Chapter 2. Weighted Voting Systems

Sections 2 and 3. The Banzhaf Power Index

John Banzhaf is an attorney and law professor. In 1965, his analysis of the power in the Nassau County NY Board of Supervisors was used in a lawsuit, which resulted in a change in the system (see page 55 for more details). The county was divided into six districts; the supervisor from each district was given votes proportional to the districts population. This resulted in a weighted voting system described by:

[58: 31, 31, 28, 21, 2, 2]

Banzhaf showed that the last three districts were "dummies": they had absolutely no power. If at least two of the first three districts voted yes, a measure passed, otherwise it failed. Note that there is no combination of districts for which the vote of any of the last three districts makes the difference between winning or losing.

More generally, the Banzhaf Power Index is a way of measuring how much of a difference a player's vote can make.

First write down all the combinations of players whose votes for a measure result in the measure passing (these are called "**winning coalitions**").

Here is an example:

[8: 5, 4, 3, 2]

There are four players, whom we'll call P1, P2, P3 and P4, whose weights are 5, 4, 3 and 2. The sum of the weights is 5+4+3+2=14. The quota is 8. {P1, P2} is an example of a winning coalition, because its total weight of 5+4=9 is greater than 8.

Winning coalitions	Weight	Critical players
P1, P2	9	P1, P2
P1, P3	8	P1, P3
P1, P2, P3	12	P1
P1, P2, P4	11	P1, P2
P1, P3, P4	10	P1, P3
P2, P3, P4	9	P2, P3, P4
P1, P2, P3, P4	14	

A **critical player** is one whose vote makes the difference between winning or losing. Let **T** be the total number of critical players. In the above example, T=12.

The **Banzhaf power index** of a player P is the number of times P is critical, divided by T.

In the example, P1 is critical 5 times, P2 and P3 are critical 3 times, and P4 is only critical 1 time. So their power indices are 5/12, 3/12, 3/12 and 1/12 (you can write these as percents: 41.6%, 25%, 25% and 83.4%).

More examples:

[15: 5, 5, 4, 2]
In this case, a motion only passes if all four players vote for it. All four players have veto power. The only winning coalition is {P1, P2, P3, P4}, and all the players are critical, so T=4 and each player has power 1/4 = 25%.

[7: 7, 3, 2]
In this case, a motion only passes if P1 votes for it. P1 is a dictator. The other two players are dummies.

Winning coalitions	Weight	Critical players
P1	7	P1
P1, P2	10	P1
P1, P3	9	P1
P1, P2, P3	12	P1

So T=4, and the power of P1 is 4/4=1=100%. The power of the other players is 0/4=0%.

In general, a dummy is a player whose power is 0% and a dictator is a player whose power is 100%.

As examples become more complicated, it is important to try to be more efficient. One thing which saves effort is to underline all critical players, instead of listing them separately. Another tactic is to organize winning coalitions by the number of members. Also it helps if you can find a description of all the winning coalitions. Here is another example:

[21: 3, 5, 9, 18]

Note that the winning coalitions are all coalitions containing P4 and one other player.

Winning coalitions, with critical players underlined:

1-player: none

2-players: {P1, P4}, {P2, P4}, {P3, P4}

3-players: {P1, P2, P4}, {P2, P3, P4}, {P1, P3, P4}

4-players: {P1, P2, P3, P4}

Number of critical players = 10

The Banzhaf power distribution is: 10%, 10%, 10%, 70%

Returning to the original Nassau County (N. Y.) Board of Supervisors, which was the weighted voting system:
[58: 31, 31, 28, 21, 2, 2]   Call the players P1, … , P6.

The winning coalitions are all coalitions with at least 2 from P1, P2, P3. P1, P2, P3 are always critical in any coalition having exactly 2 of them, and are not critical in a coalition having all 3 of them. So P1, P2, P3 have the same power.
P4, P5 and P6 are never critical, so have no power.
The Banzhaf power distribution is 1/3, 1/3, 1/3, 0, 0, 0

Read the examples in section 3. Another method of computing power is described in sections 4 and 5. Skip these.

Chapter 3. Fair Division

This Chapter is concerned with methods of dividing up goods, property, etc., among several players. We want each player to receive a **fair share.** By this we mean: if there are **N** players, each player should receive **at least 1/N** worth of the goods. However, each player may value the goods differently. So what we want is:

Each player receives what that player considers to be at least 1/N worth of the goods.

Also, we assume the players don't know the preferences of the other players.

Furthermore, we don't want to involve any outside party (a judge or arbitrator); we want the interested parties to make the decision all by themselves.

Is that really possible? We'll see that it is. We'll learn some methods in sections 2, 5 and 6 (and maybe 7); we'll skip the methods in the other sections.

Sections 2-5 are concerned with **continuous** problems; this means the "goods" are something like a piece of property, or a cake, in which we can make very small alterations in the size of a piece. Sections 6 and 7 are concerned with **discrete** problems, in which the "goods" are something like: 10 paintings, or a house, two cars and a boat---you wouldn't want to break the house or paintings down into small pieces while dividing them among players.

Section 2. The divider-chooser method.

Suppose you want to divide a piece of property (or a cake) between two players. They might value different parts of the property differently: for example, one might prefer forest and the other grasslands.

Flip a coin; the loser goes first. That player is the **divider** (we'll call him/her D). The other player is the **chooser** C. Here is the method:

D divides the property into two pieces.
C chooses one piece; the other piece goes to D.

Since D chooses second, and doesn't know the preferences of C, D should divide the property into two pieces of equal value (by his/her **own** value system). This guarantees that, no matter what piece C chooses, D will get a "fair piece" (it will have half the total value according to D). C will choose the better of the two pieces (by C's value system), so C will also get a fair piece (at least half the total value according to C).

Actually, this method is advantageous to C, but it does meet the requirements we set out.

The problem gets harder when we have more than two parties; we'll look at that situation in the rest of the sections.