Let $\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}$ be vectors in $\mathbb{R}^m$. Let A be the $m \times p$ matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_p$:

$$A = [\,\mathbf{v}_1 \quad \ldots \quad \mathbf{v}_p\,]$$

- ▶ $\mathbf{w}$ belongs to the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

MEANS

- ▶ $\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

MEANS

- ▶ There exist scalars $x_1, \ldots, x_p$ such that $\mathbf{w} = x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p$.

MEANS

- ▶ There exists a vector $\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ in $\mathbb{R}^p$ such that $A \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \mathbf{w}$.

MEANS

- ▶ The linear system $A\mathbf{x} = \mathbf{w}$ has a solution.

MEANS

- ▶ The linear system $A\mathbf{x} = \mathbf{w}$ is **consistent**.

- ▶ $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly independent.

MEANS

- ▶ **If** $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ **then** $x_1 = 0, \ldots, \ x_p = 0$.

MEANS

- ▶ **If** $A \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ **then** $\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \mathbf{0}$.

MEANS

- ▶ The homogeneous system $A\mathbf{x} = \mathbf{0}$ has **only** the trivial solution.

Any linear transformation $L\colon \mathbb{R}^m \to \mathbb{R}^n$ is given by a formula of the form $L(\mathbf{x}) = A\mathbf{x}$, where A is an $m \times n$ matrix. More specifically, the columns of A are $L(\mathbf{v}_1), \ldots, L(\mathbf{v}_n)$ where $\mathbf{v}_i$ is the column vector with 1 in the i^{th} coordinate and 0 in the other coordinates.
If L and A are related in this way, then

Ker L

IS THE SAME AS

The null space of A

IS THE SAME AS

The solution space to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

- $\mathbf{w} \in \text{Range}\, L$

MEANS

- $\mathbf{w} = L(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^n$

MEANS

- The linear system $A\mathbf{x} = \mathbf{w}$ has a solution

MEANS

- $\mathbf{w}$ belongs to the column space of A.

- L is one-to-one.

MEANS

- **If** $L(\mathbf{v}) = L(\mathbf{v}')$ **then** $\mathbf{v} = \mathbf{v}'$.

MEANS

- For each $\mathbf{w} \in \mathbb{R}^m$ there is **at most** one $\mathbf{x} \in \mathbb{R}^n$ such that $L(\mathbf{x}) = \mathbf{w}$.

MEANS

- If $\mathbf{w} \in \mathbb{R}^m$ then the linear system $A\mathbf{x} = \mathbf{w}$ has either a unique solution or no solution at all.