CONTINUOUS ANALYTIC CAPACITY, RECTIFIABILITY AND HOLOMORPHIC MOTIONS

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Abstract. We give several geometric and analytic characterizations of purely unrectifiable quasicircles in terms of various notions such as Dirichlet algebra, harmonic measure, analytic capacity and continuous analytic capacity. As an application, we show that extremal functions for continuous analytic capacity may not exist. We also construct a compact set whose continuous analytic capacity does not vary continuously under a certain holomorphic motion, thereby answering a question of Paul Gauthier. The analogous result for analytic capacity was recently established by Ransford, the author and Ai [39]. Our approach also provides a new proof of this result.

1. Introduction and statement of main results

1.1. Continuous analytic capacity and purely unrectifiable quasicircles.

Let E be a compact subset of the complex plane $\mathbb{C}$ and let $D := \widehat{\mathbb{C}} \setminus E$ denote the complement of E in the Riemann sphere $\widehat{\mathbb{C}}$. The *analytic capacity* of E is defined by

$$\gamma(E) := \sup\{|f'(\infty)| : f \in H^\infty(D), |f| \leq 1 \text{ on } D\}.$$

Here $H^\infty(D)$ denotes the space of all bounded analytic functions on D and $f'(\infty)$ is defined by

$$f'(\infty) := \lim_{z \to \infty} z(f(z) - f(\infty)).$$

Analytic capacity was introduced by Ahlfors in [1] for the study of a problem of Painlevé from 1888 asking for a geometric characterization of the compact sets $E \subset \mathbb{C}$ that are *removable* for bounded analytic functions, in the sense that all functions in $H^\infty(D)$ are constant. It is easily seen that such removable sets coincide precisely with the sets of zero analytic capacity. Despite recent advances, analytic capacity is notoriously hard to estimate in general and its properties remain quite mysterious, although there exist efficient numerical methods in some cases, see e.g. [52]. For more information on analytic capacity, we refer the reader to [18], [21], [43] and [50].

The notion of removability is closely related to Hausdorff measure. Indeed, a result generally attributed to Painlevé states that E is removable whenever its one-dimensional Hausdorff measure $\mathcal{H}^1(E)$ is zero, see e.g. [51, Theorem 2.7]. In particular, if the Hausdorff dimension of E satisfies $\dim_H(E) < 1$, then $\gamma(E) = 0$. On the other hand, a simple argument using Cauchy transforms and Frostman's

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Lemma can be used to deduce that $\gamma(E) > 0$ whenever $\dim_H(E) > 1$, see e.g. [51, Theorem 2.10]. Painlevé's problem is therefore reduced to the case of dimension precisely equal to one. Over the years, however, it became apparent that this remaining case is much more difficult, and it took more than a hundred years until a reasonable solution to Painlevé's problem was obtained.

One of the major advances toward the solution of Painlevé's problem was the proof of the so-called *Vitushkin's conjecture*:

Theorem 1.1 (Vitushkin's conjecture)**.** *Let $E \subset \mathbb{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Then $\gamma(E) = 0$ if and only if E is purely unrectifiable.*

Recall that a set E is called *purely unrectifiable* if it intersects every rectifiable curve in a set of zero length, i.e. $\mathcal{H}^1(E \cap \Gamma) = 0$ for each rectifiable curve $\Gamma \subset \mathbb{C}$.

The forward implication in Theorem 1.1 was previously known as Denjoy's conjecture and follows from the results of Calderón [14] on the L^2-boundedness of the Cauchy transform operator. The other implication was proved by David [16] in 1998. We also mention that Theorem 1.1 remains true for sets of σ-finite length, see the Postscript in [18]. However, in [25], Joyce and Morters constructed a compact set $E \subset \mathbb{C}$ with $\dim_{\mathcal{H}}(E) = 1$ which is purely unrectifiable but has positive analytic capacity. This shows that Theorem 1.1 is false in general. For arbitrary compact sets, a metric characterization of removability was obtained in a breakthrough paper by Tolsa [45] in terms of the notion of curvature of a measure, thereby providing a complete solution to Painlevé's problem.

In this article, we study the analytic capacity of purely unrectifiable sets of infinite length. This turns out to be closely related to another notion of capacity. For $E \subset \mathbb{C}$ compact, the *continuous analytic capacity* of E is defined by

$$\alpha(E) := \sup\{|g'(\infty)| : g \in A(D), |g| \leq 1 \text{ on } \mathbb{C}\}.$$

Here as before $D = \widehat{\mathbb{C}} \setminus E$, and $A(D)$ denotes the subspace of $H^\infty(D)$ consisting of continuous functions on $\widehat{\mathbb{C}}$ that are analytic on D.

The notion of continuous analytic capacity was introduced by Erokhin and Vitushkin in the 1950's to study problems of uniform rational approximation of analytic functions on compact subsets of the plane. See [48].

It follows directly from the definitions that $\alpha(E) \leq \gamma(E)$ for all E. In particular, we have that $\alpha(E) = 0$ whenever $\mathcal{H}^1(E) = 0$, in view of previous remarks. In fact, Cauchy's theorem can be used to show that $\alpha(E) = 0$ if $\mathcal{H}^1(E) < \infty$, see e.g. [51, Theorem 3.4]. More generally, this remains true if E has σ-finite $\mathcal{H}^1$ measure. On the other hand, the aforementioned argument using Cauchy transforms and Frostman's Lemma applies to continuous analytic capacity as well, so that $\alpha(E) > 0$ whenever $\dim_H(E) > 1$. The case of dimension one is much less understood for continuous analytic capacity than for analytic capacity. As far as we know, there is no known geometric characterization of sets of zero continuous analytic capacity.

In general, continuous analytic capacity and analytic capacity are not comparable. For instance, if E is a line segment or in fact any rectifiable curve, then $\gamma(E) > 0$ but $\alpha(E) = 0$. The following theorem of Tolsa, however, shows that this cannot occur if E is purely unrectifiable.

Theorem 1.2 (Tolsa [46])**.** *Let $E \subset \mathbb{C}$ be compact. Suppose that the following holds:*

$(\star)$ $\qquad \gamma(F) = 0$ *for every compact set $F \subset E$ with $\mathcal{H}^1(F) < \infty$.*

Then $\alpha(E) \approx \gamma(E)$.

Here $\alpha(E) \approx \gamma(E)$ means that there is an absolute constant C such that

$$\alpha(E) \leq \gamma(E) \leq C\alpha(E) \qquad (E \subset \mathbb{C} \text{ compact satisfying } (\star)).$$

We mention that the assumption $(\star)$ in Theorem 1.2 is in fact equivalent to E being purely unrectifiable. Indeed, if $(\star)$ holds and if $\Gamma \subset \mathbb{C}$ is any rectifiable curve, then $\gamma(E \cap \Gamma) = 0$ since $\mathcal{H}^1(E \cap \Gamma) < \infty$, so that $E \cap \Gamma$ is purely unrectifiable by Theorem 1.1 and we must have $\mathcal{H}^1(E \cap \Gamma) = 0$. Conversely, if E is purely unrectifiable and if F is any compact subset of E with $\mathcal{H}^1(F) < \infty$, then F is also purely unrectifiable and thus $\gamma(F) = 0$, again by Theorem 1.1.

In this paper, we will be mainly interested in a special family of Jordan curves called quasicircles. It turns out that for such sets, the comparability between α and γ in fact characterizes pure unrectifiability.

Theorem 1.3. *Let $J \subset \mathbb{C}$ be a quasicircle. The following are equivalent:*

- *J is purely unrectifiable,*
- *$\alpha(J') \approx \gamma(J')$ for all subarcs $J' \subset J$,*
- *$\alpha(J') = \gamma(J')$ for all subarcs $J' \subset J$.*

Theorem 1.3 is a special case of a more general theorem which we now describe. The idea is that the equality $\gamma(E) = \alpha(E)$ should hold whenever $A(D)$ is dense in $H^\infty(D)$ in some sense. This is made precise using the notion of Dirichlet algebra. For $E \subset \mathbb{C}$ compact and $D = \widehat{\mathbb{C}} \setminus E$, we say that $A(D)$ is a *Dirichlet algebra* if for every continuous function $g : E \to \mathbb{R}$ and every $\epsilon > 0$, there exists a function $f \in A(D)$ such that

$$\|g - \mathrm{Re}(f)\|_E < \epsilon,$$

where $\|\cdot\|_E$ denotes the uniform norm on E.

In 1963, Browder and Wermer characterized Dirichlet algebras on Jordan curves in terms of harmonic measure. For a Jordan curve $J \subset \mathbb{C}$, denote by Ω and Ω^* the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus J$ respectively. Let ω and ω^* be harmonic measure on Ω and Ω^* respectively, with respect to some arbitrary points, say $z_0 \in \Omega$ and $\infty \in \Omega^*$ (everything will be independent of the choice of z_0). For simplicity, let $A_J := A(\widehat{\mathbb{C}} \setminus J)$. Then A_J is the space of all continuous functions on $\widehat{\mathbb{C}}$ that are analytic on Ω and on Ω^*.

Theorem 1.4 (Browder–Wermer [13])**.** *The space A_J is a Dirichlet algebra if and only if the measures ω and ω^* are mutually singular, denoted by $\omega \perp \omega^*$.*

Much later, in 1989, Bishop, Carleson, Garnett and Jones obtained a geometric characterization of the Jordan curves for which $\omega \perp \omega^*$.

Theorem 1.5 (Bishop–Carleson–Garnett–Jones [11])**.** *For a Jordan curve $J \subset \mathbb{C}$ and ω, ω^* as above, we have that $\omega \perp \omega^*$ if and only if $\mathcal{H}^1(T_J) = 0$, where T_J denotes the set of all tangent points of J.*

The following theorem supplements Theorem 1.2, Theorem 1.4 and Theorem 1.5.

Theorem 1.6. *Let $J \subset \mathbb{C}$ be a Jordan curve. The following are equivalent:*

(1) *A_J is a Dirichlet algebra,*
(2) *$\omega \perp \omega^*$,*
(3) *$\mathcal{H}^1(T_J) = 0$, where T_J denotes the set of all tangent points of J,*

(4) *$\alpha(J') = \gamma(J')$ for all subarcs $J' \subset J$,*
(5) *$\alpha(J') \approx \gamma(J')$ for all subarcs $J' \subset J$.*

If in addition J is a quasicircle, then each of the above is equivalent to each of the following conditions:
(6) *$\omega \perp \mathcal{H}^1$ and $\omega^* \perp \mathcal{H}^1$ on J,*
(7) *J is purely unrectifiable.*

As pointed out to us by Chris Bishop, the connection between Dirichlet algebras, harmonic measures, purely unrectifiable quasicircles and (continuous) analytic capacity was already known to experts. In fact, Theorem 1.6 can be proved using classical results on harmonic measure as well as various results from Bishop's thesis [9] and from Gamelin and Garnett's paper [20]. However, as far as we know, most of the implications in Theorem 1.6 do not appear explicitly in the literature. For this reason and for the reader's convenience, we give a complete proof of Theorem 1.6 in this paper.

We now present a new application of Theorem 1.6 to the study of extremal functions for continuous analytic capacity. First, it is well-known that for any compact set $E \subset \mathbb{C}$, there exists an extremal function attaining the supremum in the definition of analytic capacity. More precisely, there exists a function f analytic on $D = \widehat{\mathbb{C}} \setminus E$ with $|f| \leq 1$ on D and $f'(\infty) = \gamma(E)$. In fact, if $\gamma(E) > 0$ and if D is connected, then f is unique and is called the *Ahlfors function* for E. By contrast, we show that extremal functions for continuous analytic capacity may not exist.

Corollary 1.7. *Let $J \subset \mathbb{C}$ be a Jordan arc such that $\mathcal{H}^1(T_J) = 0$, where T_J denotes the set of all tangent points of J. Then there is no function $g \in A_J$ with $|g| \leq 1$ on $\mathbb{C}$ and $|g'(\infty)| = \alpha(J)$.*

1.2. **Continuous analytic capacity and holomorphic motions.** Our work is mostly motivated by the following question raised by Paul Gauthier:

Question 1.8. *Does continuous analytic capacity vary continuously under a holomorphic motion?*

A *holomorphic motion* of the Riemann Sphere $\widehat{\mathbb{C}}$ is a map $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk, such that

(i) for each fixed $z \in \widehat{\mathbb{C}}$, the map $\lambda \mapsto h(\lambda, z)$ is holomorphic on $\mathbb{D}$,
(ii) for each fixed $\lambda \in \mathbb{D}$, the map $z \mapsto h(\lambda, z)$ is injective on $\widehat{\mathbb{C}}$,
(iii) $h(0, z) = z$ for all $z \in \widehat{\mathbb{C}}$.

For a holomorphic motion $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a subset $E \subset \widehat{\mathbb{C}}$, we write

$$h_\lambda(z) := h(\lambda, z) \qquad (\lambda \in \mathbb{D}, z \in \widehat{\mathbb{C}})$$

and

$$E_\lambda := h_\lambda(E),$$

so that $E_0 = E$. In this article, we shall only consider holomorphic motions that fix the point ∞, i.e. $h_\lambda(\infty) = \infty$ for all $\lambda \in \mathbb{D}$.

Holomorphic motions were introduced by Mañé, Sad and Sullivan in the 1980's, motivated by applications to holomorphic dynamics. In [30], they proved that every holomorphic motion $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is jointly continuous in (λ, z), which is part of a more general result known as the λ-lemma. For more information on holomorphic motions, we refer the reader to [4, Chapter 12] and [5].

Now, consider a real-valued function $\mathcal{F}$ defined on the collection of all compact subsets of $\mathbb{C}$. Let $E \subset \mathbb{C}$ be compact, and suppose that $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic motion of $\widehat{\mathbb{C}}$. Then, for each $\lambda \in \mathbb{D}$, the set $E_\lambda \subset \mathbb{C}$ is compact by the λ-lemma, so the function

$$\lambda \mapsto \mathcal{F}(E_\lambda) \qquad (\lambda \in \mathbb{D}) \tag{1}$$

is well-defined. The study of the regularity of the function (1) for different choices of $\mathcal{F}$ has been a unifying theme in the theory of holomorphic motions as well as the subject of an abundance of classical and more recent work. For instance, the variation of Hausdorff dimension under holomorphic motions was studied in the seminal work of Ruelle [40], who proved that the function (1) is real-analytic when $\mathcal{F} = \dim_H$ and the sets E_λ are Julia sets of hyperbolic rational maps depending holomorphically on λ. This result is considered one of the landmarks of thermonydnamic formalism. See also the work of Ransford [37] for related results.

In general, Hausdorff dimension may not vary real analytically under holomorphic motions [6]. On the other hand, it always changes continuously. The same is true for 2-dimensional Hausdorff measure, but not for s-dimensional Hausdorff measure, $0 < s < 2$. See the concluding remarks in [39]. The study of the behavior of Hausdorff dimension and area measure under holomorphic motions played a fundamental role in the influential work of Astala [3] on quasiconformal mappings. We also mention the work of Earle and Mittra in [19], who showed that the conformal modulus of quadrilaterals varies real analytically under holomorphic motions.

More recently, there has been significant interest in the study of the behavior of various capacities under holomorphic motions. In particular, it was observed that the function

$$\lambda \mapsto \gamma(E_\lambda) \qquad (\lambda \in \mathbb{D})$$

may not behave as nicely as one would expect. In [36], Pouliasis, Ransford and the author showed that there exist a compact set E with $\gamma(E) > 0$ and a holomorphic motion $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which the functions $\lambda \mapsto \gamma(E_\lambda)$ and $\lambda \mapsto \log \gamma(E_\lambda)$ are neither subharmonic nor superharmonic on $\mathbb{D}$. See also the work of Zakeri in [53] for related results and applications to holomorphic dynamics.

In fact, it turns out that the function $\lambda \mapsto \gamma(E_\lambda)$ need not be continuous either, as proved shortly after by Ransford, the author and Ai.

Theorem 1.9 (Ransford–Younsi–Ai [39])**.** *There exist a compact set $E \subset \mathbb{C}$ and a holomorphic motion $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which the function*

$$\lambda \mapsto \gamma(E_\lambda) \qquad (\lambda \in \mathbb{D})$$

is discontinuous at 0*.*

In the same paper, the authors proved that logarithmic capacity, on the other hand, does vary continuously under holomorphic motions. See also [35] for the corresponding result for condenser capacity.

The proof of Theorem 1.9 was inspired by holomorphic dynamics. Moreover, the set E and the holomorphic motion $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ were constructed so that $\gamma(E) = \gamma(E_0) > 0$ but $\gamma(E_{\lambda_n}) = 0$ for all n, for some sequence (λ_n) of small positive numbers converging to 0. In fact, a key part of the construction was to ensure that the set E together with the sets E_{λ_n} were all contained in the unit circle, so that their analytic capacity was easier to estimate. Unfortunately, this constraint on the sets E and E_{λ_n} shows that the construction in Theorem 1.9 cannot work for Question 1.8. Indeed, since the sets E and E_{λ_n} are all contained in the unit circle, we have $\alpha(E) = 0 = \alpha(E_{\lambda_n})$ for all n, and there is no longer a discontinuity at 0. This remains a fundamental obstacle and new ideas are required.

In this paper, we instead use Theorem 1.6 to answer Question 1.8 in the negative. Our approach also provides a new proof of Theorem 1.9.

Theorem 1.10. *There exist compact sets E and F as well as a holomorphic motion $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which both functions*

$$\lambda \mapsto \gamma(E_\lambda), \quad \lambda \mapsto \alpha(F_\lambda) \qquad (\lambda \in \mathbb{D})$$

are discontinuous at 0.

We also show how to combine the sets E and F in Theorem 1.10 in order to get a single compact set, and how to obtain a countable set of discontinuities.

Corollary 1.11. *Let (β_j) be any sequence in $\mathbb{D}$ satisfying the Blaschke condition*

$$\sum_{j=1}^{\infty}(1-|\beta_j|) < \infty.$$

Then there exist a compact set K and a holomorphic motion $g : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which both functions

$$\lambda \mapsto \gamma(K_\lambda), \quad \lambda \mapsto \alpha(K_\lambda) \qquad (\lambda \in \mathbb{D})$$

are discontinuous at each β_j.

We now give a sketch of the proof of Theorem 1.10, which is also inspired by holomorphic dynamics, similar to the construction in [39]. More precisely, denote by $\mathcal{M}_0$ the *main cardioid* of the Mandelbrot set, that is, the set of all parameters $c \in \mathbb{C}$ for which the polynomial $p_c(z) := z^2 + c$ has an attracting fixed point. It is well-known that the quadratic Julia sets $\mathcal{J}_c$ with $c \in \mathcal{M}_0$ are quasicircle Jordan curves and that they move holomorphically. More specifically, there is a holomorphic motion $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that for each $\lambda \in \mathbb{D}$, the map h_λ is the unique conformal map from $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto Ω_λ^* normalized by $h_\lambda(z) = z + O(1/z)$ at ∞, where Ω_λ^* is the unbounded complementary component of the quasicircle Julia set $\mathcal{J}_{\lambda/4}$. Note that the factor $1/4$ is introduced since $\mathcal{M}_0$ does not contain the unit disk but does contain the disk centered at 0 of radius $1/4$.

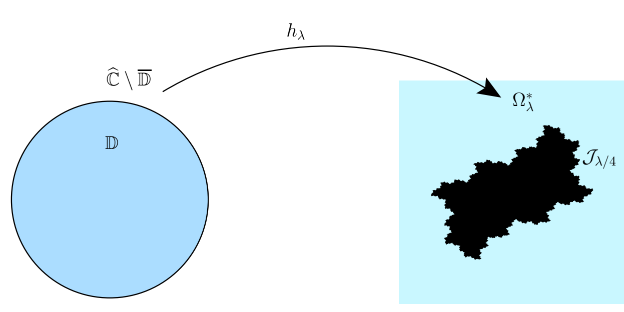

FIGURE 1. The Böttcher motion $h_\lambda(z)$.

Now, a classical theorem of Fatou states that for $\lambda \in \mathbb{D} \setminus \{0\}$, the corresponding Julia set $\mathcal{J}_{\lambda/4}$ has no tangent point. In particular, we can apply Theorem 1.6 to deduce two important consequences.

First, for such λ, the harmonic measure ω_λ^* on Ω_λ^* is mutually singular with $\mathcal{H}^1$ on $\mathcal{J}_{\lambda/4}$. In particular, if (λ_n) is a fixed sequence of non-zero complex numbers in $\mathbb{D}$ converging to 0, then for each n there is a Borel set $B_n \subset \mathcal{J}_{\lambda_n/4}$ with full harmonic measure but zero length. By definition of harmonic measure on Jordan curves, it follows that for each n, the preimage set $A_n := h_{\lambda_n}^{-1}(B_n) \subset \partial\mathbb{D}$ has full normalized Lebesgue measure in the unit circle. Letting E be a compact subset of $\cap_n A_n$ of positive measure, we see that for all n, the set $E_{\lambda_n} = h_{\lambda_n}(E)$ has zero length, hence zero analytic capacity. On the other hand, we have $\gamma(E_0) = \gamma(E) > 0$ by Theorem 1.1. It follows that the function

$$\lambda \mapsto \gamma(E_\lambda) \qquad (\lambda \in \mathbb{D})$$

is discontinuous at 0.

Secondly, Theorem 1.6 implies that for $\lambda \in \mathbb{D} \setminus \{0\}$, we have $\alpha(\mathcal{J}_{\lambda/4}) = \gamma(\mathcal{J}_{\lambda/4})$. But $\gamma(\mathcal{J}_{\lambda/4}) = 1$, which easily follows from the fact that $h_\lambda : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \Omega_\lambda^*$ is conformal with normalization $h_\lambda(z) = z + O(1/z)$ at ∞. Therefore, taking $F = \partial\mathbb{D}$, we get that $\alpha(F_\lambda) = \alpha(\mathcal{J}_{\lambda/4}) = 1$ for all $\lambda \in \mathbb{D} \setminus \{0\}$. On the other hand, note that $\alpha(F_0) = \alpha(F) = 0$, since $F = \partial\mathbb{D}$ has finite length. It follows that the function

$$\lambda \mapsto \alpha(F_\lambda) \qquad (\lambda \in \mathbb{D})$$

is discontinuous at 0.

As a consequence of the proof of Theorem 1.10, we obtain the following result of independent interest.

Corollary 1.12. *Let c be a non-zero parameter in the main cardioid $\mathcal{M}_0$ of the Mandelbrot set, and let $\mathcal{J}_c$ be the corresponding Julia set. Denote by ω_c and ω_c^* the harmonic measures on the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus \mathcal{J}_c$ respectively. Then*

- *$\mathcal{J}_c$ has no tangent point,*
- *$\omega_c \perp \omega_c^*$,*
- *$\omega_c \perp \mathcal{H}^1$ and $\omega_c^* \perp \mathcal{H}^1$ on $\mathcal{J}_c$,*
- *$\mathcal{J}_c$ is purely unrectifiable,*
- *$\alpha(\mathcal{J}_c) = \gamma(\mathcal{J}_c) = 1$,*
- *$\alpha(J') = \gamma(J')$ for all subarcs $J' \subset \mathcal{J}_c$.*

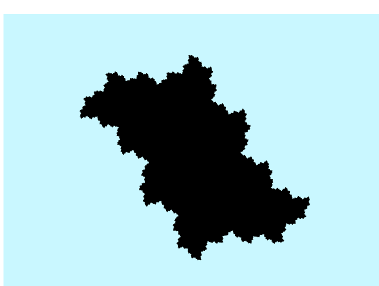

FIGURE 2. A Julia set $\mathcal{J}_c$ for $c \in \mathcal{M}_0$.

Calculating the exact value of continuous analytic capacity is in general quite difficult, even for simple compact sets. As far as we know, Corollary 1.12 provides the first examples of fractal sets for which the precise value of the continuous analytic capacity is known (and positive).

The rest of the article is structured as follows. Section 2 contains various preliminaries and notation that will be needed throughout the paper. In Section 3, we show that the comparability between α and γ for all subarcs of a Jordan curve J is equivalent to A_J being a Dirichlet algebra, completing the proof of the equivalence of (1),(2),(3),(4) and (5) in Theorem 1.6. In Section 4, we prove that (3) implies (6) in Theorem 1.6, namely that $\mathcal{H}^1(T_J)=0$ implies that ω and ω^* are both mutually singular with $\mathcal{H}^1$ on J, provided that J is a quasicircle. In Section 5, we show that the singularity of ω and ω^* with $\mathcal{H}^1$ on a quasicircle implies pure unrectifiability. In Section 6, we discuss the fact that pure unrectifiability implies the comparability of continuous analytic capacity and analytic capacity for all subarcs, completing the proof of Theorem 1.6. In Section 7, we prove Corollary 1.7. Lastly, Section 8 and Section 9 are devoted to holomorphic motions. In Section 8, we prove Theorem 1.10, and Section 9 contains the proof of Corollary 1.11.

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2. Notation and preliminaries

2.1. **Notation.** The following notation will be used throughout the paper. We denote the complex plane by $\mathbb{C}$, the Riemann sphere (extended complex plane) by $\widehat{\mathbb{C}}$ and the open unit disk by $\mathbb{D}$. For $z_0 \in \mathbb{C}$ and $r>0$, we denote by

$$B(z_0,r) := \{z \in \mathbb{C} : |z-z_0| < r\}$$

and

$$\overline{B}(z_0,r) := \{z \in \mathbb{C} : |z-z_0| \leq r\}$$

the open disk and closed disk respectively centered at z_0 of radius r. In particular, we have $\mathbb{D} = B(0,1)$.

Let A be a subset of $\mathbb{C}$. For $s \geq 0$ and $0 < \delta \leq \infty$, we define

$$\mathcal{H}^s_\delta(A) := \inf\left\{ \sum_j \operatorname{diam}(A_j)^s : A \subset \bigcup_j A_j, A_j \subset \mathbb{C}, \operatorname{diam}(A_j) \leq \delta \right\}.$$

Here $\operatorname{diam}(A_j)$ denotes the diameter of the set A_j:

$$\operatorname{diam}(A_j) := \sup_{z,w \in A_j} |z-w|.$$

The *s-dimensional Hausdorff measure* of A is

$$\mathcal{H}^s(A) := \sup_{\delta>0} \mathcal{H}^s_\delta(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A).$$

The *Hausdorff dimension* of A is the unique positive number $\dim_H(A)$ such that

$$\mathcal{H}^s(A) = \begin{cases} \infty & \text{if } s < \dim_H(A) \\ 0 & \text{if } s > \dim_H(A). \end{cases}$$

We will also sometimes use *length* to denote 1-dimensional Hausdorff measure $\mathcal{H}^1$.

2.2. **Jordan curves and harmonic measure.** A curve $J \subset \mathbb{C}$ parametrized by a continuous function $\eta : [0,1] \to \mathbb{C}$ is called a *Jordan arc* if η is injective, in other words, if the curve J is simple (non self-intersecting). We say that J is *rectifiable* if $\mathcal{H}^1(J) < \infty$.

If instead η is injective when restricted to $(0,1]$ and if $\eta(0) = \eta(1)$, we call J a *Jordan curve*.

If J is a Jordan arc and if $t_0 \in (0,1)$, we say that J has a *tangent* at $\eta(t_0)$ if there exists an angle θ such that

$$\arg\left(\eta(t) - \eta(t_0)\right) \to \begin{cases} \theta & \text{if } t \to t_0^+ \\ \theta + \pi & \text{if } t \to t_0^-. \end{cases}$$

This is independent of the choice of the parametric representation. The set of tangent points of J will be denoted by T_J. Note that this definition remains valid if J is a Jordan curve, and in this case we can also consider whether $\eta(0) = \eta(1)$ is a tangent point.

Let $J \subset \mathbb{C}$ be a Jordan curve and denote by Ω and Ω^* the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus J$ respectively, so that $\infty \in \Omega^*$. Fix a point $z_0 \in \Omega$, and consider conformal maps $f : \mathbb{D} \to \Omega$ and $g : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \Omega^*$ with $f(0) = z_0$ and $g(\infty) = \infty$. Then f and g extend to homeomorphisms on the closure of their respective domain by Carathéodory's theorem, and we can consider the pushforward measures $\omega := f_*(\sigma), \omega^* := g_*(\sigma)$, where σ is the normalized Lebesgue measure on $\partial\mathbb{D}$. This defines Borel probability measures on J satisfying

$$\omega(E) = \sigma(f^{-1}(E))$$

and

$$\omega^*(E) = \sigma(g^{-1}(E))$$

for every Borel set $E \subset J$. The measures ω and ω^* are called *harmonic measures* for Ω and Ω^* with respect to z_0 and ∞ respectively. If $J \subset \mathbb{C}$ is a Jordan arc, then $\widehat{\mathbb{C}} \setminus J$ has only one component, but since J has two sides there are two measures ω and ω^* which give the harmonic measure of sets on each of the two sides of J.

Let μ_1 and μ_2 be two positive Borel measures on $\mathbb{C}$. We say that μ_1 is *absolutely continuous* with respect to μ_2, and write $\mu_1 \ll \mu_2$, if $\mu_1(E) = 0$ whenever $E \subset \mathbb{C}$ is a Borel set with $\mu_2(E) = 0$. If $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$, then we say that μ_1 and μ_2 are *mutually absolutely continuous* and write $\mu_1 \ll \mu_2 \ll \mu_1$.

On the other hand, we say that μ_1 and μ_2 are *mutually singular*, and write $\mu_1 \perp \mu_2$, if there exist two disjoint Borel sets $A, B \subset \mathbb{C}$ such that μ_1 is concentrated on A and μ_2 is concentrated on B, meaning that

$$\mu_1(E) = \mu_1(E \cap A)$$

and

$$\mu_2(E) = \mu_2(E \cap B)$$

for every Borel set $E \subset \mathbb{C}$.

For example, it is well-known that harmonic measures on the same domain but for different points are always mutually absolutely continuous, see e.g. [38, Theorem 4.3.6].

2.3. **Quasicircles.** We will also need some preliminaries from the theory of quasiconformal mappings.

Let $K \geq 1$, let U, V be domains in $\widehat{\mathbb{C}}$ and let $f : U \to V$ be an orientation-preserving homeomorphism. We say that f is K-*quasiconformal* on U if it belongs to the Sobolev space $W^{1,2}_{loc}(U)$ and satisfies the Beltrami equation

$$\overline{\partial} f = \mu \, \partial f$$

almost everywhere on U for some measurable function μ with $\|\mu\|_\infty \leq \frac{K-1}{K+1}$. If $f : U \to V$ is orientation-reversing, then we say that f is K-*quasiconformal* if its complex conjugate $\overline{f}$ is K-quasiconformal.

A Jordan curve $J \subset \mathbb{C}$ is called a *quasicircle* if it is the image of the unit circle under a quasiconformal mapping of the whole plane. There are numerous characterizations of quasicircles (see e.g. [23]), but in this paper we shall only need the following due to Ahlfors [2, Chapter IV, Section D].

Theorem 2.1. *Let $J \subset \mathbb{C}$ be a Jordan curve with complementary components Ω and Ω^*. Then J is a quasicircle if and only if it admits a quasiconformal reflection, that is, an orientation-reversing quasiconformal mapping $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

(i) *$R(\Omega) = \Omega^*$ and $R(\Omega^*) = \Omega$,*
(ii) *$R(R(z)) = z$ for all $z \in \widehat{\mathbb{C}}$,*
(iii) *$R(z) = z$ for all $z \in J$.*

Moreover, if this is the case, then given any $w \in J$ we can choose $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that it is bilipschitz on any compact subset E of $\mathbb{C} \setminus \{w\}$:

$$\frac{1}{C}|z_1 - z_2| \leq |R(z_1) - R(z_2)| \leq C|z_1 - z_2| \qquad (z_1, z_2 \in E),$$

where C depends only on E and J.

2.4. **Properties of analytic capacity and continuous analytic capacity.** We conclude this section with a list of properties of analytic capacity and continuous analytic capacity. Recall from the introduction that for $E \subset \mathbb{C}$ compact and $D := \widehat{\mathbb{C}} \setminus E$, the analytic capacity of E is

$$\gamma(E) = \sup\{|f'(\infty)| : f \in H^\infty(D), |f| \leq 1 \text{ on } D\}$$

and the continuous analytic capacity of E is

$$\alpha(E) = \sup\{|g'(\infty)| : g \in A(D), |g| \leq 1 \text{ on } \mathbb{C}\},$$

where $H^\infty(D)$ is the space of all bounded analytic functions on D and $A(D)$ is the subspace of $H^\infty(D)$ consisting of all continuous functions on $\widehat{\mathbb{C}}$ that are analytic on D.

In the following, E, F and E_n, $n \in \mathbb{N}$, all denote compact subsets of $\mathbb{C}$. Also, we denote by Ω_E and Ω_F the unbounded components of $\widehat{\mathbb{C}} \setminus E$ and $\widehat{\mathbb{C}} \setminus F$ respectively.

(P1) $\alpha(E) \leq \gamma(E)$.
(P2) For $c, d \in \mathbb{C}$, $\gamma(cE + d) = |c|\gamma(E)$ and $\alpha(cE + d) = |c|\alpha(E)$.
(P3) If $E \subset F$, then $\gamma(E) \leq \gamma(F)$ and $\alpha(E) \leq \alpha(F)$.
(P4) If $E_1 \supset E_2 \supset E_3 \ldots$, then $\gamma(\cap_n E_n) = \lim_{n \to \infty} \gamma(E_n)$.

(P5) $\gamma(E) = \gamma(\partial\Omega_E)$.
(P6) If $\gamma(E) > 0$ and if $\Omega_E = \widehat{\mathbb{C}} \setminus E$ (i.e. $\widehat{\mathbb{C}} \setminus E$ is connected), then there is a unique function $f \in H^\infty(\Omega_E)$ with $|f| \leq 1$ on Ω_E and $f'(\infty) = \gamma(E)$, called the *Ahlfors function* for E. If in addition E is connected, then f is the unique conformal map from Ω_E onto $\mathbb{D}$ with $f(\infty) = 0$ and $f'(\infty) > 0$.
(P7) If $f : \Omega_E \to \Omega_F$ is conformal with $f(z) = az + b + O(1/z)$ at ∞, then $\gamma(F) = |a|\gamma(E)$.
(P8) For $a, b \in \mathbb{C}$, $\gamma([a, b]) = |a - b|/4$.
(P9) For $z_0 \in \mathbb{C}$ and $r > 0$, $\gamma(\overline{B}(z_0, r)) = r$.
(P10) If E is connected, then $\gamma(E) \geq \operatorname{diam}(E)/4$ and $\gamma(E) = c(E)$, where $c(E)$ denotes the logarithmic capacity of E.
(P11) $\gamma(E) \leq \mathcal{H}^1(E)$. In particular, if $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
(P12) $\gamma(E) = 0$ if and only if E is removable for the class H^∞, i.e., every bounded analytic function on $\widehat{\mathbb{C}} \setminus E$ is constant.
(P13) $\alpha(E) = 0$ if and only if E is removable for the class A, i.e., every continuous function on $\widehat{\mathbb{C}}$ analytic on $\widehat{\mathbb{C}} \setminus E$ is constant.
(P14) If $\Omega_E = \widehat{\mathbb{C}} \setminus E$ and if E is bounded by finitely many pairwise disjoint analytic Jordan curves, then $\alpha(E) = \gamma(E)$. In particular, we have $\alpha(\overline{B}(z_0, r)) = r$ for all $z_0 \in \mathbb{C}, r > 0$.
(P15) If E is contained in the real line, then $\gamma(E) = \mathcal{H}^1(E)/4$.
(P16) If E is contained in a rectifiable curve, then $\gamma(E) = 0$ if and only if $\mathcal{H}^1(E) = 0$.
(P17) If $\mathcal{H}^1(E) < \infty$, then $\alpha(E) = 0$. More generally, if E has σ-finite length, then $\alpha(E) = 0$.
(P18) If $\dim_H(E) > 1$, then $\alpha(E) > 0$ and so in particular $\gamma(E) > 0$.
(P19) If $\gamma(E) = 0$, then $\gamma(E \cup F) = \gamma(F)$. If $\alpha(E) = 0$, then $\alpha(E \cup F) = \alpha(F)$.
(P20) There is an absolute constant C such that

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F))$$

and

$$\alpha(E \cup F) \leq C(\alpha(E) + \alpha(F)).$$

(P21) If $\phi : \mathbb{C} \to \mathbb{C}$ is bilipschitz, there is a constant C depending only on ϕ such that

$$\frac{1}{C}\gamma(E) \leq \gamma(\phi(E)) \leq C\gamma(E)$$

and

$$\frac{1}{C}\alpha(E) \leq \alpha(\phi(E)) \leq C\alpha(E).$$

Properties (P1) to (P13) are standard and the proofs can be found in various textbooks such as [18], [21] and [43]. Property (P14) is due to Ahlfors and Garnett, see [21, Chapter I, Theorem 4.1]. Property (P15) is due to Pommerenke, see [21, Chapter I, Theorem 6.2]. Property (P16) follows from Property (P11) and Theorem 1.1. Property (P17) is essentially due to Besicovitch [8], see e.g. [51, Theorem 3.4] and [51, Corollary 3.5]. For Property (P18), see e.g. [51, Theorem 2.1]. The proof of Property (P19) will be given in Section 9. Properties (P20) and (P21) are deep results due to Tolsa, see [45], [46] and [47]. It is not known if we can take $C = 1$ in Property (P20).

3. Continuous analytic capacity and Dirichlet algebras

In this section, we prove that (1),(2),(3),(4) and (5) in Theorem 1.6 are equivalent for a Jordan curve $J \subset \mathbb{C}$:

(1) A_J is a Dirichlet algebra,
(2) $\omega \perp \omega^*$,
(3) $\mathcal{H}^1(T_J) = 0$, where T_J denotes the set of all tangent points of J,
(4) $\alpha(J') = \gamma(J')$ for all subarcs $J' \subset J$,
(5) $\alpha(J') \approx \gamma(J')$ for all subarcs $J' \subset J$.

Here A_J is the space of all continuous functions on $\widehat{\mathbb{C}}$ analytic on $\widehat{\mathbb{C}} \setminus J$, and as before ω and ω^* denote the harmonic measures on the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus J$ respectively.

Recall from the introduction that (1),(2) and (3) are already known to be equivalent, by Theorem 1.4 and Theorem 1.5. It therefore suffices to show that (1) implies (4) and that (5) implies (1), since (4) trivially implies (5).

We first need some background results on Dirichlet algebras. Recall that A_J is a Dirichlet algebra if for every continuous function $g : J \to \mathbb{R}$ and every $\epsilon > 0$, there exists a function $f \in A_J$ such that

$$\|g - \mathrm{Re}(f)\|_J < \epsilon,$$

where $\|\cdot\|_J$ denotes the uniform norm on J. Dirichlet algebras turn out to be related to other types of approximation.

Definition 3.1. Let $D := \widehat{\mathbb{C}} \setminus J$. We say that A_J is *pointwise boundedly dense* in $H^\infty(D)$ if there is a constant $C > 0$ depending only on J such that for every $f \in H^\infty(D)$, there is a sequence $(f_n) \subset A_J$ such that $\|f_n\|_D \leq C\|f\|_D$ and $f_n(z) \to f(z)$ as $n \to \infty$ for all $z \in D$.

Note that if A_J is pointwise boundedly dense in $H^\infty(D)$, then the sequence (f_n) in the definition in fact converges to f locally uniformly on D, by Vitali's convergence theorem for analytic functions.

Definition 3.2. We say that A_J is *strongly pointwise boundedly dense* in $H^\infty(D)$ if A_J is pointwise boundedly dense in $H^\infty(D)$ with $C = 1$.

It is quite remarkable that the three types of approximation are in fact equivalent.

Theorem 3.3. *For $J \subset \mathbb{C}$ a Jordan curve or a Jordan arc and $D = \widehat{\mathbb{C}} \setminus J$, the following are equivalent:*

(i) *A_J is a Dirichlet algebra.*
(ii) *A_J is pointwise boundedly dense in $H^\infty(D)$.*
(iii) *A_J is strongly pointwise boundedly dense in $H^\infty(D)$.*

The implication (i) $\Rightarrow$ (ii) is due to Hoffman [49] (see also [20, Theorem 9.1]), the implication (ii) $\Rightarrow$ (iii) is due to Davie [17, Theorem 1.3] (see also [20, Theorem 6.8]) and the implication (iii) $\Rightarrow$ (i) is due to Gamelin and Garnett [20, Theorem 9.1]. See also [9] and [10] for alternate proofs.

We can now prove the implication (1) $\Rightarrow$ (4) in Theorem 1.6.

Proof. Let $J \subset \mathbb{C}$ be a Jordan curve, and suppose that A_J is a Dirichlet algebra. Let $J' \subset J$ be a subarc. We have to show that $\alpha(J') = \gamma(J')$. In view of Property (P1) from Section 2, it suffices to show that $\alpha(J') \geq \gamma(J')$.

For this, first note that since A_J is a Dirichlet algebra, we have that $\mathcal{H}^1(T_J) = 0$, by the equivalence of (1) and (3) in Theorem 1.6. Now, every tangent point of J' is clearly also a tangent point of J, hence $\mathcal{H}^1(T_{J'}) = 0$. Again by the equivalence of (1) and (3) in Theorem 1.6, it follows that $A_{J'}$ is a Dirichlet algebra. Note that the equivalence of (1),(2) and (3) in Theorem 1.6 holds not only for Jordan curves but also for Jordan arcs, see [11] and [13]. In particular, by Theorem 3.3, the space $A_{J'}$ is strongly pointwise boundedly dense in $H^\infty(D')$, where $D' := \widehat{\mathbb{C}} \setminus J'$.

Now, let $\epsilon > 0$, and let $f \in H^\infty(D')$ with $|f| \leq 1$ on D' and $|f'(\infty)| \geq \gamma(J') - \epsilon/2$. Since $A_{J'}$ is strongly pointwise boundedly dense in $H^\infty(D')$, there is a sequence $(f_n) \subset A_{J'}$ with $|f_n| \leq 1$ on D' and $f_n \to f$ locally uniformly on D'. In particular, for n large enough, we have $|f_n'(\infty)| \geq |f'(\infty)| - \epsilon/2$, and thus

$$\alpha(J') \geq |f_n'(\infty)| \geq |f'(\infty)| - \frac{\epsilon}{2} \geq \gamma(J') - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we obtain $\alpha(J') \geq \gamma(J')$, as required. □

For the proof of the implication (5) ⇒ (1) in Theorem 1.6, we need a result of Gamelin and Garnett relating the notions of Dirichlet algebra and continuous analytic capacity. Building upon the work of Vitushkin on uniform rational approximation, Gamelin and Garnett proved the following result.

Theorem 3.4 (Gamelin–Garnett [20]). *Let $J \subset \mathbb{C}$ be a Jordan curve. We have that A_J is a Dirichlet algebra if and only if there is a σ-curvilinear null set E such that for all $z \in J \setminus E$,*

$$\liminf_{\delta \to 0} \frac{\alpha(\overline{B}(z,\delta) \cap J)}{\delta} > 0.$$

A *curvilinear null set* is a bounded subset of a C^2 curve with zero length, and a *σ-curvilinear null set* is a countable union of curvilinear null sets. See [20, Theorem 10.1].

We can now prove that (5) implies (1) in Theorem 1.6, thereby completing the proof of the equivalence of (1),(2),(3),(4) and (5).

Proof. Suppose that (5) holds, so there exists an absolute constant C such that $\alpha(J') \geq C\gamma(J')$ for all subarcs J' of J. We show that this implies

$$\liminf_{\delta \to 0} \frac{\alpha(\overline{B}(z,\delta) \cap J)}{\delta} > 0$$

for all $z \in J$, so that we can take $E = \emptyset$ in Theorem 3.4 and deduce that A_J is a Dirichlet algebra.

Let $z \in J$, and let $\delta > 0$ be small enough so that the ball $\overline{B}(z,\delta)$ does not contain J. Then there exists a subarc J' of J contained in the ball $\overline{B}(z,\delta)$ whose diameter is at least δ. By Property (P3) and Property (P10) of Section 2, we obtain

$$\alpha(\overline{B}(z,\delta) \cap J) \geq \alpha(J') \geq C\gamma(J') \geq C\frac{\operatorname{diam}(J')}{4} \geq C\frac{\delta}{4}.$$

Hence

$$\liminf_{\delta \to 0} \frac{\alpha(\overline{B}(z,\delta) \cap J)}{\delta} \geq \frac{C}{4} > 0$$

as required.

□

4. Tangent points and singularity of ω, ω^* with $\mathcal{H}^1$

In this section, we prove that (3) $\Rightarrow$ (6) in Theorem 1.6, namely that if J is a quasicircle with $\mathcal{H}^1(T_J) = 0$, then $\omega \perp \mathcal{H}^1$ and $\omega^* \perp \mathcal{H}^1$ on J. Recall that T_J is the set of tangent points of J and that ω and ω^* are the harmonic measures on Ω and Ω^*, the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus J$ respectively. As we shall see, the implication (3) $\Rightarrow$ (6) in Theorem 1.6 follows from classical results of Pommerenke and Makarov on harmonic measures and length.

Following [34, Section 6.6], define

$$S := \{w \in J : \text{there is an open triangle of vertex } w \text{ in } \Omega\},$$

and similarly S^* with Ω^* instead of Ω. Clearly, we have $T_J \subset S \cap S^*$. In fact, the sets T_J and $S \cap S^*$ are equal up to a set of zero length, which we denote by

$$S \cap S^* \doteq T_J.$$

We need the following theorem of Makarov:

Theorem 4.1 (Makarov compression theorem [27],[28]). *There are partitions*

$$J = S \cup B_0 \cup B_1$$

and

$$J = S^* \cup B_0^* \cup B_1^*$$

where $\mathcal{H}^1(B_0) = 0, \omega(B_1) = 0$ *and* $\mathcal{H}^1(B_0^*) = 0, \omega^*(B_1^*) = 0$.

It follows that ω is concentrated on the set $S \cup B_0$ and, similarly, that ω^* is concentrated on the set $S^* \cup B_0^*$. Moreover, the sets $S \cup B_0$ and $S^* \cup B_0^*$ can be shown to have σ-finite length. This is a version of the celebrated result of Makarov on the fact that the dimension of harmonic measure on simply connected domains is always equal to 1. See [34, Section 6.6].

We shall also need the following result of Pommerenke:

Proposition 4.2. *If J is a quasicircle, then*

$$T_J \doteq S \doteq S^*.$$

See [34, Proposition 6.28].

We can now prove the implication (3) $\Rightarrow$ (6) in Theorem 1.6.

Proof. Let $J \subset \mathbb{C}$ be a quasicircle, and suppose that (3) holds in Theorem 1.6, i.e. $\mathcal{H}^1(T_J) = 0$. By Proposition 4.2, we get that $\mathcal{H}^1(S) = 0$ and $\mathcal{H}^1(S^*) = 0$. Now, by Theorem 4.1, we have a partition

$$J = S \cup B_0 \cup B_1$$

where $\mathcal{H}^1(B_0) = 0$ and $\omega(B_1) = 0$. Since $\mathcal{H}^1(S) = 0$, it follows that $\mathcal{H}^1$ restricted to J is concentrated on B_1. On the other hand, the measure ω is concentrated on $S \cup B_0$, a set disjoint from B_1. Thus $\omega \perp \mathcal{H}^1$ on J, and similarly $\omega^* \perp \mathcal{H}^1$ on J. $\square$

5. Singularity of ω, ω^* with $\mathcal{H}^1$ and pure unrectifiability

In this section, we prove the implication (6) $\Rightarrow$ (7) in Theorem 1.6, namely that if J is a quasicircle and if $\omega \perp \mathcal{H}^1$ and $\omega^* \perp \mathcal{H}^1$ on J, then J is purely unrectifiable. In fact, the proof will show that the singularity of only one of the harmonic measures with $\mathcal{H}^1$ is required.

The implication (6) $\Rightarrow$ (7) is a simple consequence of the following local F. and M. Riesz theorem for quasicircles.

Theorem 5.1. *Let $J \subset \mathbb{C}$ be a quasicircle and let $\Gamma \subset \mathbb{C}$ be a rectifiable curve. Then $\omega \ll \mathcal{H}^1 \ll \omega$ on $J \cap \Gamma$.*

Remark. In fact, we have $\omega \ll \mathcal{H}^1$ on $J \cap \Gamma$ even if J is only assumed to be the boundary of a simply connected domain, by a famous theorem of Bishop and Jones [12]. For quasicircles, however, the proof is considerably simpler, as we shall see. On the other hand, we mention that there exist a Jordan curve J and a rectifiable curve Γ such that $\mathcal{H}^1$ is *not* absolutely continuous with respect to ω on $J \cap \Gamma$, see [22, Chapter VI, Exercise 4].

Before proving Theorem 5.1, we explain how it can be used to deduce the implication (6) $\Rightarrow$ (7) in Theorem 1.6. Suppose that $J \subset \mathbb{C}$ is a quasicircle and that $\omega \perp \mathcal{H}^1$ on J. Let $\Gamma \subset \mathbb{C}$ be a rectifiable curve. By Theorem 5.1, we have $\omega \ll \mathcal{H}^1 \ll \omega$ on $J \cap \Gamma$. Since $\omega \perp \mathcal{H}^1$ on J and $\omega \ll \mathcal{H}^1$ on $J \cap \Gamma$, it easily follows that $\omega(J \cap \Gamma) = 0$ and thus $\mathcal{H}^1(J \cap \Gamma) = 0$, since $\mathcal{H}^1 \ll \omega$ on $J \cap \Gamma$. This shows that J is purely unrectifiable, as required.

It remains to prove Theorem 5.1. For this, we need the following lemma from Bishop's thesis [9, Lemma 7.1], whose proof uses the quasiconformal reflection from Theorem 2.1.

Lemma 5.2. *Let $J \subset \mathbb{C}$ be a quasicircle with complementary components Ω, Ω^* and let $\Gamma \subset \mathbb{C}$ be a rectifiable curve. Then there is a rectifiable Jordan curve Γ' such that $J \cap \Gamma = J \cap \Gamma'$. Moreover, we can construct Γ' so that either of the following holds:*

(i) $\Gamma' \subset \overline{\Omega}$ *and* $\Omega' \subset \Omega$
(ii) $\Gamma' \subset \overline{\Omega^*}$ *and* $\Omega \subset \Omega'$.

Here Ω' is the bounded complementary component of Γ'.

We can now prove Theorem 5.1.

Proof. Let $J \subset \mathbb{C}$ be a quasicircle and let $\Gamma \subset \mathbb{C}$ be a rectifiable curve.

First, let Γ' be as in Lemma 5.2 so that (i) holds. Then $J \cap \Gamma = J \cap \Gamma'$, with $\Gamma' \subset \overline{\Omega}$ and $\Omega' \subset \Omega$. Since harmonic measures on the same domain are all mutually absolutely continuous ([38, Theorem 4.3.6]), we may assume that the base point $z_0 \in \Omega$ for ω belongs to Ω'. Let E be a Borel subset of $J \cap \Gamma = J \cap \Gamma'$. Since $\Omega' \subset \Omega$, it follows from the Subordination Principle for harmonic measures ([38, Corollary 4.3.9]) that $\omega'(E) \leq \omega(E)$, where ω' is the harmonic measure for Ω' with respect to z_0. In particular, if $\omega(E) = 0$, then $\omega'(E) = 0$. Also, by the F. and M. Riesz theorem ([22, Chapter VI, Theorem 1.2]), the measures ω' and $\mathcal{H}^1$ are mutually absolutely continuous on Γ', thus $\mathcal{H}^1(E) = 0$. This shows that $\mathcal{H}^1 \ll \omega$ on $J \cap \Gamma$.

Conversely, let Γ' be as in Lemma 5.2 so that (ii) holds. Then $J \cap \Gamma = J \cap \Gamma'$, but this time $\Gamma' \subset \overline{\Omega^*}$ and $\Omega \subset \Omega'$. Again, denote by ω' the harmonic measure for Ω' with respect to $z_0 \in \Omega$, the base point for ω. Let E be a subset of $J \cap \Gamma = J \cap \Gamma'$ with $\mathcal{H}^1(E) = 0$. Again by the F. and M. Riesz theorem, we get that $\omega'(E) = 0$. But $\omega(E) \leq \omega'(E)$, again by the Subordination Principle, so that $\omega(E) = 0$. This shows that $\omega \ll \mathcal{H}^1$ on $J \cap \Gamma$, as required. □

Remark. We mention that Theorem 5.1 is essentially equivalent to the case $d = 1$ of a more general result of Azzam ([7, Theorem 1.11]) on harmonic measure and

$\mathcal{H}^d$ on uniform domains with uniform complements in $\mathbb{R}^{d+1}$. Note that a simply connected domain in the plane is uniform with uniform complement if and only if it is bounded by a quasicircle, see e.g. [23, Section 3.5].

6. Pure unrectifiability and the comparability of α and γ

In this section, we complete the proof of Theorem 1.6 by proving the implication (7) $\Rightarrow$ (5). As mentioned in the introduction, this follows from the following result due to Tolsa:

Theorem 6.1 (Tolsa [46]). *Let $E \subset \mathbb{C}$ be a purely unrectifiable compact set. Then $\gamma(F) \approx \alpha(F)$ for every compact subset F of E.*

See [46, Corollary 3.7].

Theorem 6.1 is a profound result culminating from the work of Tolsa in [44], [45] and [46]. We briefly discuss the main ingredients of the proof.

Let μ be a positive Radon measure on $\mathbb{C}$. The *linear density* of μ at $x \in \mathbb{C}$ is

$$\Theta_\mu(x) := \lim_{r \to 0} \frac{\mu(B(x,r))}{r},$$

if the limit exists, and the *upper linear density* of μ at $x \in \mathbb{C}$ is

$$\Theta_\mu^*(x) := \limsup_{r \to 0} \frac{\mu(B(x,r))}{r}.$$

Let M be the maximal radial Hardy–Littlewood operator

$$M\mu(x) := \sup_{r>0} \frac{\mu(B(x,r))}{r}$$

and for $x \in \mathbb{C}$, define

$$c_\mu(x)^2 := \int \int c(x,y,z)^2 \, d\mu(y) \, d\mu(z),$$

where $c(x,y,z)$ is the *Menger curvature* of $x,y,z \in \mathbb{C}$, defined by

$$c(x,y,z) := \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circle passing through x,y,z. By convention, we set $R(x,y,z) = \infty$ and $c(x,y,z) = 0$ if x,y,z lie on a same line.

We also define the potential

$$U_\mu(x) := M\mu(x) + c_\mu(x) \qquad (x \in \mathbb{C}).$$

In [44] and [45], Tolsa observed that the potential U_μ can be used to estimate analytic capacity.

Theorem 6.2 (Tolsa). *For $E \subset \mathbb{C}$ compact, we have*

$$\gamma(E) \approx \sup\{\mu(E) : \operatorname{supp}(\mu) \subset E, U_\mu(x) \leq 1 \textit{ for all } x \in \mathbb{C}\}.$$

See [44, Equation (1.6)] and [45, Theorem 1.1].

Shortly after, Tolsa proved a similar result for continuous analytic capacity.

Theorem 6.3 (Tolsa). *For $E \subset \mathbb{C}$ compact, we have*

$$\alpha(E) \approx \sup\{\mu(E) : \operatorname{supp}(\mu) \subset E, \Theta_\mu(x) = 0 \textit{ for } \mu\textit{-a.e } x \in E \textit{ and } U_\mu(x) \leq 1 \textit{ for all } x \in \mathbb{C}\}.$$

See [46, Equation (2.2)].

We can now explain how Theorem 6.1 follows from Theorem 6.2 and Theorem 6.3:

Proof. Let $E \subset \mathbb{C}$ be a purely unrectifiable compact set and let $F \subset E$ be compact. By Theorem 6.2, there is a Radon measure μ supported on F such that $\mu(F) \approx \gamma(F)$ and $U_\mu(x) \leq 1$ for all $x \in \mathbb{C}$. For $n \in \mathbb{N}$, define

$$F_n := \{x \in F : \Theta^*_\mu(x) > 1/n\}.$$

Then $\mathcal{H}^1(F_n) < \infty$ by [29, Theorem 6.9]. Also, if K is any compact subset of F_n, then K is purely unrectifiable, so that $\gamma(K) = 0$ by Theorem 1.1 and thus $\mu(K) = 0$ by Theorem 6.2. It follows from the regularity of μ that $\mu(F_n) = 0$, for each n. Hence $\Theta_\mu(x) = 0$ for almost every $x \in F$ with respect to μ. By Theorem 6.3, we obtain

$$\alpha(F) \gtrsim \mu(F) \approx \gamma(F)$$

and thus $\alpha(F) \approx \gamma(F)$ as required. □

This completes the proof of Theorem 1.6.

We conclude this section with a remark due to Chris Bishop regarding Theorem 1.6.

Remark. Theorem 1.6 can also be proved without using Tolsa's theorem. In fact, as mentioned in the introduction, it can be proved using only classical results on harmonic measure as well as results from Bishop's thesis [9] and Gamelin and Garnett's paper [20]. For a Jordan curve $J \subset \mathbb{C}$, as in saw in Section 3, the equivalence of (1),(2),(3),(4) and (5) in Theorem 1.6 follows from the Browder–Wermer theorem (Theorem 1.4), the Bishop–Carleson–Garnett–Jones theorem (Theorem 1.5) as well as Theorem 3.4 from Gamelin and Garnett's paper [20]. In [9, Chapter II], Bishop gives a new proof of the Browder–Wermer theorem.

Suppose now that J is a purely unrectifiable Jordan curve. We first claim that $\mathcal{H}^1(T_J) = 0$. Indeed, if $\mathcal{H}^1(T_J) > 0$, then one can construct a rectifiable curve $\Gamma \subset \mathbb{C}$ with $\mathcal{H}^1(T_J \cap \Gamma) > 0$ ([9, Chapter I, Section 9]), and J is not purely unrectifiable. Now, it follows from McMillan's twist point theorem (see [9, Chapter I, Theorem 2.1]) that almost every point of J (with respect to harmonic measure) is a twist point. By Makarov's theorem, both harmonic measures are mutually singular with $\mathcal{H}^1$ on the set of twist points. This shows that (7) implies (6) in Theorem 1.6.

Suppose now that $\omega \perp \mathcal{H}^1$ and $\omega^* \perp \mathcal{H}^1$ on J. Then, again by McMillan's twist point theorem, both ω and ω^* are concentrated on the set of twist points of J. But $\omega \perp \omega^*$ on that set, by [9, Chapter I, Theorem 2.2]. This shows that (6) implies (2) in Theorem 1.6.

Summarizing, we obtain that Theorem 1.6 can be stated in a stronger form.

Theorem 6.4. *For arbitrary Jordan curves, we have that* $(1-5)$ *are equivalent and* $(7) \Rightarrow (6) \Rightarrow (1-5)$. *For quasicircles, we have* $(1-5) \Rightarrow (6) \Rightarrow (7)$.

7. Proof of Corollary 1.7

In this section, we prove Corollary 1.7.

Proof. Let $J \subset \mathbb{C}$ be a Jordan arc such that $\mathcal{H}^1(T_J) = 0$, where T_J denotes the set of all tangent points of J. We have to show that there is no continuous function $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ analytic on $\widehat{\mathbb{C}} \setminus J$ such that $|g| \leq 1$ on $\mathbb{C}$ and $|g'(\infty)| = \alpha(J)$.

Suppose for a contradiction that an extremal function g exists. Multiplying by a constant of modulus 1 if necessary, we may assume that $g'(\infty) = \alpha(J)$. Now, recall from Section 3 that the implication (3) $\Rightarrow$ (4) in Theorem 1.6 holds not only for Jordan curves, but also for Jordan arcs. It follows that $\alpha(J) = \gamma(J)$, so that g is also extremal for $\gamma(J)$. By Property (P6) from Section 2, we get that $g : \widehat{\mathbb{C}} \setminus J \to \mathbb{D}$ is conformal, with $g(\infty) = 0$ and $g'(\infty) > 0$. But since J is a Jordan arc, every conformal map from $\widehat{\mathbb{C}} \setminus J$ onto $\mathbb{D}$ must be discontinuous at each point of J which is not an endpoint, by the classical theory of boundary behavior of conformal maps. This contradicts the fact that g is continuous everywhere. It follows that there is no extremal function for $\alpha(J)$, as required. □

8. Proof of Theorem 1.10

In this section, we show how to use Theorem 1.6 in order to prove Theorem 1.10. More specifically, we construct a holomorphic motion h of $\widehat{\mathbb{C}}$ and compact sets E and F for which both functions

$$\lambda \mapsto \gamma(E_\lambda), \quad \lambda \mapsto \alpha(F_\lambda) \qquad (\lambda \in \mathbb{D})$$

are discontinuous at 0.

Recall from the introduction that a map $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic motion of $\widehat{\mathbb{C}}$ if

(i) for each fixed $z \in \widehat{\mathbb{C}}$, the map $\lambda \mapsto h(\lambda, z)$ is holomorphic on $\mathbb{D}$,
(ii) for each fixed $\lambda \in \mathbb{D}$, the map $z \mapsto h(\lambda, z)$ is injective on $\widehat{\mathbb{C}}$,
(iii) $h(0, z) = z$ for all $z \in \widehat{\mathbb{C}}$.

For $\lambda \in \mathbb{D}$, the sets E_λ and F_λ are defined by

$$E_\lambda = h_\lambda(E), F_\lambda = h_\lambda(F)$$

where $h_\lambda(z) = h(\lambda, z)$, so that $E_0 = E$ and $F_0 = F$. Note that for $\lambda \in \mathbb{D}$, the map $h_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism by the λ-lemma. In fact, it is well-known that each $h_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal, see e.g. [4, Theorem 12.3.2]. In particular, the sets E_λ and F_λ are compact subsets of $\mathbb{C}$, since we assume that $h_\lambda(\infty) = \infty$.

As mentioned in the introduction, the holomorphic motion $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ in Theorem 1.10 stems from the theory of holomorphic dynamics. We need some preliminaries.

For a quadratic polynomial $p_c(z) := z^2 + c$, we define the *filled Julia set* $\mathcal{K}_c$ of p_c by

$$\mathcal{K}_c := \{z \in \mathbb{C} : p_c^m(z) \nrightarrow \infty \text{ as } m \to \infty\},$$

where p_c^m denotes the m-th iterate of the polynomial p_c. The *Julia set* $\mathcal{J}_c$ of p_c is defined as the boundary of the filled Julia set:

$$\mathcal{J}_c := \partial \mathcal{K}_c,$$

and the *Mandelbrot set* $\mathcal{M}$ is defined as the set of all parameters $c \in \mathbb{C}$ for which the orbit of 0 under p_c remains bounded:

$$\mathcal{M} := \{c \in \mathbb{C} : p_c^m(0) \nrightarrow \infty \text{ as } m \to \infty\}.$$

Note that a parameter c belongs to $\mathcal{M}$ if and only if the corresponding Julia set $\mathcal{J}_c$ is connected. For $c \notin \mathcal{M}$, the Julia set $\mathcal{J}_c$ is a Cantor set. The Mandelbrot set contains a main cardioid $\mathcal{M}_0$ defined as the set of all parameters c for which the polynomial p_c has an attracting fixed point. It is easy to see that $\mathcal{M}_0$ contains

$B(0, 1/4)$. For background on Julia sets and the Mandelbrot set, the reader may consult [31] and [32].

The main cardioid $\mathcal{M}_0$ is especially interesting for Theorem 1.10 since it is well-known that quadratic Julia sets with parameters belonging to $\mathcal{M}_0$ move holomorphically. More precisely, for each $c \in \mathcal{M}_0$, there is a unique conformal map $B_c : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus \mathcal{K}_c$ with normalization $B_c(z) = z + O(1/z)$ at ∞, called the *Böttcher map*. The map $B(c, z) := B_c(z)$ is holomorphic in both variables. Note that B_0 is the identity since $\mathcal{J}_0$ is the unit circle.

In order to work on the unit disk rather than $\mathcal{M}_0$, we make the change of variable $c = \lambda/4$. For $\lambda \in \mathbb{D}$, we denote by Ω_λ and Ω_λ^* the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus \mathcal{J}_{\lambda/4}$ respectively. Then the map $h : \mathbb{D} \times (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) \to \widehat{\mathbb{C}}$ defined by

$$h(\lambda, z) := B_{\lambda/4}(z) \qquad (\lambda \in \mathbb{D}, z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$$

gives a holomorphic motion of $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. By a famous theorem of Słodkowski (see [41] or [4, Theorem 12.3.2]), the holomorphic motion h extends to a holomorphic motion of the whole sphere, which we denote by the same letter and refer to as the *Böttcher motion*. For each $\lambda \in \mathbb{D}$, the map $h_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal. Moreover, it maps $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ conformally onto Ω_λ^*, and the unit circle homeomorphically onto $\mathcal{J}_{\lambda/4}$. See Figure 3.

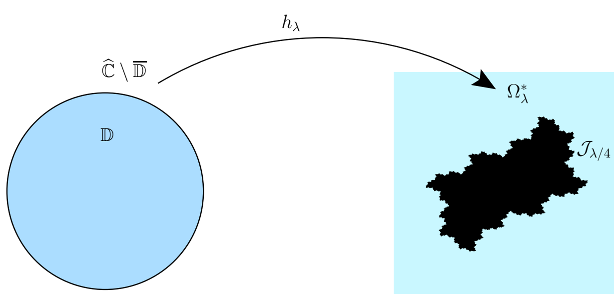

FIGURE 3. The Böttcher motion $h_\lambda(z)$.

In particular, this shows that the Julia sets $\mathcal{J}_{\lambda/4}$ for $\lambda \in \mathbb{D}$ are all quasicircles, and Theorem 1.6 applies. In fact, the following result due to Fatou shows that condition (3) in Theorem 1.6 is satisfied.

Lemma 8.1 (Fatou). *For $c \in \mathcal{M}_0 \setminus \{0\}$, the corresponding Julia set $\mathcal{J}_c$ has no tangent point.*

See [42, Chapter 5, Section 3, Theorem 1] for a proof.

Remark. The fact that condition (3) in Theorem 1.6 is satisfied for the Julia sets $\mathcal{J}_c$ with $c \in \mathcal{M}_0 \setminus \{0\}$ also follows from a general theorem of Hamilton [24, Theorem 2], who proved that the set of tangent points of a Jordan curve rational Julia set always has zero $\mathcal{H}^1$ measure unless the Julia set is a circle.

We can now proceed with the proof of Theorem 1.10.

Proof. Let $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the Böttcher holomorphic motion. Recall that for each $\lambda \in \mathbb{D}$, the map $h_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal. Moreover, it maps the unit circle homeomorphically onto the quasicircle $\mathcal{J}_{\lambda/4}$, and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ conformally onto Ω_λ^*, the unbounded complementary component of $\mathcal{J}_{\lambda/4}$. In addition, each h_λ has normalization $h_\lambda(z) = z + O(1/z)$ at ∞.

Now, by Lemma 8.1, all the conditions in Theorem 1.6 are true for the Julia sets $\mathcal{J}_{\lambda/4}$, $\lambda \in \mathbb{D} \setminus \{0\}$. This has two important consequences.

First, by (6) in Theorem 1.6, we have that for each $\lambda \in \mathbb{D} \setminus \{0\}$, the harmonic measure ω_λ^* on Ω_λ^* is mutually singular with $\mathcal{H}^1$ on $\mathcal{J}_{\lambda/4}$. Let (λ_n) be any sequence of non-zero complex numbers in $\mathbb{D}$ converging to 0. Then for each n, there is a Borel set $B_n \subset \mathcal{J}_{\lambda_n/4}$ with $\omega_\lambda^*(B_n) = 1$ but $\mathcal{H}^1(B_n) = 0$. By the definition of harmonic measure in Section 2, it follows that for each n, the preimage set $A_n := h_{\lambda_n}^{-1}(B_n) \subset \partial\mathbb{D}$ has full normalized Lebesgue measure in the unit circle. In particular, the intersection $\cap_n A_n$ also has full Lebesgue measure. Let E be any compact subset of $\cap_n A_n$ with positive Lebesgue measure. Then for each n, we have

$$\mathcal{H}^1(E_{\lambda_n}) = \mathcal{H}^1(h_{\lambda_n}(E)) \leq \mathcal{H}^1(h_{\lambda_n}(A_n)) = \mathcal{H}^1(B_n) = 0.$$

In particular, by Property (P11) in Section 2, we have that $\gamma(E_{\lambda_n}) = 0$ for all n. On the other hand, it follows from Property (P16) in Section 2 that $\gamma(E_0) = \gamma(E) > 0$, since $E \subset \partial\mathbb{D}$ has positive Lebesgue measure. This shows that the function

$$\lambda \mapsto \gamma(E_\lambda) \qquad (\lambda \in \mathbb{D})$$

is discontinuous at 0.

Secondly, by (4) in Theorem 1.6, we have $\alpha(\mathcal{J}_{\lambda/4}) = \gamma(\mathcal{J}_{\lambda/4})$ for all $\lambda \in \mathbb{D} \setminus \{0\}$. But $\gamma(\mathcal{J}_{\lambda/4}) = \gamma(\overline{\mathbb{D}}) = 1$, by Property (P7) and Property (P9) in Section 2. Letting $F := \partial\mathbb{D}$, we obtain

$$\alpha(F_\lambda) = \alpha(\mathcal{J}_{\lambda/4}) = \gamma(\mathcal{J}_{\lambda/4}) = 1,$$

for all $\lambda \in \mathbb{D} \setminus \{0\}$. On the other hand, it follows from Property (P17) in Section 2 that $\alpha(F_0) = \alpha(F) = 0$, since $\mathcal{H}^1(F) < \infty$. Thus the function

$$\lambda \mapsto \alpha(F_\lambda) \qquad (\lambda \in \mathbb{D})$$

is discontinuous at 0, as required. □

Remark. The construction of the set E in the proof relies on the fact that for $\lambda \in \mathbb{D} \setminus \{0\}$, the harmonic measure ω_λ^* is mutually singular with $\mathcal{H}^1$ on $\mathcal{J}_{\lambda/4}$. This consequence of Theorem 1.6 also follows from a theorem of Zdunik, as mentioned to us by Saeed Zakeri. In [54, Theorem 1], Zdunik proved that if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree at least 2, then the measure of maximal entropy μ on $\mathcal{J}_f$ is mutually singular with $\mathcal{H}^\alpha$, where α is the dimension of μ, except for the case when f is critically finite with parabolic orbifold. For quadratic polynomials $z \mapsto z^2 + c$ with $c \in \mathcal{M}_0$, the measure of maximal entropy μ coincides with harmonic measure at ∞, and $\alpha = 1$ by Makarov's theorem. Note that the case $c = 0$ is the only quadratic polynomial with parabolic orbifold.

9. Proof of Corollary 1.11

In this section, we prove Corollary 1.11. The first step is to combine the holomorphic motions $\{E_\lambda\}$ and $\{F_\lambda\}$ from Theorem 1.10 in order to obtain a single compact set K' and a holomorphic motion $g' : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which both functions

$$\lambda \mapsto \gamma(K'_\lambda), \quad \lambda \mapsto \alpha(K'_\lambda) \qquad (\lambda \in \mathbb{D})$$

are discontinuous at 0. Here $K'_\lambda := g'_\lambda(K')$, where $g'_\lambda(z) = g'(\lambda, z)$. In the second part of the proof, we modify K' and g' in order to get a discontinuity at each point of a given Blaschke sequence.

9.1. **First step: Combining holomorphic motions.** Let E, F and $h : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be as in Theorem 1.10, so that h is a holomorphic motion of $\widehat{\mathbb{C}}$ and both functions

$$\lambda \mapsto \gamma(E_\lambda), \quad \lambda \mapsto \alpha(F_\lambda) \qquad (\lambda \in \mathbb{D})$$

are discontinuous at 0, where $E_\lambda = h_\lambda(E)$ and $F_\lambda = h_\lambda(F)$. By construction, the set E is a compact subset of $F = \partial\mathbb{D}$ with $\gamma(E) > 0$, and there is a sequence (λ_n) in $\mathbb{D} \setminus \{0\}$ converging to 0 such that

$$\gamma(E_{\lambda_n}) = 0 \qquad (n \in \mathbb{N})$$

and

$$\alpha(F_{\lambda_n}) = 1 = \gamma(F_{\lambda_n}) \qquad (n \in \mathbb{N}).$$

Now, take $d > 0$ large enough so that for each $\lambda \in \mathbb{D}$, the translate $E_\lambda + d$ is disjoint from F_λ. We shall fix the value of d later. Let $K' := (E + d) \cup F$, and define $g' : \mathbb{D} \times K' \to \widehat{\mathbb{C}}$ by

$$g'(\lambda, z) = \begin{cases} h(\lambda, z - d) + d & \text{if } z \in E + d \\ h(\lambda, z) & \text{if } z \in F. \end{cases}$$

It is easy to see that g' is a holomorphic motion of K'. By Słodkowski's theorem, we can extend g' to a holomorphic motion of $\widehat{\mathbb{C}}$, which we still denote by the same letter. Note that for $\lambda \in \mathbb{D}$, we have

$$K'_\lambda = g'_\lambda((E + d) \cup F) = g'_\lambda(E + d) \cup g'_\lambda(F) = (E_\lambda + d) \cup F_\lambda.$$

Lemma 9.1. *For K' and $g' : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as above, we have that both functions*

$$\lambda \mapsto \gamma(K'_\lambda), \quad \lambda \mapsto \alpha(K'_\lambda) \qquad (\lambda \in \mathbb{D})$$

are discontinuous at 0.

The proof of Lemma 9.1 involves estimating $\gamma((E_\lambda + d) \cup F_\lambda)$ and $\alpha((E_\lambda + d) \cup F_\lambda)$. In general, estimating the analytic capacity or the continuous analytic capacity of the union of two sets is quite difficult. For this reason, we shall need several auxiliary results.

First, we need Property (P19) from Section 2, which we formulate as a lemma.

Lemma 9.2. *Let E and F be compact subsets of $\mathbb{C}$.*

(i) *If $\gamma(E) = 0$, then $\gamma(E \cup F) = \gamma(F)$.*
(ii) *If $\alpha(E) = 0$, then $\alpha(E \cup F) = \alpha(F)$.*

For the proof, we need the following proposition.

Proposition 9.3. *Let $U \subset \mathbb{C}$ be open and let $E \subset \mathbb{C}$ be compact.*

(i) *If $\gamma(E) = 0$, then every bounded analytic function on $U \setminus E$ can be extended to a bounded analytic function on U.*
(ii) *If $\alpha(E) = 0$, then every bounded continuous function on U which is analytic on $U \setminus E$ is in fact analytic on the whole open set U.*

Note that here it is not assumed that E is contained in U. In particular, it may happen that $E \cap \partial U \neq \emptyset$.

The proof of (i) in Proposition 9.3 can be found in [43, Proposition 1.18]. In fact, essentially the same argument can be used to prove (ii), as we will show. First, we need some preliminaries on the Cauchy transform and Vitushkin's localization operator.

Let μ be a complex Borel measure on $\mathbb{C}$ with compact support. The *Cauchy transform* of μ is defined by

$$\mathcal{C}\mu(z) := \int \frac{1}{\zeta - z} d\mu(\zeta).$$

It is easy to see that the above integral converges for almost every $z \in \mathbb{C}$ with respect to Lebesgue measure. Furthermore, the Cauchy transform $\mathcal{C}\mu$ is analytic outside the support of μ and satisfies $\mathcal{C}\mu(\infty) = 0$ and $\mathcal{C}\mu'(\infty) = -\mu(\mathbb{C})$.

The definition of the Cauchy transform also makes sense if μ is a compactly supported distribution. In this case, we define

$$\mathcal{C}\mu := -\frac{1}{z} * \mu.$$

The following elementary lemma is standard, see e.g. [15, Theorem 18.5.4].

Lemma 9.4. *We have*

$$\overline{\partial}\frac{1}{\pi z} = \delta_0$$

in the sense of distributions, where δ_0 is the Dirac delta at the origin. As a consequence, if μ is a compactly supported distribution on $\mathbb{C}$, then

$$\overline{\partial}(\mathcal{C}\mu) = -\pi\mu.$$

Also, if $f \in L^1_{loc}(\mathbb{C})$ is analytic in a neighborhood of infinity and if $f(\infty) = 0$, then

$$\mathcal{C}(\overline{\partial}f) = -\pi f.$$

Definition 9.5. Let $f \in L^1_{loc}(\mathbb{C})$ and $\phi \in C^\infty_c(\mathbb{C})$. We define *Vitushkin's localization operator* V_ϕ by

$$V_\phi f := \phi f + \frac{1}{\pi}\mathcal{C}(f\overline{\partial}\phi).$$

The same definition holds more generally if f is a distribution.

The following lemma gives an equivalent formula for Vitushkin's localization operator.

Lemma 9.6. *Let $f \in L^1_{loc}(\mathbb{C})$ and $\phi \in C^\infty_c(\mathbb{C})$. Then*

$$V_\phi f = -\frac{1}{\pi}\mathcal{C}(\phi\overline{\partial}f)$$

in the sense of distributions.

Proof. By Lemma 9.4, we have

$$\overline{\partial}(V_\phi f) = f\overline{\partial}\phi + \phi\overline{\partial}f + \frac{1}{\pi}\overline{\partial}(\mathcal{C}(f\overline{\partial}\phi)) = \phi\overline{\partial}f = \overline{\partial}\left(-\frac{1}{\pi}\mathcal{C}(\phi\overline{\partial}f)\right).$$

But both $V_\phi f$ and $-\frac{1}{\pi}\mathcal{C}(\phi\overline{\partial}f)$ are analytic in a neighborhood of ∞ and vanish at that point, hence these two distributions must be equal, again by Lemma 9.4. □

We can now prove Proposition 9.3.

Proof. Let $U \subset \mathbb{C}$ be open and let $E \subset \mathbb{C}$ be compact. To prove (i), suppose that $\gamma(E) = 0$, and let f be any bounded analytic function on $U \setminus E$. We may assume that U is bounded. Consider a grid of squares $\{Q_j\}$ of side length l covering the plane. Let $\{\phi_j\} \subset C_c^\infty(\mathbb{C})$ be a partition of unity subordinated to $\{2Q_j\}$, i.e. $0 \leq \phi_j \leq 1$, $\operatorname{supp}(\phi_j) \subset 2Q_j$ for each j and

$$\sum_j \phi_j \equiv 1$$

on $\mathbb{C}$. Now, define f to be zero on $\mathbb{C} \setminus (U \setminus E)$. Then $V_{\phi_j} f$ is identically zero except for finitely many j's and

$$f = -\frac{1}{\pi}\mathcal{C}(\overline{\partial} f) = -\frac{1}{\pi}\sum_j \mathcal{C}(\phi_j \overline{\partial} f) = \sum_j V_{\phi_j} f,$$

where we used Lemma 9.4 and Lemma 9.6. Also, for each j, we have

$$\operatorname{supp}(\overline{\partial}(V_{\phi_j} f)) = \operatorname{supp}(\overline{\partial}(\mathcal{C}(\phi_j \overline{\partial} f))) \subset \operatorname{supp}\phi_j \cap \operatorname{supp}\overline{\partial} f \subset 2Q_j \cap (E \cup \partial U).$$

Hence $V_{\phi_j} f$ is analytic outside $2Q_j \cap E$ whenever j is such that $2Q_j \cap \partial U = \emptyset$. Moreover, it is easy to see that $V_{\phi_j} f$ is bounded. Since $\gamma(2Q_j \cap E) = 0$, it follows from Property (P12) in Section 2 that $V_{\phi_j} f$ is constant. But $V_{\phi_j} f$ vanishes at ∞, so it must be identically zero. This is true for all j such that $2Q_j \cap \partial U = \emptyset$, thus

$$f = \sum_{j: 2Q_j \cap \partial U \neq \emptyset} V_{\phi_j} f.$$

In particular, the function f is analytic on U except maybe in a $4l$-neighborhood of ∂U. Since l is arbitrary, we obtain that f is analytic on the whole open set U. Finally, it must be bounded since E has empty interior.

The proof of (ii) is similar, except that in this case we assume that f is continuous on U and define f to be zero on $\mathbb{C} \setminus U$. The same argument as above shows that $V_{\phi_j} f$ is analytic outside $2Q_j \cap E$ whenever j is such that $2Q_j \cap \partial U = \emptyset$. Moreover, for such j, the function $V_{\phi_j} f$ is continuous on $\widehat{\mathbb{C}}$, as is easily seen from the definition of Vitushkin's localization operator

$$V_{\phi_j} f = \phi_j f + \frac{1}{\pi}\mathcal{C}(f \overline{\partial} \phi_j).$$

If $\alpha(E) = 0$, then $\alpha(2Q_j \cap E) = 0$ for each j and we deduce from Property (P13) in Section 2 that $V_{\phi_j} f \equiv 0$, for each j such that $2Q_j \cap \partial U = \emptyset$. It follows that f is analytic on the whole open set U.

□

We can now proceed with the proof of Lemma 9.2.

Proof. Let E and F be compact subsets of $\mathbb{C}$. First, suppose that $\gamma(E) = 0$. Note that $\gamma(E) \leq \gamma(E \cup F)$, by Property (P3) from Section 2. For the reverse inequality, let $f \in H^\infty(\widehat{\mathbb{C}} \setminus (E \cup F))$ with $|f| \leq 1$ on $\widehat{\mathbb{C}} \setminus (E \cup F)$. Since $\gamma(E) = 0$, we can use (i) in Proposition 9.3 with $U := \widehat{\mathbb{C}} \setminus F$ to deduce that f has an extension to a bounded analytic function on $\widehat{\mathbb{C}} \setminus F$, which we denote by the same letter. Note that the extension also satisfies $|f| \leq 1$, since E has empty interior. It follows that $|f'(\infty)| \leq \gamma(F)$. Taking the supremum over such f gives $\gamma(E \cup F) \leq \gamma(F)$, as required.

If $\alpha(E) = 0$, then the same argument but using (ii) in Proposition 9.3 shows that $\alpha(E \cup F) = \alpha(F)$. □

In addition to Lemma 9.2, the following result of Pommerenke will be needed for the proof of Lemma 9.1.

Theorem 9.7 (Pommerenke (1960))**.** *Let $F_1, \dots, F_n$ be compact subsets of $\mathbb{C}$. Then for every $\epsilon > 0$, there exists $\delta > 0$ with the following property:*

If $E_k := F_k + d_k$ for some $d_k \in \mathbb{C}$, $k = 1, \dots, n$, and if all distances between E_i and E_j $(i \neq j)$ are greater than δ, then

$$\left| \gamma\left(\bigcup_{k=1}^{n} E_k \right) - \sum_{k=1}^{n} \gamma(E_k) \right| < \epsilon.$$

See [33] or [26, Chapter 6, Section 5].

We can finally proceed with the proof of Lemma 9.1. Recall that $g' : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic motion and K' is a compact set such that

$$K'_\lambda = (E_\lambda + d) \cup F_\lambda \qquad (\lambda \in \mathbb{D}).$$

Here

$$\gamma(E_{\lambda_n}) = 0 \qquad (n \in \mathbb{N})$$

and

$$\alpha(F_{\lambda_n}) = 1 = \gamma(F_{\lambda_n}) \qquad (n \in \mathbb{N})$$

for some sequence $(\lambda_n) \subset \mathbb{D} \setminus \{0\}$ converging to 0. Also, the compact set E is contained in $F = \partial\mathbb{D}$ and satisfies $\gamma(E) > 0$.

Proof. We first show that the function

$$\lambda \mapsto \alpha(K'_\lambda) \qquad (\lambda \in \mathbb{D})$$

is discontinuous at 0. For this, note that by Property (P1) and Property (P2) from Section 2, we have

$$\alpha(E_{\lambda_n} + d) = \alpha(E_{\lambda_n}) \leq \gamma(E_{\lambda_n}) = 0 \qquad (n \in \mathbb{N}).$$

It then follows from (ii) in Lemma 9.2 that

$$\alpha(K'_{\lambda_n}) = \alpha((E_{\lambda_n} + d) \cup F_{\lambda_n}) = \alpha(F_{\lambda_n}) = 1 \qquad (n \in \mathbb{N}).$$

On the other hand, we have that $\mathcal{H}^1(K') = \mathcal{H}^1((E+d) \cup F) < \infty$, hence $\alpha(K') = 0$ by Property (P17) from Section 2. This shows that the function

$$\lambda \mapsto \alpha(K'_\lambda) \qquad (\lambda \in \mathbb{D})$$

is discontinuous at 0.

It remains to prove that

$$\lambda \mapsto \gamma(K'_\lambda) \qquad (\lambda \in \mathbb{D})$$

is discontinuous at 0. For this, first observe that similarly as above, we have $\gamma(K'_{\lambda_n}) = 1$ for all $n \in \mathbb{N}$. Moreover, note that all the previously obtained estimates for $\gamma(K'_{\lambda_n})$, $\alpha(K'_{\lambda_n})$ and $\alpha(K')$ are independent of d as long as it is large enough. We now choose the value of d. By Theorem 9.7 with $n = 2$, $\epsilon := \gamma(E) > 0$, $F_1 := E$, $F_2 := F$, $d_1 := d$, $d_2 := 0$, we can take $d > 0$ large enough so that

$$|\gamma((E+d) \cup F) - \gamma(E+d) - \gamma(F)| < \gamma(E).$$

Then

$$\begin{aligned}\gamma((E+d)\cup F) &> \gamma(E+d)+\gamma(F)-\gamma(E)\\ &= \gamma(E)+\gamma(F)-\gamma(E)\\ &= \gamma(F)=1,\end{aligned}$$

where we used Property (P2), Property (P5) and Property (P9) from Section 2. Summarizing, we have $\gamma(K'_{\lambda_n})=1$ for all $n\in\mathbb{N}$ but $\gamma(K')=\gamma((E+d)\cup F)>1$. It follows that the function

$$\lambda\mapsto\gamma(K'_\lambda)\qquad(\lambda\in\mathbb{D})$$

is discontinuous at 0, as required. □

Remark. For the proof, only the following version of Lemma 9.2 was needed:

If E and F are disjoint compact subsets of $\mathbb{C}$ and if $\gamma(E)=0$, then $\gamma(E\cup F)=\gamma(F)$ and $\alpha(E\cup F)=\alpha(F)$.

The above is much easier to prove. However, we decided to give a proof of the more general Lemma 9.2 since (ii) appears to be new and may be of independent interest.

9.2. **Second step: Countably many discontinuities.** We now complete the proof of Corollary 1.11 by showing how to modify the compact set K' and the holomorphic motion $g':\mathbb{D}\times\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ from Lemma 9.1 in order to obtain a discontinuity at each point of a given Blaschke sequence.

Proof. Let (β_j) be a sequence in $\mathbb{D}$ satisfying the Blaschke condition

$$\sum_{j=1}^{\infty}(1-|\beta_j|)<\infty.$$

Let K' and $g':\mathbb{D}\times\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be as in Lemma 9.1, so that both functions

$$\lambda\mapsto\gamma(K'_\lambda),\quad \lambda\mapsto\alpha(K'_\lambda)\qquad(\lambda\in\mathbb{D})$$

are discontinuous at 0.

Let $b:\mathbb{D}\to\mathbb{D}$ be the Blaschke product

$$b(z):=\prod_{j=1}^{\infty}\frac{-|\beta_j|}{\beta_j}\frac{z-\beta_j}{1-\overline{\beta_j}z}\qquad(z\in\mathbb{D}).$$

Then b is well-defined and analytic on $\mathbb{D}$ since (β_j) satisfies the Blaschke condition. Let $K:=K'_{b(0)}$ and define $g:\mathbb{D}\times\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ by

$$g(\lambda,z):=g'(b(\lambda),(g'_{b(0)})^{-1}(z))\qquad(\lambda\in\mathbb{D},z\in\widehat{\mathbb{C}}).$$

It is easy to see that g defines a holomorphic motion of $\widehat{\mathbb{C}}$ with

$$K_\lambda:=g_\lambda(K)=K'_{b(\lambda)}\qquad(\lambda\in\mathbb{D}).$$

Hence, if

$$\psi(\lambda):=\gamma(K_\lambda)\qquad(\lambda\in\mathbb{D})$$

and

$$\phi(\lambda):=\gamma(K'_\lambda)\qquad(\lambda\in\mathbb{D}),$$

then $\psi=\phi\circ b$. We now show that ψ is discontinuous at each β_j. Suppose for a contradiction that ψ is continuous at some β_j. Let $B\subset\mathbb{D}$ be a closed disk centered at β_j sufficiently small so that B contains no other zero of b. Note that $b(B)$ is a neighborhood of 0. Let (λ_n) be any sequence in $b(B)$ converging to 0. We show that $\phi(\lambda_n) \to \phi(0)$ as $n \to \infty$, contradicting the fact that ϕ is discontinuous at 0. In order to see this, let (λ_{n_k}) be an arbitrary subsequence of (λ_n). For each k, we have $\lambda_{n_k} = b(\eta_k)$ for $\eta_k \in B$. Passing to a subsequence if necessary, we may assume that (η_k) converges to a point in B, say η_0. By continuity, we have $b(\eta_0) = 0$, so that η_0 must be equal to β_j, by the choice of B. We obtain

$$\begin{aligned}\phi(0) = \psi(\beta_j) &= \lim_{k\to\infty} \psi(\eta_k)\\ &= \lim_{k\to\infty} \phi(b(\eta_k))\\ &= \lim_{k\to\infty} \phi(\lambda_{n_k}).\end{aligned}$$

This shows that every subsequence of $(\phi(\lambda_n))$ has a subsequence converging to $\phi(0)$. Equivalently, the whole sequence $(\phi(\lambda_n))$ converges to $\phi(0)$. As mentioned, this contradicts the fact that ϕ is discontinuous at 0, and we deduce that the function

$$\lambda \mapsto \gamma(K_\lambda) \qquad (\lambda \in \mathbb{D})$$

is discontinuous at each β_j. The same argument works with γ replaced by α, as required. □

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