A "Grammar Lesson" for Linear Algebra

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Some Incorrect Statements Often Encountered in Student Proofs

$\mathbf{v}_1, \ldots, \mathbf{v}_p$ *are linearly independent so* $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ *with* $x_1 = \cdots = x_p = 0$.

Prove that $z_1 A\mathbf{v}_1 + \cdots + z_p A\mathbf{v}_p = 0$ *where* $z_1 = \cdots = z_p = 0$.
[In other words, prove that if $z_1 = \cdots = z_p = 0$ then $z_1 A\mathbf{v}_1 + \cdots + z_p A\mathbf{v}_p = 0$. Certainly this would be true in any case, without any assumptions at all, but it doesn't prove that $A\mathbf{v}_1, \ldots, A\mathbf{v}_p$ are linearly independent.]

The row rank of a matrix is the number of non-zero rows.
[This is true only if the matrix is in row echelon form.]

$\mathbf{v}_1, \ldots, \mathbf{v}_p$ *are linearly independent means that* $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ *has the trivial solution.*
[It would have the trivial solution in any case.]

$A\mathbf{w}$ *is spanned by* $A\mathbf{v}_1, \ldots, A\mathbf{v}_p$.
[A set of vectors doesn't span a vector, it spans a subspace.]

The equation $a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p = \mathbf{0}$ *is linearly independent.*
[A set of vectors can be linearly independent, an equation cannot be.]

The dimension spanned by $A\mathbf{v}_1, \ldots, A\mathbf{v}_p$ *is the number of matrices that span it.*
[The dimension is a number, it can't be spanned by anything. Furthermore, $A\mathbf{v}_i$ are vectors in $\mathbb{R}^m$ — it's not useful to refer to them as matrices. Finally **the dimension of a subspace is NOT defined as the number of vectors that span it!** It's defined as the number of vectors in a **basis.**]

The column rank are the columns of a matrix that spans the column space.

If $A\mathbf{v}_1, \ldots, A\mathbf{v}_p$ *are linearly independent,* $\Rightarrow$ $x_1 A\mathbf{v}_1 + \cdots + x_p A\mathbf{v}_p = \mathbf{0}$ *for all* $x_1 = \cdots = x_p = 0$.

Given that $\mathbf{w}$ *is in the subspace spanned by* $\mathbf{v}_1, \ldots, \mathbf{v}_p$, $\Rightarrow$ $\mathbf{w}$ *is a basis.*
[How could one vector be a basis? It would have to be a one-dimensional space.]

If $A\mathbf{v}_1, \ldots, A\mathbf{v}_p$ *is linearly independent then there exists constants such that* $a_1 = \cdots = a_p = 0$ *and* $a_1 A\mathbf{v}_1 + \cdots + a_p A\mathbf{v}_p = 0$.
[That would be true whether the vectors are linearly independent are not.]

[In reference to four vectors in $\mathbb{R}^3$] *The dimension of* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ *is four.*
[First of all, a set of four vectors can't be a subspace so it can't have a dimension.]
[This set of vectors **spans** a subspace (of $\mathbb{R}^3$) and the **subspace** spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ has a dimension, but that the dimension of the subspace can't be larger than three because the subspace is contained in the three-dimensional space $\mathbb{R}^3$.]

SOME CORRECT STATEMENTS

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent then $a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p = \mathbf{0}$ implies $a_1 = \cdots = a_p = 0$.

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent then $a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p = \mathbf{0}$ is possible **only** if $a_1 = \cdots = a_p = 0$.

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent then $a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p = \mathbf{0}$ is never true except when $a_1 = \cdots = a_p = 0$.

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent then $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ has no non-trivial solution.

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent then $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$ has **only** the trivial solution.

> It may seem that I'm just being picky about the language here. But in fact you are writing down things that are just plain not true, and because you are getting the language wrong you are screwing up the logical structure of your proofs, so you wind up without anything worth even partial credit.