors in R, and so there must exist $\mathfrak{p} \in \mathrm{Ass}\, R$ such that $\mathfrak{p} \supset \mathfrak{a} = \mathrm{ann}\, M$. Since R then contains a submodule isomorphic to $R/\mathfrak{p}$, in order to prove that $\mathrm{Hom}_R(M,\, R) \neq 0$ it suffices to prove that $\mathrm{Hom}_R(M,\, R/\mathfrak{p}) \neq 0$. Now $\mathfrak{p}M \neq M$, otherwise $M_\mathfrak{p} = 0$ by Nakayama's Lemma, contrary to the fact that $\mathfrak{p} \in \mathrm{Supp}\, M$ since $\mathfrak{p} \supset \mathfrak{a}$. It thus suffices to show that $\mathrm{Hom}_R(M/\mathfrak{p}M,\, R/\mathfrak{p}) \neq 0$. Therefore there is no loss of generality in supposing that M is an integral domain and $\mathfrak{p} = 0 \in \mathrm{Supp}\, M$. With this assumption, let K be the quotient field of R and choose a basis $u_1, \dots, u_r$ for $M_\mathfrak{p}$. Let $\alpha\colon M \to M_\mathfrak{p}$ be the canonical map and let $s \in R$ be such that $s\alpha(M) \subseteq Ru_1 \oplus \cdots \oplus Ru_r$. Then the composition of one of the projection mappings from $Ru_1 \oplus \cdots \oplus Ru_r$ into R with $s\alpha$ yields a non-trivial homomorphism in $\mathrm{Hom}_R(M,\, R)$.

Conversely, if there exists $\varphi \neq 0 \in \mathrm{Hom}_R(M,\, R)$ then for every $a \in \mathrm{ann}\, M$, $a\varphi(M) = 0$ so that $\mathrm{ann}\, M$ consists of zero divisors and $\mathrm{grade}\, M = 0$.

Now suppose that $\mathrm{grade}\, M \geq 1$ and let $x \in \mathfrak{a} = \mathrm{ann}\, M$ be regular in R. It is easily seen that $\mathrm{grade}_R\, M = 1 + \mathrm{grade}_{R/xR}\, M/xM$. The theorem is therefore a consequence of the following lemma:

Lemma. Let x be a regular element in R such that $xM = 0$. Let $\mathcal{C}$ be the category of R/xR-modules, identified as the full subcategory of the category of R-modules consisting of those modules M such that $xM = 0$. Then for $i \geq 1$, the restriction of $\operatorname{Ext}^i_R(\_\,, R)$ to $\mathcal{C}$ is naturally isomorphic to $\operatorname{Ext}^{i-1}_{R/xR}(\_\,, R/xR)$.

PROOF: $\{\operatorname{Ext}^{i+1}_R\}_{i\geq 0}$ is clearly an exact co-connected sequence of functors on $\mathcal{C}$.

(1) From the fact that $\operatorname{Hom}_R(\_\,, R) = 0$ on $\mathcal{C}$ we get

$$0 = \operatorname{Hom}_R(\_\,, R) \to \operatorname{Hom}_R(\_\,, R/xR) \to \operatorname{Ext}^1_R(\_\,, R) \xrightarrow[0]{x} \operatorname{Ext}^1(\_\,, R) \to \dots$$

for modules in $\mathcal{C}$, so that

$$\operatorname{Hom}_{R/xR}(\_\,, R/xR) \approx \operatorname{Hom}_R(\_\,, R/xR) \approx \operatorname{Ext}^1_R(\_\,, R)$$

for modules in $\mathcal{C}$.

(2) Since proj. $\dim R/xR \leq 1$, $\operatorname{Ext}^{i+1}_R(R/xR, R) = 0$ for $i > 0$. Thus $\operatorname{Ext}^{i+1}_R(\_\,, R)$ vanishes on free R/xR-modules. The theorem now follows from the characterization of derived functors in terms of universal properties. ☑

Corollary. If $\mathfrak{a}$ is an idea such that $\operatorname{grade} R/\mathfrak{a} \geq 2$, then any map $\varphi\colon \mathfrak{a} \to R$ is given by mujltiplication by a unique element of R.

PROOF: This follows from the exact sequence

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, R) = 0 \to \operatorname{Hom}_R(R, R) \to \operatorname{Hom}_R(\mathfrak{a}, R) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, R) = 0.$$ ☑

Auslander-Buchsbaum, Annals **68**(1958), pp. 625–57.

Lemma. If

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to F_{-1} \to 0$$

is an exact sequence of finitely generated free R-modules, then

$$\sum_{-1}^{n} \operatorname{rank} F_i = 0.$$

Proposition 6.2. If M has a finite resolution

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

where the F_i are finitely generated free R-modules, then M is faithful if and only if $\operatorname{grade} M = 0$.

(In other words, if M has a finite free resolution then the annihilator of M is non-trivial if and only if it contains a regular element.)

PROOF: $(\Rightarrow)$: Trivial.

$(\Leftarrow)$: Let $\mathfrak{p} \in \operatorname{Ass} R$. Localizing the above resolution at $\mathfrak{p}$, we see that $M_{\mathfrak{p}}$ has finite projective dimension over $R_{\mathfrak{p}}$. But $\operatorname{depth} R_{\mathfrak{p}} = 0$, so that $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$. Thus

$$\operatorname{rank}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \sum_0^n \operatorname{rank} F_i,$$

so that $\operatorname{rank}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ is the same for all $\mathfrak{p} \in \operatorname{Ass} R$. But if M has grade 0 then $\operatorname{ann} M$ consists of zero divisors, so there exists $\mathfrak{p} \in \operatorname{Ass} R$ with $\operatorname{ann} M \subseteq \mathfrak{p}$, so that $M_{\mathfrak{p}} \neq 0$. Thus $M_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p} \in \operatorname{Ass} R$, i. e.

$$\operatorname{Ass} R \subseteq \operatorname{Supp} M.$$

For any $\mathfrak{p} \in \operatorname{Ass} R$, then, $M_{\mathfrak{p}}$ is a non-zero free $R_{\mathfrak{p}}$-module. Thus if $\mathfrak{a} = \operatorname{ann} M$, we conclude that $\mathfrak{a}_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Ass} R$, i. e.

$$\operatorname{Supp} \mathfrak{a} \cap \operatorname{Ass} R = \varnothing.$$

But $\operatorname{Ass} \mathfrak{a} \subseteq \operatorname{Ass} R$, so this implies that $\operatorname{Ass} \mathfrak{a} = \varnothing$. Thus $\mathfrak{a} = 0$, so M is faithful. ☑

The Koszul Complex

Reference. Matsumura, Commutative Algebra, §18.D, p. 132.

Construction. Let R be a commutative noetherian ring and $a_1, \dots, a_n \in R$. Let E be the free exterian algebra over R on the symbols $T_1, \dots, T_n$. Thus $E_0 = R$, $E_1 = RT_1 \oplus \cdots \oplus RT_n$, $E_i \approx R^{\binom{n}{i}}$, $x^2 = 0$ for $x \in E_1$, and for $x \in E_i$, $y \in E_j$, $xy = (-1)^{ij} yx$.

Define a graded R-linear map $d \colon E \to E$ with degree 1 by defining $d(T_i) = a_i$, $d(xy) = (dx)y + (-1)^i x(dy)$ for $x \in E_i$, $y \in E_j$. Note that $d(R) = d(E_0) = 0$ and

$$d(T_{i_1} \cdots T_{i_k}) = \sum_{r=1}^k (-1)^{r-1} a_{i_r} T_{i_1} \cdots \widehat{T_{i_r}} \cdots T_{i_k}.$$

We define the Koszul Ring or Koszul Complex for R with respect to $a_1, \dots, a_n$ to be this graded ring E, and denote it by $R{<}T_i;\ dT_i = a_i{>}$.

Lemma. (E, d) is a chain complex.

PROOF: Use induction on i to show that $d^2(D_i) = 0$. ☑

Lemma. Let $\mathcal{Z} = \operatorname{Ker} d$ and $\mathcal{B} = d(E)$. Then $\mathcal{Z}$ is a ring and $\mathcal{B}$ an ideal in $\mathcal{Z}$, so that $H(E) = \mathcal{Z}/\mathcal{B}$ is a graded ring.

Lemma. Let $\mathfrak{a} = (a_1, \ldots, a_n)$.

(1) $H_0(E) = R/\mathfrak{a}$
(2) $H_n(E) \approx \operatorname{ann}_R \mathfrak{a}$
(3) $H_q(E) = 0$ for $q < 0$ or $q > n$.
(4) $\mathfrak{a} H(E) = 0$.

PROOF: (4) Since $\mathfrak{a} = d(E_1) \subseteq \mathcal{B}$ and $\mathcal{B}\mathcal{Z} \subseteq \mathcal{B}$, this is clear. ☑

Theorem. Let R be noetherian, let $\mathfrak{a} = (a_1, \ldots, a_n)$, and let $r = \operatorname{depth}_{\mathfrak{a}} R = \operatorname{grade} R/\mathfrak{a}$. Let $E = R\!<\!T_i;\ dT_i = a_i\!>$. Then $H_{n-r}(E) \neq 0$ and $H_q(E) = 0$ for $q > n - r$.

PROOF: By induction on r. If $r = \operatorname{depth}_{\mathfrak{a}} R = 0$ then $H_n(E) = \operatorname{ann} \mathfrak{a} \neq 0$, and by construction $H_q = 0$ for $q > n$.

If $\operatorname{depth}_{\mathfrak{a}} R > 0$, let x_1 be a regular element in $\mathfrak{a}$. Let $\bar{R} = R/(x_1)$, $\bar{E} = E/x_1 E$. Since x_1 is regular on E there is an exact sequence of complexes

$$0 \longrightarrow E \xrightarrow{x_1} E \longrightarrow E/x_1E \longrightarrow 0 .$$

Now let q be the largest integer such that $H_q(E) \neq 0$. Then we get

$$0 = H_{q+1}(E) \to H_{q+1}(\bar{E}) \to H_q(E) \xrightarrow{x_1} H_q(E) \to \cdots .$$

Now $\mathfrak{a} H_q(E) = 0$ so, since $x_1 \in \mathfrak{a}$, this implies

$$H_{q+1}(\bar{E}) \approx H_q(E).$$

On the other hand, for $p > q + 1$, we have

$$H_p(E) = 0 \to H_p(\bar{E}) \to H_{p-1}(E) = 0.$$

Thus $H_{q+1}(\bar{E}) \neq 0$ and $H_p(\bar{E}) = 0$ for $p > q + 1$. Since $\operatorname{depth}_{\bar{\mathfrak{a}}} \bar{R} = r - 1$, it now follows by induction that $q + 1 = n - (r - 1)$ so that $q = n - r$. ☑

Theorem. Let R be noetherian, and suppose that $\mathfrak{a} = (a_1, \ldots, a_n) \subseteq J(R)$. Let $E = R\!<\!T_i;\ dT_i = a_i\!>$. Then the following are equivalent:

(1) $a_1, \ldots, a_n$ is a regular R-sequence.

(2) $H_i(E) = 0$ for $i \geq 1$.

(3) $H_1(E) = 0$.

(4) Any sequence of n elements generating $\mathfrak{a}$ is a regular R-sequence.

(5) $\operatorname{grade} R/\mathfrak{a} = 1$.

PROOF: By the previous theorem, (1) $\Rightarrow$ (5) $\Rightarrow$ (2) $\Rightarrow$ (3).

(4) $\Rightarrow$ (1): Easy from the previous theorem and from (3) $\Rightarrow$ (1).

(3) $\Rightarrow$ (1): By induction on n. Let $E' = R\langle T_1, \dots, T_{n-1};\, dT_i = a_i\rangle$. Consider the "chain maps" $i\colon E' \hookrightarrow E$ and $j\colon E \to E'$, where j is defined by the conditions $j(E') = 0$, $j(T_{k_1} \dots T_{k_s} T_n) = T_{k_1} \dots T_{k_s}$. This gives a short exact sequences of chain complexes

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E' \longrightarrow 0\,.$$

Since j has degree -1, we get a long exact sequence

$$\cdots \to H_1(E') \xrightarrow{\delta_1} H_1(E') \xrightarrow{i_*} H_1(E) \xrightarrow{j_*} H_0(E') \xrightarrow{\delta_0} H_0(E) \to \cdots\,.$$

We now compute δ_1. An element in $H_q(E')$ has the form $[z]$, where $z \in E'_q$ and $dz = 0$. Now $z = j(zT_n)$ and $\delta_q[z] = [d(zT_n)]$. But

$$d(zT_n) = dz\,T_n + (-1)^q z\,dT_n = (-1)^q a_n z \in E',$$

so that, up to sign, δ_q is just multiplication by a_n: $\delta_q[z] = (-1)^q a_n[z]$.

Now $H_1(E) = 0$ by hypothesis, so δ_1 is surjective and δ_0 monic. In other words $a_n H_1(E') = H_1(E')$, and multiplication by a_n is monic on $H_0(E')$. Therefore $H_1(E') = 0$ by Nakayama's Lemma. Therefore by the induction hypothesis, $a_1, \dots, a_{n-1}$ is a regular R-sequence. Since $H_0(E') = R/(a_1, \dots, a_{n-1})$ and a_n is regular on $H_0(E')$, it follows that $a_1, \dots, a_n$ is a regular R-sequence. ☑

Corollary. Suppose that R is noetherian and let $a_1, \dots, a_n \in J(R)$. If $a_1, \dots, a_n$ is a regular R-sequence, then every permutation of it is also a regular R-sequence.

This corollary may seem not very surprising. However it is not valid without the hypothesis that the sequence be contained in $J(R)$.

Example [Kaplansky, Commutative Rings, §3.1, Exercise 7, p. 102]. Let $R = K[X, Y, Z]$, where K is a field. The elements X, $Y - XY$, $Z - XZ$ form a regular R-sequence, but in the order $Y - XY$, $Z - XZ$, X they do not.

Corollary. Suppose that R is noetherian. Let $\mathfrak{a} = (a_1, \dots, a_n) \subseteq J(R)$ and suppose that $\operatorname{grade} R/\mathfrak{a} = n$. Then E is a projective resolution for $R/\mathfrak{a}$. In consequence,

$$\operatorname{Tor}_i^R(R/\mathfrak{a},\, R/\mathfrak{a}) \approx (R/\mathfrak{a})^{\binom{n}{i}} \approx \operatorname{Ext}_R^i(R/\mathfrak{a},\, R/\mathfrak{a}).$$

PROOF: By construction, E_q is a free R-module for each q. And by the theorem, if $\operatorname{grade} R/\mathfrak{a} = n$ then E is exact in degrees larger than 0 and $H_0(E) = R/\mathfrak{a}$. Thus E is a projective resolution for $R/\mathfrak{a}$. Therefore $\operatorname{Tor}^R(R/\mathfrak{a}, R/\mathfrak{a})$ is the homology of $E/\mathfrak{a}E = E \otimes_R R/\mathfrak{a}$. But since $d(D) \subseteq \mathfrak{a}E$, the differentiation on this complex is trivial. ☑