CHAPTER 4

Calculus

Calculus deals with the instantaneous rate of change of quantities. Calculus is important because most of the laws of science do not provide direct information about the values of variables but only about their rate of change. For instance, velocity is the rate of change of position, acceleration is the rate of change of velocity, and Newton's Law of Motion tells you how velocity will change in various situations, i.e., it tells you about acceleration. That's why you see the concept of the derivative used throughout science, in physics, chemistry, biology, economics, even psychology. Calculus provides a language, a conceptual framework for describing relationships that would be difficult to discuss in any other language. Calculus gives you the ability to find the effects of changing conditions on the system being investigated. By studying these changing conditions, you can learn how to control the system and make it do what you want it to do. Calculus, by giving engineers and you the ability to model and control systems, provides for extraordinary power over the material world.

How does calculus model change? What is calculus like? The fundamental idea of calculus is to study change by studying "instantaneous" change, which you may look at as changes over tiny intervals of time. It turns out that such changes tend to be much simpler than changes over longer intervals of time. This means instantaneous change is much easier to model. Calculus was invented by Newton and at about the same time by Leibniz.

Calculus is a gateway to further mathematics.

1. The notion of a function

A key object in all of mathematics is the notion of function. A function is what you find on calculators. You can enter an ***input***, push a button labeled with a function name, and obtain an ***output*** that is displayed. The output is predictable and always the same for the same input and the same button.

Definition 1.1. *A* ***function*** *f accepts an input x and assigns to x an output or function value at x, or the* ***image*** *or* ***value*** *of x under f. The image of x is written as $f(x)$ or as y.*

As a mathematical term, *function* was coined by Gottfried Wilhelm Leibniz in 1694, and he studied quantities related to a curve, such as a curve's steepness at a specific point. The word function was later adopted by Leonhard Euler during the mid-18th century for computational expressions such as $f(x) = \sin x + x^3$.

Definition 1.2. *The* ***graph*** *of a function f is the set of all ordered pairs $(x, f(x))$ for all x that can be used:*

$$\text{graph of } f = \{(x, f(x)) \mid \text{all } x \text{ accepted by } f\}$$

[Read: "The graph of f is the set of all pairs $(x, f(x))$ such that x is accepted by the function."]
Functions whose inputs and outputs are real numbers are called ***real-valued functions of a real variable*** *and their graphs can be plotted using a Cartesian coordinate system. These are the familiar graphs or plots of functions, the ordered pairs being the Cartesian coordinates of the points of the graph.*

Example 1.3. The area A of a square depends on the length of its sides x. The rule that connects x and A is given by the equation $A = A(x) = x^2$. With each positive number x there is associated one value of A. We can say that A is a function of x. For instance, if the sides of the square have length 2, we have that $A(2) = 4$. The image of 3 is 9. This function accepts all values $x \geq 0$ (a length cannot be negative). The graph of $A(x)$ is half a parabola.

Exercise 1.4. When you turn on a hot water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

Exercise 1.5. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.

Example 1.6. Let $f(x) = 3x - 5$. For instance $f(0) = 3 \cdot 0 - 5 = -5$ and $f(1) = 3 \cdot 1 - 5 = -2$. The graph of this function f is a straight line.

Example 1.7. Let $f(x) = x - x^2 = \frac{1}{4} - (x - \frac{1}{4})^2$. We have that $f(0) = 0$, $f(-2) = -2 - (-2)^2 = -6$, $f(\frac{1}{2}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. The graph is a parabola with highest point $(\frac{1}{2}, \frac{1}{4})$.

Exercise 1.8. Temperatures readings T (in F) were recorded every two hours from midnight to noon in Atlanta, Georgia, on March 18, 1996. The time t was measured in hours from midnight. The pairs (t, T) are as follows:

$$\{(0, 58), (2, 57), (4, 53), (6, 50), (8, 51), (10, 57), (12, 61)\}.$$

Use the readings to sketch a rough graph of T as a function of t. Use this graph to estimate the temperature at 11 am.

Example 1.9. Let $f(x) = -2x^3 + x^2 - 4x + 3$. Then

(1) $f(0) = -2 \cdot 0^3 + 0^2 - 4 \cdot 0 + 3 = 3$.
(2) $f(-3) = -2(-3)^3 + (-3)^2 - 4(-3) + 3 = (-2)(-27) + 9 + 12 + 3 = 54 + 9 + 12 + 3 = 78$.

(3) $f(-x) = -2(-x)^3 + (-x)^2 - 4(-x) + 3 = (-2)(-x^3) + x^2 + 4x + 3 = 2x^3 + x^2 + 4x + 3$.
(4) $f(x+2) = -2(x+2)^3 + (x+2)^2 - 4(x+2) + 3 = -2(x^3 + 3x^2 \cdot 2 + 3x \cdot 2^2 + 2^3) + (x^2 + 4x + 4) - 4(x+2) + 3 = -2(x^3 + 6x^2 + 12x + 8) + x^2 + 4x + 4 - 4x - 8 + 3 = -2x^3 - 12x^2 - 24x - 16 + x^2 - 1 = -2x^3 - 11x^2 - 24x - 17$.
(5) $f(2a) = -2(2a)^3 + (2a)^2 - 4(2a) + 3 = (-2)8a^3 + 4a^2 - 8a + 3 = -16a^3 + 4a^2 - 8a + 3$.
(6) $f(a+b) = -2(a+b)^3 + (a+b)^2 - 4(a+b) + 3 = (-2)(a^3 + 3a^2b + 3ab^2 + b^3) + (a^2 + 2ab + b^2) - 4a - 4b + 3 = -2a^3 - 6a^2b - 6ab^2 - 2b^3 + a^2 + 2ab + b^2 - 4a - 4b + 3$.
(7) $f(1 + x^2) =?$. This is the case of $f(a + b)$ where $a = 1$ and $b = x^2$. Thus $f(1+x^2) = -2 - 6x^2 - 6x^4 - 2x^6 + 1 + 2x^2 + x^4 - 4 - 4x^2 + 3 = -2x^6 - 5x^4 - 8x^2 - 2$.

Exercise 1.10. If $f(x) = 2x^2 + 3x - 4$, find $f(0)$, $f(2)$, $f(\sqrt{2})$, $f(1 + \sqrt{2})$, $f(-x)$, $f(x + 1)$, $2f(x)$ and $f(2x)$.

Exercise 1.11. If $f(x) = x^3 - 3x - 4$, find $f(0)$, $f(2)$, $f(\sqrt{2})$, $f(-x)$, $f(x - 1)$, $f(a)$ and $f(2b)$.

Exercise 1.12. If $f(x) = 2x^3 - 3x^2 - 4x + 5$, find $f(0)$, $f(2)$, $f(\sqrt{2})$, $f(-x)$, $f(x+2)$, $f(a + b)$ and $f(2x)$.

Exercise 1.13. If $f(x) = x^3 + x^2 + x + 1$, find $f(0)$, $f(-1)$, $f(\sqrt{2})$, $f(1+a)$, $f(-x)$, $f(x + 1)$, and $f(-2x)$.

Exercise 1.14. If $f(x) = 2x^2 + 3x - 4$, find $f(2)$, $f(2+h)$, $\frac{1}{h}(f(2+h) - f(2))$, $f(1)$, $f(1 + h)$, $\frac{1}{h}(f(1+h) - f(1))$, $\frac{1}{h}(f(x+h) - f(x))$

2. Velocity and Acceleration

Imagine a car that is going from town A to town B by following different kinds of roads (highway, trail ...etc). Say that the distance (with the chosen itinerary) between the two towns is Δs km, and the car spends Δt hours on the road. Figure 1 illustrates the trip of the car.

The ***average velocity***, $\bar{v}$, of the car is equal to the total distance divided by the total time, that is:

$$\bar{v} = \frac{\Delta s}{\Delta t}.$$

where $\bar{v}$ is expressed in km/h. Since the state of the road is not uniform over the whole trip, the velocity v_1 at time t_1 is not the same as the velocity v_2 at time t_2. At each time t of the trip, there is a velocity $v(t)$. We say here that $v(t)$ is the ***instantaneous velocity*** of the car at time t.

Knowing the average velocity $\bar{v}$ does not give us the velocity at a specific time t. And knowing the velocity at time t_1 does not tell us what was or what will be the velocity at another time t_2.

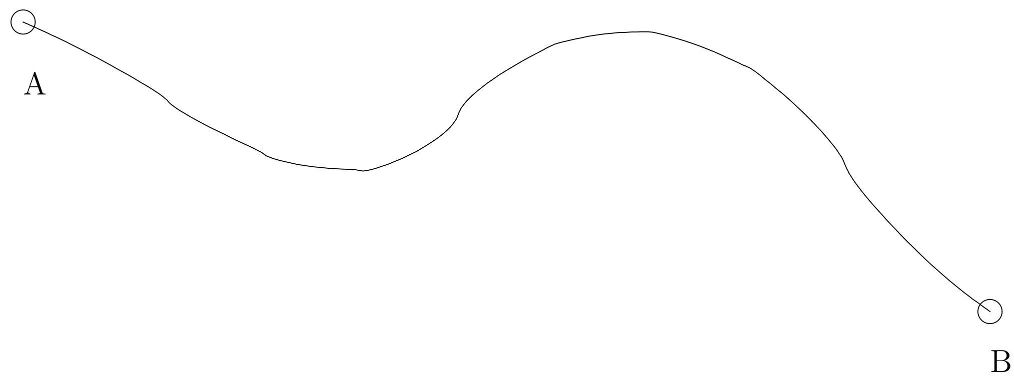

Figure 1. *Car traveling from town A to town B.*

How does the speedometer of a car manage to show the current (instantaneous) velocity at every time?

- The car keeps track of the revolutions of the wheels.
- The car multiplies the number of revolutions by $2\pi\cdot$ radius of the wheel to obtain the distance traveled.
- It is a fact that the velocity of a car is essentially the same in short time intervals, say in 15 second intervals.
- At time t the car computes the distance covered in say 15 seconds and computes the *average velocity* in this short interval, converts the result to miles/hour and displays the result.
- Taking shorter time intervals, say 5 second intervals, or 1 second intervals, the resulting instantaneous velocity becomes more truthful.

To obtain an instantaneous velocity for an arbitrary linear (along a straight line) motion we imitate the method of the car.

Example 2.1. The position of a car s [feet] is given by the following values with respect to the time t [seconds]. We have the pairs

$$(t, s) = \{(0,0), (1,10), (2,32), (3,70), (4,119), (5,178)\}.$$

First find the average velocity for the time period beginning when $t = 2$ and lasting (i) 3 sec, (ii) 2 sec and (iii) 1 sec. Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 2$.

Answer: The average velocity of the car starting at $t = 2$ and lasting 3 seconds is given by the distance traveled divided by the elapsed time. It means that we start at $t = 2$ and finish at $t = 5$. Moreover the traveled distance is given by $178 - 32 = 146$. The corresponding average velocity is then $\frac{146}{3} = 48.66666$. The answers to (ii) and (iii) are respectively 43.5 and 38.

Definition 2.2. *Suppose that $s = s(t)$ describes the motion of a moving object along a straight line. We fix a moment t and consider later moments $t + h$. Between times*

t and $t+h$ the object has covered the distance $s(t+h)-s(t)$ and the time elapsed is $(t+h)-t=h$. (Figure 2.)

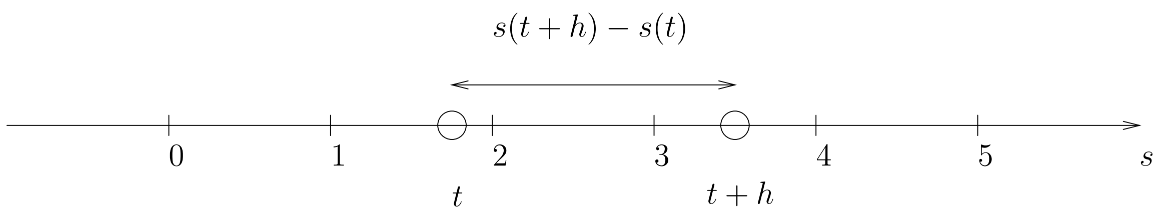

FIGURE 2. Linear Motion

The average velocity in the time interval $[t,t+h]$ is

$$\frac{s(t+h)-s(t)}{h}$$

*If the time interval h is very small, we can take $\frac{s(t+h)-s(t)}{h}$ to be a close approximation to the instantaneous velocity at time t of the moving object. The smaller the elapsed time h, the better the approximation. Contrary to the situation of a mechanical reality like a car, in thought we can make h smaller and smaller and end up with a value $v(t)$ that is called the **limit value** of $\frac{s(t+h)-s(t)}{h}$ **as** h **approaches** 0. We write*

$$v(t)=\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}.$$

*This value $v(t)$ is the **instantaneous velocity** of the moving object in question.*

EXAMPLE 2.3. Suppose that your math book is dropped from the roof of a building 100 meters above the ground (by accident of course!). Find the instantaneous velocity of the book after 1 second.

Galileo Galilei discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling:

$$s(t)=4.9t^2,$$

where $s(t)$ is the distance traveled by the book in meters and t is the time in seconds. The average velocity is given by:

$$\text{average velocity } = \frac{\text{distance traveled}}{\text{time elapsed}}.$$

The average velocity over the brief time interval of a tenth of a second from $t=1$ to $t=1.1$ is

$$\frac{4.9(1.1)^2-4.9}{0.1}=10.29 \text{ [sec]}.$$

To find the instantaneous velocity let us compute the average velocity over successively smaller time periods:

$$\begin{array}{rll} 1 \leq t \leq 1.1 & \text{average velocity} & = 10.29 \text{ [s]} \\ 1 \leq t \leq 1.05 & \text{average velocity} & = 10.045 \text{ [s]} \\ 1 \leq t \leq 1.01 & \text{average velocity} & = 9.849 \text{ [s]} \\ 1 \leq t \leq 1.001 & \text{average velocity} & = 9.8049 \text{ [s]} \end{array}$$

The instantaneous velocity is the limit value of these average velocities as the time interval goes to 0 we guess to be $v(1) = 9.8$ [m/s^2].
More generally, fix a time t and let h be the duration of a small time interval. The average velocity between t and $t + h$ is

$$\begin{aligned} \frac{s(t+h) - s(t)}{h} &= \frac{4.9(t+h)^2 - 4.9t^2}{h} \\ &= 4.9\frac{t^2 + 2th + h^2 - t^2}{h} \\ &= 4.9\frac{h(2t+h)}{h} = 4.9(2t+h). \end{aligned}$$

In this simplified form we can see clearly what happens if h gets smaller and smaller: It disappears and we found

$$v(t) = \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h\to 0} 4.9(2t+h) = 4.9 \cdot 2t = 9.8t \text{ [m/s]}.$$

Example 2.4. ***Uniform linear motion.*** Consider an object moving on a straight line according to the equation $s(t) = at + b$ where s is the position in meters of the object, t is the time in second and a and b are some constants. Figure 3 shows the position of this object with respect to time.
The velocity of the object is the rate of change of its position with respect to the time. Between time t_1 and t_2 ($t_2 > t_1$), the object goes from position $r(t_1)$ to position $r(t_2)$. The average velocity between position $(t_1, s(t_1))$ and position $(t_2, s(t_2))$ is:

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{at_2 + b - (at_1 + b)}{t_2 - t_1} = a. \tag{2.5}$$

So, the average velocity $\frac{\Delta s}{\Delta t}$ does not depend on times t_1 and t_2 and the rate of change of the position of the object with respect to time is always the same, that means that the velocity of the object is constant and is equal to a [meters/seconds].

Remark 2.6. *Notice that the expression of the average velocity in Figure 3 is the same as the expression for the slope of the straight line that we see in the picture. The fact that the motion is uniform reappears geometrically as a graph that is a straight line.*

Exercise 2.7. The displacement (in meters) of an object moving in a straight line is given by $s(t) = t^2 - 8t + 18$, where t is measured in seconds.

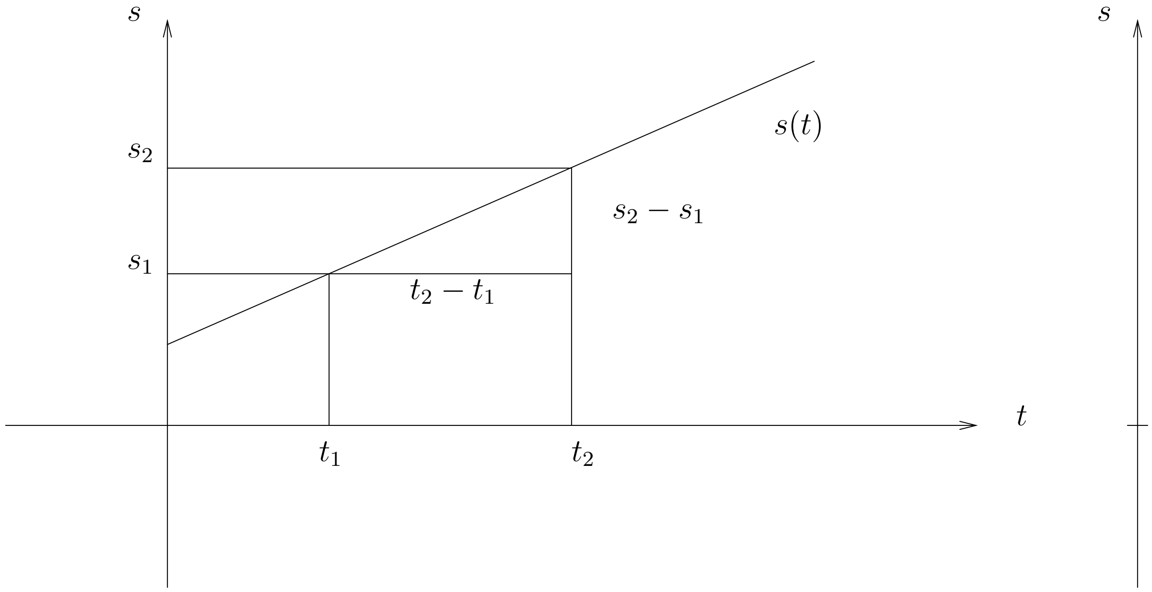

FIGURE 3. $s(t) = at + b$.

- Find the average velocities over the following time intervals: $[3, 4]$, $[3.5, 4]$, $[4, 5]$, $[4.5, 5]$.
- Find the instantaneous velocity when $t = 4$.
- Draw the graph of s as a function of t.

EXERCISE 2.8.

(1) Let $s(t) = 2t^2 - 1$.
   (a) Compute the average velocity between time 4 and $4 + h$.
   (b) Find the instantaneous velocity $v(4)$ at time $t = 4$.
   (c) Compute the average velocity between time 2 and $2 + h$.
   (d) Find the instantaneous velocity $v(2)$ at time $t = 2$.
   (e) Compute the average velocity between time t and $t + h$.
   (f) Find the instantaneous velocity $v(t)$ at time t.

(2) Let $s(t) = t^2 - t$.
   (a) Compute the average velocity between time 3 and $3 + h$.
   (b) Find the instantaneous velocity $v(3)$ at time $t = 3$.
   (c) Compute the average velocity between time 2 and $2 + h$.
   (d) Find the instantaneous velocity $v(2)$ at time $t = 2$.
   (e) Compute the average velocity between time t and $t + h$.
   (f) Find the instantaneous velocity $v(t)$ at time t.

The acceleration is the rate of change of the velocity. To find the instantaneous rate of change of the velocity we imitate how we obtained the instantaneous velocity.

Definition 2.9. *Let $v = v(t)$ be a function giving the velocity of a moving object as a function of time. The* ***average acceleration*** *in the time interval $[t, t+h]$ is*

$$\frac{v(t+h) - v(t)}{h}$$

If the time interval h is very small, we can take $\frac{v(t+h)-v(t)}{h}$ to be a close approximation to the instantaneous acceleration at time t of the moving object. The smaller the elapsed time h, the better the approximation. We make h smaller and smaller and end up with a value $a(t)$ that is called the ***limit value*** *of $\frac{v(t+h)-v(t)}{h}$* ***as h approaches*** *0. We write*

$$a(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}.$$

This value $a(t)$ is the ***instantaneous acceleration*** *of the moving object in question.*

Exercise 2.10. The velocity of a car v [feet/sec] is given by the following values with respect to the time t [seconds]. We have the pairs

$$(t, v) = \{(0,0), (1,10), (2,32), (3,70), (4,119), (5,178)\}.$$

First find the average acceleration for the time period beginning when $t = 2$ and lasting (i) 3 sec, (ii) 2 sec and (iii) 1 sec. Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 2$.

Example 2.11. Suppose that your math book is dropped from the roof of a building 100 meters above the ground (by accident of course!). Find the instantaneous velocity of the book after 1 second.

We previously found that the instantaneous velocity of the unfortunate book is

$$v(t) = 9.8t$$

The average acceleration is given by:

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time elapsed}}.$$

The average velocity over the brief time interval of a tenth of a second from $t = 1$ to $t = 1.1$ is

$$\frac{9.8 \cdot 1.1 - 9.8 \cdot 1.0}{0.1} = 9.8 \text{ [m/sec}^2\text{]}.$$

More generally, fix a time t and let h be the duration of a small time interval. The average acceleration between t and $t + h$ is

$$\begin{aligned} \frac{v(t+h) - v(t)}{h} &= \frac{9.8(t+h) - 9.8t}{h} \\ &= 9.8\frac{h}{h} = 9.8. \end{aligned}$$

In this simplified form we can see that the value is the same for any h and we found

$$a(t) = \lim_{h\to 0} \frac{v(t+h) - v(t)}{h} = \lim_{h\to 0} 9.8 = 9.8 \text{ [m/s}^2\text{]}.$$

EXERCISE 2.12. The velocity (in meters per second) of a moving object is given by $v(t) = 2t^2 - 9t + 8$.

(1) Find the average accelerations over the following time intervals: $[3,4]$, $[3.5,4]$, $[4,5]$, $[4.5,5]$.
(2) Find the instantaneous acceleration when $t = 4$.
(3) Draw the graph of v as a function of t.

Figure 4 shows an example of non-uniform motion of an object with respect to time. Here the position s is plotted against the time t.

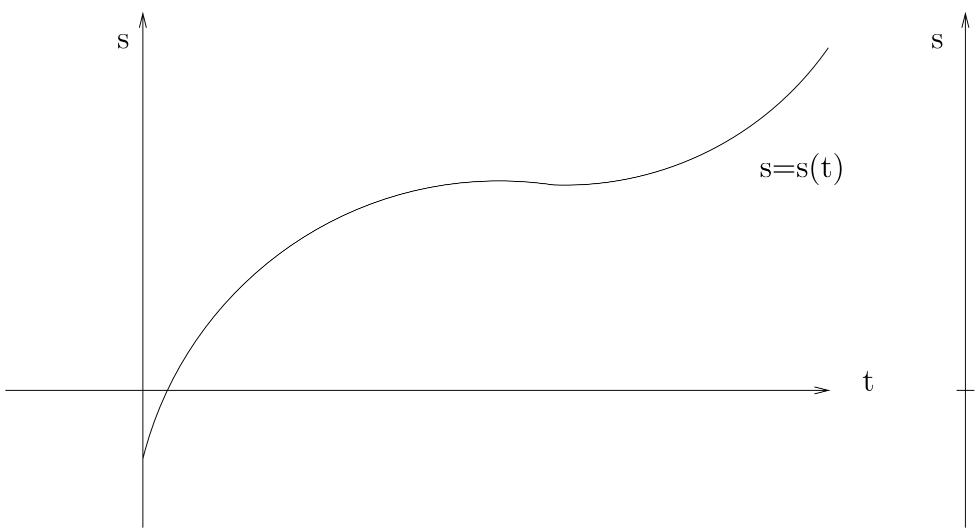

FIGURE 4. *Example of motion of an object with respect to time.*

The rate of change of the position of the object at time t_0 is given by the slope of the tangent to the curve at the point $(t_0, s(t_0))$. An approximation of this slope is given by the line connecting $(t_0, s(t_0))$ to another position of the object at a time $t_0 + h$ close to t_0, and is equal to $(s(t_0+h) - s(t_0))/h$. The smaller the time difference h, the closer we are to the instantaneous velocity at time t_0. Letting h get smaller and smaller is the passage to the limit. If $v(t_0)$ is the velocity of the object at time t_0, we have:

$$v(t_0) = \lim_{h\to 0} \frac{s(t_0+h) - s(t_0)}{h} \tag{2.13}$$

3. Differentiation

Finding the instantaneous velocity and the instantaneous acceleration really meant going through the same process twice. There are numerous other examples from real life that ask for the instantaneous rate of chage of one variable with respect to another and dealing with these one by one just means doing the same thing over and over again. Typically, mathematics attempts to unify the concrete cases and deals with the abstraction that comprises all the special examples. We have a two varaibles, we now call them x and y where y is a function of x: $y = f(x)$. Figure 3 shows an example.

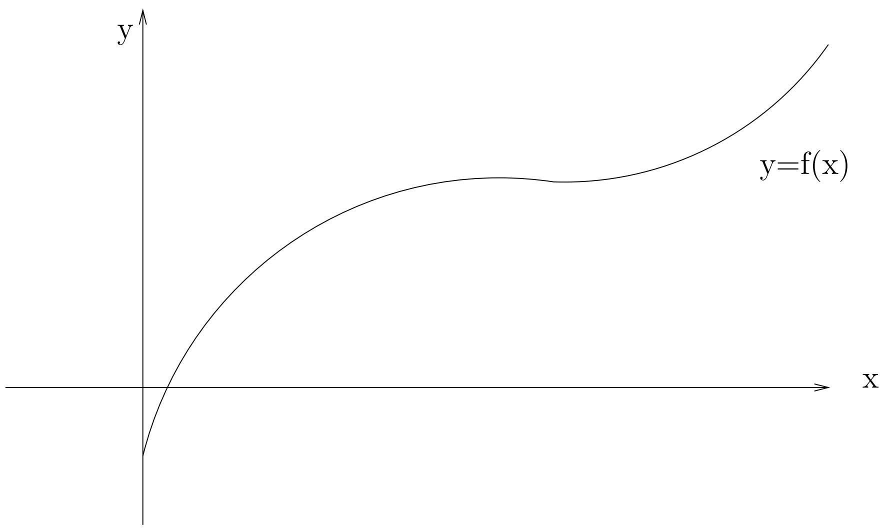

FIGURE 5. The general situation: $y = f(x)$.

The average rate of change of the value y between x_0 and $x_0 + h$ is given by

$$\frac{f(x_0 + h) - f(x_0)}{h}.$$

For small h, this is an approximation to the instantaneous rate of change of y at the point $(x_0, f(x_0))$ of the graph of f. The smaller the x-difference h, the more accurate the approximation will be. Ultimately, the idea is that h becomes smaller and smaller and this is the process of passage to the limit. If we denote by $y'(t_0)$ the instantaneous rate of change of y with respect to x, then we have

$$y'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}. \tag{3.1}$$

DEFINITION 3.2. *The* ***derivative*** $y' = f'(x)$ *of the function* $y = f(x)$ *at value* x *is defined as*

$$y'(x) = f'(x) = \lim_{h \to 0} \tfrac{f(x+h) - f(x)}{h}.$$

The process of finding $f'(x)$ *is called* ***differentiation****.*

$$f'(t) = \frac{dy}{dx}.$$

The new notation is due to Leibniz who thought of the derivative as the rate of change in y with respect to x as the quotient of an "infinitesimal" change dy in y corresponding to an infinitesimal change dx in x.

REMARK 3.3. With the general concept of differenciation and derivative at our disposal we can say that instantaneous velocity is the derivative $v(t) = s'(t)$ of the position function $s(t)$ and acceleration is the derivative $v'(t) = s''(t)$ of the velocity function. $s''(t)$ is called the ***second derivative*** of $s(t)$.
Looking at a function $y = f(x)$ geometrically, the derivative $y' = f'(x)$ is the slope of the tangent line at the point $(x, f(x))$ of the graph of f.

Luckily there are general formulas for finding derivatives.

THEOREM 3.4. *The derivative of the function $f(x) = x^2$ is given by $f'(x) = 2x$.More generally, we have that the derivative of $f(x) = x^n$ is given by $f'(x) = nx^{n-1}$.*

THEOREM 3.5. *The derivative of the general polynomial function*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is given by

$$f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2 a_2 x + a_1$$

In particular,

(1) *if $f(x) = a_0$, then $f'(x) = 0$,*
(2) *if $f(x) = a_1 x$, then $f'(x) = a_1$.*

EXAMPLE 3.6.

(1) *Let $f(x) = 3x^3 - 5x^2 + 3$. Then $f'(x) = 3 \cdot 3x^2 - 5 \cdot 2x^1 = 9x^2 - 10x$.*
(2) *Let $f(x) = x^4 - 3x^3 + x^2 + 3x - 10$. Then $f'(x) = 4x^3 - 3 \cdot 3x^2 + 2x^1 + 3 = 4x^3 - 9x^2 + 2x + 3$.*
(3) *Let $f(x) = (x-3)^3 + 3(x+3)^2 - 10x + 3$. We first find the standard form of $f(x)$:*

$$\begin{aligned} f(x) &= x^3 - 9x^2 + 27x - 27 + 3x^2 + 18x + 27 - 10x + 3 \\ &= x^3 - 6x^2 + 35x + 3 \end{aligned}$$

Hence $f'(x) = 3x^2 - 18x + 35$.

4. Newton's Law of Motion and Free Fall

Newton's Law of Motion says that

$$F(t) = m \cdot a(t)$$

where t is time, F is the force acting on an object, m is the mass of the object, and a is its acceleration. This is correct if the object moves on a straight line. In general, the quantities F and a have to be "vectors" incorporating a direction.

Free Fall

For this problem, we let the x-axis be the horizontal surface of the earth and let the y-axis be perpendicular to the x-axis and y is the elevation above ground level.
An object is ejected in the y direction with an initial velocity v_0 and at an initial elevation y_0. Let $y = y(t)$ describe the position of the object at time t and this is the function that we would like to know. We assume that $t = 0$ is the moment when the process starts. Hence

$$y(0) = y_0.$$

The instantaneous velocity of the object is $v(t) = y'(t)$, hence

$$y'(0) = v(0) = v_0.$$

The force due to gravity acting on our object is $F = mg$ where $g = 9.81$ [m/sec^2]. This is a fact of physics. Newton's Law of Motion says that

$$mg = -ma = -mv'(t) \quad \text{or equivalently} \quad v'(t) = -g.$$

Surprise: The mass m of the object is irrelevant: feathers fall as fast as metal balls. The reason this is not so in reality is the air resistance that we disregard.
We have $v'(t) = -g$ and therefor

$$v(t) = -gt + c \quad \text{where } c \text{ is an arbitrary constant.}$$

It follows from $v(0) = v_0$ that

$$y'(t) = v(t) = -gt + v_0.$$

Hence

$$y(t) = -\tfrac{1}{2}gt^2 + v_0t + d, \quad \text{where } d \text{ is an arbitrary constant.}$$

It follows from $y(0) = y_0$ that

$$y(t) = -\tfrac{1}{2}gt + v_0t + y_0,$$

and this is the formula that we previously accepted as a fact in connection with the flight of the cannon ball.

5. Outdoing the illustrious Archimedes

Archimedes, considered one of the three greatest mathematicians ever was able to prove the following theorem.

THEOREM 5.1. *The straight line ℓ cuts the parabola $\mathcal{P}$ at the points A and B. The point C is on the parabola half-way bewteen A and B. Then the area of the region btween the straight line and the parabola is $4/3$ times the area of the triangle ΔABC.*

The following theorem enables us to compute areas of regions between a curve and the x-axis.

THEOREM 5.2. ***The Fundamental Theorem of Calculus*** *Let $f(x)$ be a (sufficiently nice) function defined for $a \le x \le b$ and assume that $f(x) \ge 0$ for $a \le x \le b$. Then there is a well-defined "region under the curve between a and b". Let $A = A(x)$ be the area of the region under the curve between a and x. Then*

$$A'(x) = f(x).$$

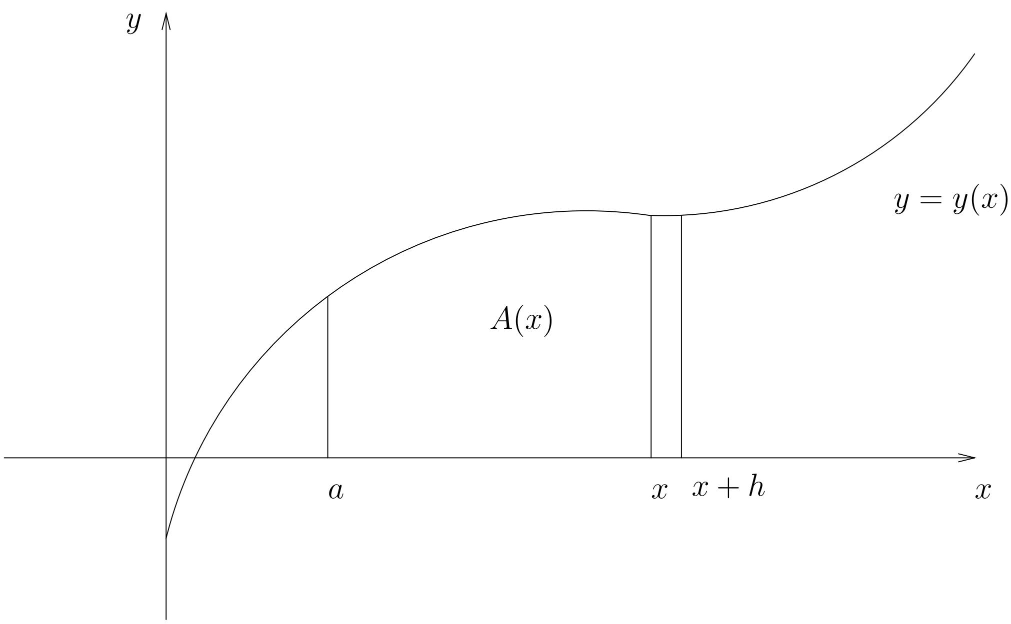

FIGURE 6. The Fundamental Theorem of Calculus

PROOF. See Figure 6. For a very small h,

$$\frac{A(x+h) - A(x)}{h} \approx \frac{f(x) \cdot h}{h} = f(x).$$

Hence

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

□

Finding area under a curve requires to compute a function if its derivative is known.

EXAMPLE 5.3. *We know: if $p(x) = x^n$, then $p'(x) = nx^{n-1}$. Suppose that $f'(x) = x^n$. We guess that $f(x) = \frac{1}{n+1}x^{n+1} + c$ where c is an arbitrary constant. Now differentiate the result to see that the derivative of f is equal to the given derivative.*

(1) *If $f'(x) = x$, then $f(x) = \frac{1}{2}x^2 + c$.*
(2) *If $f'(x) = x^2$, then $f(x) = \frac{1}{3}x^3 + c$.*
(3) *If $f'(x) = x^3$, then $f(x) = \frac{1}{4}x^4 + c$.*

(4) *If* $f'(x) = 2x^3$, *then* $f(x) = 2\frac{1}{4}x^4 + c = \frac{1}{2}x^4 + c$.
(5) *If* $f'(x) = 2x^3 - x^2 + 3$, *then* $f(x) = 2\frac{1}{4}x^4 - \frac{1}{3}x^3 + 3x + c$.

EXAMPLE 5.4. *Suppose that* f *is a function of* x *such that* $f'(x) = -3x^2 + \frac{1}{2}x + 4$. *We guess that* $f(x) = (-3)\frac{1}{3}x^3 + \frac{1}{2}\frac{1}{2}x^2 + 4x + c = -x^3 + \frac{1}{4}x^2 + 4x + c$ *where* c *is an arbitrary constant. Now differentiate* $f(x)$ *to see that its derivative is equal to the given derivative.*

EXAMPLE 5.5. *Suppose that* f *is a function of* x *such that* $f'(x) = -3(x^2 + \frac{1}{2})^2(x+4)$. *Find* $f(x)$.
Here the function $f'(x)$ *is not in standard form and the previous procedure is not immediately applicable. We must first simplify algebraically.*

$$\begin{aligned} f'(x) &= -3(x^4 + x^2 + \frac{1}{4})(x+4) \\ &= (-3x^4 - 3x^2 - \frac{3}{4})(x+4) \\ &= -3x^5 - 3x^3 - \frac{3}{4}x - 12x^4 - 12x^2 - 3 \\ &= -3x^5 - 12x^4 - 3x^3 - 12x^2 - \frac{3}{4}x - 3. \end{aligned}$$

We now guess that $f(x) = -3\frac{1}{6}x^6 - 12\frac{1}{5}x^5 - 3\frac{1}{4}x^4 - 12\frac{1}{3}x^3 - \frac{3}{4}\frac{1}{2}x^2 - 3x + c = -\frac{1}{2}x^6 - \frac{12}{5}x^5 - \frac{3}{4}x^4 - 4x^3 - \frac{3}{8}x^2 - 3x + c$ *where* c *is an arbitrary constant. Check by differentiating.*

EXAMPLE 5.6. *Find the area between* $x = 1$ *and* $x = 3$ *under the curve* $f(x) = x^2 + 1$. *Let* $A(x)$ *be the area function. Then* $A(1) = 0$ *and* $A'(x) = f(x) = x^2 + 1$. *Hence* $A(x) = \frac{1}{3}x^3 + x + c$ *where* c *is some unknown constant. Since* $A(1) = 0$ *we find that* $\frac{1}{3}1^3 + 1 + c = 0$, *hence* $c = -\frac{1}{3} - 1 = -\frac{4}{3}$. *We now know the exact area function* $A(x) = \frac{1}{3}x^3 + x - \frac{4}{3}$ *and the desired area is* $A(3) = \frac{1}{3}3^3 + 3 - \frac{4}{3} = 9 + 3 - \frac{4}{3} = \frac{32}{3}$.

EXERCISE 5.7. Consider a parabola. In a suitably chosen Cartesian coordinate system the equation of the parabola is $y = x^2$.

(1) Find the most general function $A(x)$ such that $A'(x) = x^2$.
(2) Find the area of the region between the parabola, the x-axis and the lines $x = -1$ and $x = +1$.
(3) Find the area of the region between the parabola, the x-axis and the lines $x = -2$ and $x = +1$.
(4) Find the area of the region between the parabola, the x-axis and the lines $x = 0$ and $x = +5$.
(5) Find the area of the region between the parabola, the x-axis and the lines $x = a$ and $x = b$ where $a < b$.

EXERCISE 5.8. Find $f(x)$ when

(1) $f'(x) = 4x^3 + 6x - 3$

(2) $f'(x) = (x+2)(x-3)$
(3) $f'(x) = x^4 - x^3$
(4) $f'(x) = x^2(x^2 - 3x + 1)$

Answers for Exercise 5.8: (1) $f(x) = x^4 + 3x^2 - 3x + c$; (2) $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + c$; (3) $f(x) = \frac{1}{5}x^5 - \frac{1}{4}x^4 + c$; (4) $f(x) = \frac{1}{5}x^5 - x^4 + \frac{1}{3}x^3 + c$.

6. Additional Exercises

EXERCISE 6.1.

(1) If $f(x) = -x^2 - x^4$, then $f'(x) =$.
(2) If $f(x) = (x+1)^2$, then $f'(x) =$
(3) If $f(x) = 2x(x+3)$, then $f'(x) =$

EXERCISE 6.2. Let $s(t) = 10 - t^2$ describe the motion of an object moving on a straight line.

(1) Compute the average velocity of the moving object between time t and $t+h$.
(2) Compute instantaneous velocity $v(t) = s'(t)$ of the moving object.
(3) Compute the instantaneous acceleration $a(t) = v'(t)$ of the moving object.

EXERCISE 6.3. If $s(t) = (1+t^2)(1-t^2)$ describes the motion of a moving particle on a straight line, then what is the instantaneous velocity of the particle?

EXERCISE 6.4.

(1) If $f'(x) = -2x^2 + 1$, then $f(x) =?$
(2) If $f'(x) = (x-1)x$ and $f(0) = 3$, then $f(x) =?$
(3) If $f'(x) = 3(x+1)^2$, then $f(x) =?$

EXERCISE 6.5. If $v(t) = 10 - t$ is the velocity of an object moving along a straight line and if the motion of the particle is described by the function $s = s(t)$ where $s(0) = 2$, then $s(t) =?$

7. Answers to exercises

Answers to Exercise 1.10. $f(0) = -4$, $f(2) = 10$, $f(\sqrt{2}) = 3\sqrt{2}$, $f(1+\sqrt{2}) = 5+7\sqrt{2}$, $f(-x) = 2x^2 - 3x - 4$, $f(x+1) = 2x^2 + 7x + 1$, $f(2x) = 8x^2 + 6x - 4$.
Answers to Exercise 1.11. $f(0) = -4$, $f(2) = -2$, $f(\sqrt{2}) = -4 - \sqrt{2}$, $f(-x) = -x^3 + 3x - 4$, $f(x-1) = x^3 - 3x^2 + 3x - 2$, $f(a) = a^3 - 3a - 4$, $f(2b) = 8b^3 - 6b - 4$.
Answers to Exercise 1.12. $f(0) = 5$, $f(2) = 1$, $f(\sqrt{2}) = -1$, $f(-x) = -2x^3 - 3x^2 + 4x + 5$, $f(x+2) = 2x^3 + 9x^2 + 8x + 9$, $f(a+b) = 2a^3 + 6a^2b + 6ab^2 + 2b^3 - 3a^2 - 6ab - 3b^2 - 4a - 4b + 5$, $f(2x) = 16x^3 - 12x^2 - 8x + 5$.
Answers to Exercise 1.13. $f(0) = 1$, $f(-1) = 0$, $f(\sqrt{2}) = 3\sqrt{2} + 3$, $f(1+a) = a^3 + 4a^2 + 6a + 4$, $f(-x) = -x^3 + x^2 - x + 1$, $f(x+1) = x^3 + 4x^2 + 6x + 4$, $f(-2x) = -8x^3 + 4x^2 - 2x + 1$.

Answers to Exercise 1.14. $f(2) = 10$, $f(2+h) = 2h^2+11h+10$, $\frac{1}{h}(f(2+h) - f(2)) = 2h+11$, $f(1) = 1$, $f(1+h) = 2h^2+7h+1$, $\frac{1}{h}(f(1+h) - f(1)) = 2h+7$, $\frac{1}{h}(f(x+h) - f(x)) = 2h + 4x + 3$.
Answer to Exercise 2.7 (1) (2) 0; (3) $s(t) = (t-4)^2 + 2$, its a parabola with lowest point $(4, 2)$.
Answers to Exercise 2.8 (1)(a) $16+2h$, (1)(b) 16, (1)(c) $8+2h$, (1)(d) 8, (1)(e) $4t+2h$, (1)(f) $4t$.
(2)(a) $5 + h$, (2)(b) 5, (2)(c) $7 + h$, (2)(d) 7, (2)(e) $2t + h - 1$, (2)(f) $2t - 1$.
Answers to Exercise 2.12 (1) Compute $\frac{1}{1}(2 \cdot 4^2 - 9 \cdot 4 + 8 - (2 \cdot 3^2 - 9 \cdot 3 + 8))$, etc., (2) $a(4) = \lim_{h \to 0} 7 + 2h = 7$, (3) $v(t) = 2\left(t - \frac{9}{4}\right)^2 - \frac{17}{8}$. The graph ($v$ versus t) is a parabola with lowest point $(9/4, -17/8)$.
Answers to Exercises 5.7 (1) $A(x) = \frac{1}{3}x^3 + c$, (2) $A(1) - A(-1) = 2/3$, (3) $A(1) - A(-2) = 3$, (4) $A(5) - A(0) = 125/3$, (5) $A(b) - A(a) = \frac{1}{3}(b^3 - a^3)$
Answers to Exercises 6.1 (1) $f'(x) = -2x - 4x^3$, (2) $f'(x) = 2x + 2$, (3) $f'(x) = 4x + 6$
Answers to Exercises 6.2 (1) $-2t - h$, (2) $v(t) = -2t$, (3) $a(t) = -2$.
Answer to Exercises 6.3 $v(t) = s'(t) = -4t^3$.
Answers to Exercises 6.4 (1) $f(x) = -\frac{2}{3}x^3 + x + c$, (2) $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3$, (3) $f(x) = x^3 + 3x^2 + 3x + 1 + c = x^3 + 3x^2 + 3x + c'$
Answer to Exercises 6.5 $s(t) = 2 + 10t - \frac{1}{2}t^2$.