

Diagram of a Proof

E. L. Lady

The role of an **if/then** statement in a proof is very different depending on whether it occurs in the hypothesis or the conclusion.

If the statement “If P then Q” is what has **to be proved**, then P gets added the hypothesis of the theorem (by using the word “Suppose”), and the goal of the proof is to prove Q.

If the statement “If P then Q” is part of what’s **given**, this tells us that if we want to prove Q, it will be sufficient to prove P. Therefore instead of directing our efforts to proving Q, we can concentrate on proving P.

Problem. Let A is an $m \times n$ matrix and let $\mathbf{v}_1, \dots, \mathbf{v}_p$ be vectors in \mathbb{R}^n . Prove that if the vectors $A\mathbf{v}_1, \dots, A\mathbf{v}_p$ in \mathbb{R}^m are linearly independent, then the vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly independent.

Given:

$A\mathbf{v}_1, \dots, A\mathbf{v}_p$ are linearly independent.

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If $a_1 A\mathbf{v}_1 + \dots + a_p A\mathbf{v}_p = \mathbf{0}$
then $a_1 = \dots = a_p = 0$.

To be proved:

$\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly independent.

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If $a_1 \mathbf{v}_1 + \dots + a_p \mathbf{v}_p = \mathbf{0}$
then $a_1 = \dots = a_p = 0$.

Now $a_1 \mathbf{v}_1 + \dots + a_p \mathbf{v}_p = \mathbf{0}$ gets added to the list of what’s given by using the word “suppose.”

And to succeed in proving $a_1 = \dots = a_p = 0$ it is good enough to prove that $a_1 A\mathbf{v}_1 + \dots + a_p A\mathbf{v}_p = \mathbf{0}$, so that becomes the new goal.

Revised given:

$A\mathbf{v}_1, \dots, A\mathbf{v}_p$ are linearly independent.
Suppose $a_1 \mathbf{v}_1 + \dots + a_p \mathbf{v}_p = \mathbf{0}$.

Desired end of proof:

Thus $a_1 A\mathbf{v}_1 + \dots + a_p A\mathbf{v}_p = \mathbf{0}$.
Therefore $a_1 = \dots = a_p = 0$
because $A\mathbf{v}_1, \dots, A\mathbf{v}_p$ are linearly independent.

It is now obvious how to complete the proof.

PROOF: Suppose that $A\mathbf{v}_1, \dots, A\mathbf{v}_p$ are linearly independent. And suppose that $a_1 \mathbf{v}_1 + \dots + a_p \mathbf{v}_p = \mathbf{0}$. Multiplying both sides by A and invoking the distributive law yields $a_1 A\mathbf{v}_1 + \dots + a_p A\mathbf{v}_p = \mathbf{0}$. Therefore $a_1 = \dots = a_p = 0$ because $A\mathbf{v}_1, \dots, A\mathbf{v}_p$ are linearly independent.

We have shown that if $a_1 \mathbf{v}_1 + \dots + a_p \mathbf{v}_p = \mathbf{0}$ then a_1, \dots, a_p must all be 0. This shows that $\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly independent.