A HOLOMORPHIC MOTION WITH NON-SUBHARMONIC HAUSDORFF DIMENSION

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ABSTRACT. We present an example of a holomorphic motion of a Cantor set $\lambda \mapsto A_\lambda$ for which neither the Hausdorff dimension $\dim_{\mathrm{H}}(A_\lambda)$ nor the lower Minkowski dimension $\underline{\dim}_{\mathrm{M}}(A_\lambda)$ is a subharmonic function of λ. This answers negatively a question of Fuhrer–Ransford–Younsi, and distinguishes these dimensions from packing and upper Minkowski dimensions, for which subharmonicity holds. All four dimensions are computed explicitly.

1. INTRODUCTION

In what follows, we write $\mathbb{D}$ and $\mathbb{C}$ for the open unit disk and the complex plane, respectively.

Definition 1.1. Let A be a subset of $\mathbb{C}$. A *holomorphic motion* of A is a map $f : \mathbb{D} \times A \to \mathbb{C}$ such that:

(i) for each fixed $z \in A$, the map $\lambda \mapsto f(\lambda, z)$ is holomorphic on $\mathbb{D}$;
(ii) for each fixed $\lambda \in \mathbb{D}$, the map $z \mapsto f(\lambda, z)$ is injective on A;
(iii) for all $z \in A$, we have $f(0, z) = z$.

We write $f_\lambda(z) := f(\lambda, z)$ and $A_\lambda := f_\lambda(A)$. We sometimes abuse terminology and call the set-valued map $\lambda \mapsto A_\lambda$ a holomorphic motion of A.

Definition 1.2. A positive function $u : \mathbb{D} \to [0, \infty)$ is called *inf-harmonic* if there exists a family $\mathcal{H}$ of harmonic functions on $\mathbb{D}$ such that $u(\lambda) = \inf_{h \in \mathcal{H}} h(\lambda)$ for all $\lambda \in \mathbb{D}$.

For more details on holomorphic motions and inf-harmonic functions, we refer the reader to [2]. In that paper, the authors studied the variation of dimension under holomorphic motions and proved the following results. We denote by $\dim_{\mathrm{H}}, \dim_{\mathrm{P}}, \underline{\dim}_{\mathrm{M}}, \overline{\dim}_{\mathrm{M}}$ the Hausdorff, packing, lower Minkowski and upper Minkowski dimensions respectively. In the first result dim denotes either $\overline{\dim}_{\mathrm{M}}$ or $\dim_{\mathrm{P}}$.

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Theorem 1.3 (Fuhrer–Ransford–Younsi)**.** *Let* dim *denote either of* $\dim_{\mathrm{P}}$ *or* $\overline{\dim}_{\mathrm{M}}$*, and let* $f : \mathbb{D} \times A \to \mathbb{C}$ *be a holomorphic motion of a set* $A \subset \mathbb{C}$*, which we assume to be bounded in the case* $\dim = \overline{\dim}_{\mathrm{M}}$*. Then either* $\dim(A_\lambda) = 0$ *for all* $\lambda \in \mathbb{D}$*, or* $\lambda \mapsto 1/\dim(A_\lambda)$ *is an inf-harmonic function on* $\mathbb{D}$*. In particular,* $\lambda \mapsto \dim(A_\lambda)$ *is a subharmonic function on* $\mathbb{D}$*.*

The authors also proved that, if $\dim_{\mathrm{H}}(A_0) > 0$, then the function $\lambda \mapsto 1/\dim_{\mathrm{H}}(A_\lambda)$ is the supremum of a family of inf-harmonic functions, and asked whether it is in fact inf-harmonic; see [2, Question 11.1]. They remarked that a positive answer would imply that $\lambda \mapsto \dim_{\mathrm{H}}(A_\lambda)$ is a subharmonic function, and that even this weaker statement was still an open problem.

The goal of this note is to answer both of these questions in the negative, not only for $\dim_{\mathrm{H}}$, but also for $\underline{\dim}_{\mathrm{M}}$. Our main result is as follows.

Theorem 1.4. *There exist a compact set* $A \subset \mathbb{C}$ *and a holomorphic motion* $f : \mathbb{D} \times A \to \mathbb{C}$ *such that, writing* $M(\lambda) := \max\{|\operatorname{Re}\lambda|, |\operatorname{Im}\lambda|\}$*, we have*

$$\dim_{\mathrm{H}}(A_\lambda) = \underline{\dim}_{\mathrm{M}}(A_\lambda) = \frac{1}{3 + M(\lambda)} \qquad (\lambda \in \mathbb{D}), \tag{1.1}$$

and

$$\dim_{\mathrm{P}}(A_\lambda) = \overline{\dim}_{\mathrm{M}}(A_\lambda) = \frac{1}{3 - M(\lambda)} \qquad (\lambda \in \mathbb{D}). \tag{1.2}$$

In particular, as both $\dim_{\mathrm{H}}(A_\lambda)$ *and* $\underline{\dim}_{\mathrm{M}}(A_\lambda)$ *attain a strict maximum at* $\lambda = 0$*, neither is subharmonic, and the reciprocal of neither is inf-harmonic.*

We give the construction and its verification in the following sections, and we make some general comments on the example at the end.

2. Construction

Set

$$N_0 := 0, \qquad N_k := k! \quad (k \geq 1),$$

and define

$$\sigma_n := i^{\,k-1} \qquad \text{if } N_{k-1} < n \leq N_k,$$

so the sequence (σ_n) contains the values $1, i, -1, -i$ alternating cyclically in constant blocks. We also define

$$S_0 := 0, \qquad S_n := \sum_{j=1}^{n} \sigma_j \quad (n \geq 1).$$

Lemma 2.1. *For each* $\lambda \in \mathbb{C}$*, we have*

$$\limsup_{n\to\infty} \operatorname{Re}\left(\lambda \frac{S_n}{n}\right) = \max\{|\operatorname{Re}\lambda|, |\operatorname{Im}\lambda|\} \tag{2.1}$$

and

$$\liminf_{n\to\infty} \operatorname{Re}\left(\lambda \frac{S_n}{n}\right) = -\max\{|\operatorname{Re}\lambda|, |\operatorname{Im}\lambda|\}. \tag{2.2}$$

Proof. It suffices to prove (2.1), since (2.2) then follows by replacing λ with $-\lambda$.

Note that for $k \geq 2$ we have

$$\begin{aligned} \left| \frac{S_{N_k}}{N_k} - i^{\,k-1} \right| &= \left| \frac{S_{N_{k-1}} - N_{k-1} i^{\,k-1}}{N_k} \right| \\ &\leq \frac{2N_{k-1}}{N_k} = \frac{2}{k}. \end{aligned} \tag{2.3}$$

In particular each of $1, i, -1, -i$ is the limit of some subsequence of S_n/n, from which it easily follows that

$$\max\{|\operatorname{Re}\lambda|, |\operatorname{Im}\lambda|\} \leq \limsup_{n\to\infty} \operatorname{Re}\left(\lambda \frac{S_n}{n}\right).$$

For the reverse inequality, note that every σ_j belongs to $\{1, i, -1, -i\}$, and thus

$$\operatorname{Re}\left(\lambda \frac{S_n}{n}\right) = \frac{1}{n}\sum_{j=1}^{n} \operatorname{Re}(\lambda\sigma_j) \leq \max\{|\operatorname{Re}\lambda|, |\operatorname{Im}\lambda|\},$$

as required. □

Now, for $n \geq 0$ and $\lambda \in \mathbb{D}$, define

$$R_n(\lambda) := 2^{-3n-\lambda S_n}.$$

Let $\Omega := \{0,1\}^{\mathbb{N}}$, equipped with the product topology. For $\lambda \in \mathbb{D}$ and $\varepsilon = (\varepsilon_n)_{n\geq 1} \in \Omega$, define

$$\pi_\lambda(\varepsilon) := \sum_{n=1}^{\infty} \varepsilon_n R_n(\lambda).$$

Lemma 2.2. (i) *For each $\lambda \in \mathbb{D}$, the function $\pi_\lambda : \Omega \to \mathbb{C}$ is continuous and injective.*

(ii) *For each $\varepsilon \in \Omega$, the function $\lambda \mapsto \pi_\lambda(\varepsilon)$ is holomorphic on $\mathbb{D}$.*

Proof. Note that $R_0 = 1$. For $n \in \mathbb{N}$ and $\lambda \in \mathbb{D}$, we have

$$\frac{R_n(\lambda)}{R_{n-1}(\lambda)} = 2^{-3-\lambda\sigma_n}.$$

Since $|\sigma_n| = 1$ and $|\lambda| < 1$, this implies that

$$\frac{1}{16} \leq \left| \frac{R_n(\lambda)}{R_{n-1}(\lambda)} \right| \leq \frac{1}{4}. \tag{2.4}$$

In particular $|R_n(\lambda)| \leq 4^{-n}$. It follows that the series

$$\pi_\lambda(\varepsilon) := \sum_{n=1}^{\infty} \varepsilon_n R_n(\lambda)$$

converges uniformly in both $\lambda \in \mathbb{D}$ and $\varepsilon \in \Omega$. Thus the function $\pi_\lambda : \Omega \to \mathbb{C}$ is continuous, and, for each fixed $\varepsilon \in \Omega$, the function $\lambda \mapsto \pi_\lambda(\varepsilon)$ is holomorphic on $\mathbb{D}$.

It remains to show that π_λ is injective. Let $\varepsilon, \eta \in \Omega$ with $\varepsilon \neq \eta$, and suppose that they first differ at position m. By (2.4), we have

$$\sum_{n>m} |R_n(\lambda)| \leq |R_m(\lambda)| \sum_{j=1}^{\infty} 4^{-j} = \frac{1}{3}|R_m(\lambda)|. \tag{2.5}$$

As $|\varepsilon_m - \eta_m| = 1$ and $|\varepsilon_n - \eta_n| \leq 1$ for every $n > m$, we obtain

$$\begin{aligned} |\pi_\lambda(\varepsilon) - \pi_\lambda(\eta)| &= \left| (\varepsilon_m - \eta_m) R_m(\lambda) + \sum_{n>m} (\varepsilon_n - \eta_n) R_n(\lambda) \right| \\ &\geq |R_m(\lambda)| - \sum_{n>m} |R_n(\lambda)| \geq \frac{2}{3}|R_m(\lambda)|. \end{aligned}$$

Therefore $\pi_\lambda(\varepsilon) \neq \pi_\lambda(\eta)$ and $\pi_\lambda : \Omega \to \mathbb{C}$ is injective, as required. □

Now define

$$A := \pi_0(\Omega) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 8^{-n} : \varepsilon_n \in \{0, 1\} \right\}.$$

Geometrically, A is a Cantor set in $\mathbb{R}$ constructed as follows. Starting with $[0, 1/7]$, we retain the leftmost and rightmost one-eighths of the interval, removing the open middle three-quarters. We repeat this procedure in each remaining interval. The set A is the intersection of the resulting decreasing sequence of compact sets.

Define $f : \mathbb{D} \times A \to \mathbb{C}$ by

$$f\big(\lambda, \pi_0(\varepsilon)\big) := \pi_\lambda(\varepsilon).$$

By Lemma 2.2, the map f defines a holomorphic motion of A. Moreover, for each $\lambda \in \mathbb{D}$, the set $A_\lambda = \pi_\lambda(\Omega)$ is compact, since Ω is compact and π_λ is continuous.

3. Hausdorff dimension

In this section we prove the desired formula for the Hausdorff dimension of the sets A_λ from Theorem 1.4, namely

$$\dim_{\mathrm{H}}(A_\lambda) = \frac{1}{3 + M(\lambda)} \qquad (\lambda \in \mathbb{D}), \tag{3.1}$$

where $M(\lambda) := \max\{|\operatorname{Re}\lambda|, |\operatorname{Im}\lambda|\}$.

Lemma 3.1. *For each $\lambda \in \mathbb{D}$, we have*

$$\dim_{\mathrm{H}}(A_\lambda) = \liminf_{n\to\infty} \frac{n \log 2}{-\log |R_n(\lambda)|}.$$

Equation (3.1) easily follows from Lemma 3.1. Indeed, by definition of $R_n(\lambda)$, we have

$$-\log|R_n(\lambda)| = (3n + \mathrm{Re}(\lambda S_n))\log 2.$$

Combining this with (2.1), we obtain

$$\begin{aligned}\dim_{\mathrm{H}}(A_\lambda) &= \liminf_{n\to\infty} \frac{1}{3 + \mathrm{Re}(\lambda S_n/n)} \\ &= \frac{1}{3 + \limsup_{n\to\infty} \mathrm{Re}(\lambda S_n/n)} \\ &= \frac{1}{3 + \max\{|\operatorname{Re}\lambda|, |\operatorname{Im}\lambda|\}} \\ &= \frac{1}{3 + M(\lambda)},\end{aligned}$$

as required.

For the proof of Lemma 3.1, we introduce the following definition.

Definition 3.2. Fix $\lambda \in \mathbb{D}$. A *cylinder* of length n is the subset of A_λ obtained by fixing the first n binary digits $\varepsilon_1, \ldots, \varepsilon_n$ in

$$\pi_\lambda(\varepsilon) := \sum_{j=1}^{\infty} \varepsilon_j R_j(\lambda).$$

We will need the following properties of cylinders.

Proposition 3.3. *Fix $\lambda \in \mathbb{D}$ and write $\ell_n := |R_n(\lambda)|$.*

(i) *Each cylinder of length n has diameter at most $\ell_n/3$.*
(ii) *The distance between two distinct cylinders of length n is at least $2\ell_n/3$.*

Proof. By definition, the diameter of any cylinder of length n is at most

$$\sum_{m>n} \ell_m \le \frac{1}{3}\ell_n,$$

by (2.5). This proves (i).

For (ii), let $\pi_\lambda(\varepsilon)$ and $\pi_\lambda(\eta)$ belong to distinct cylinders of length n. Then ε and η first differ at some position $m \le n$. As in the proof of Lemma 2.2, we obtain

$$|\pi_\lambda(\varepsilon) - \pi_\lambda(\eta)| \ge |R_m(\lambda)| - \sum_{j>m} |R_j(\lambda)| \ge \frac{2}{3}\ell_m \ge \frac{2}{3}\ell_n,$$

as required. □

Proof of Lemma 3.1. Fix $\lambda \in \mathbb{D}$, and again write $\ell_n = |R_n(\lambda)|$. Let

$$D := \liminf_{n\to\infty} \frac{n\log 2}{-\log \ell_n}.$$

We have to show that $\dim_{\mathrm{H}}(A_\lambda) = D$.

First note that there are 2^n cylinders of length n, and that each has diameter at most $\ell_n/3$ by Proposition 3.3. Fix $s > D$, and choose t with $D < t < s$. There are infinitely many n for which

$$\frac{n \log 2}{-\log \ell_n} < t.$$

Along this subsequence,

$$2^n \left(\frac{\ell_n}{3}\right)^s \leq 3^{-s} \ell_n^{s-t} \longrightarrow 0.$$

The cylinder covers therefore give the estimate $\dim_{\mathrm{H}}(A_\lambda) \leq s$. As this holds for each $s > D$, we conclude that $\dim_{\mathrm{H}}(A_\lambda) \leq D$.

For the reverse inequality, we consider the push-forward under π_λ of the product probability measure on Ω that gives probability $1/2$ to each digit. The resulting probability measure μ on A_λ assigns mass 2^{-n} to every cylinder of length n.

Fix $0 < s < D$. For all sufficiently large n, we have

$$2^{-n} \leq \ell_n^s. \tag{3.2}$$

For a sufficiently small radius $r > 0$, choose n such that

$$\frac{\ell_n}{3} \leq r < \frac{\ell_{n-1}}{3}. \tag{3.3}$$

By Proposition 3.3, the distance between two different cylinders of length $n-1$ is at least

$$\frac{2}{3}\ell_{n-1} > 2r.$$

A ball of radius r thus meets at most one cylinder of length $n-1$. Combining (3.2) and (3.3), we obtain

$$\mu(B(z,r)) \leq 2^{-(n-1)} = 2\,2^{-n} \leq 2\ell_n^s \leq 2(3r)^s. \tag{3.4}$$

This holds for all centers $z \in \mathbb{C}$ and all sufficiently small r. It follows from the Mass Distribution Principle [1, Lemma 1.2.8] that $\dim_{\mathrm{H}}(A_\lambda) \geq s$. Since this holds for every $0 < s < D$, we deduce that $\dim_{\mathrm{H}}(A_\lambda) \geq D$, as required. □

4. Lower and upper Minkowski dimensions

In this section we prove the formulas for the lower and upper Minkowski dimensions of the sets A_λ from Theorem 1.4, namely

$$\underline{\dim}_{\mathrm{M}}(A_\lambda) = \frac{1}{3 + M(\lambda)} \qquad (\lambda \in \mathbb{D}) \tag{4.1}$$

and

$$\overline{\dim}_{\mathrm{M}}(A_\lambda) = \frac{1}{3 - M(\lambda)} \qquad (\lambda \in \mathbb{D}), \tag{4.2}$$

where $M(\lambda) := \max\{|\operatorname{Re}\lambda|, |\operatorname{Im}\lambda|\}$.

Lemma 4.1. *For each $\lambda \in \mathbb{D}$, we have*

$$\underline{\dim}_{\mathrm{M}}(A_\lambda) = \liminf_{n\to\infty} \frac{n \log 2}{-\log \ell_n}$$

and

$$\overline{\dim}_{\mathrm{M}}(A_\lambda) = \limsup_{n\to\infty} \frac{n \log 2}{-\log \ell_n},$$

where $\ell_n = |R_n(\lambda)|$.

Equations (4.1) and (4.2) easily follow from Lemma 4.1, using the relation

$$\frac{n \log 2}{-\log \ell_n} = \frac{1}{3 + \mathrm{Re}(\lambda S_n/n)}$$

and Lemma 2.1.

Proof of Lemma 4.1. Fix $\lambda \in \mathbb{D}$, and again set $\ell_n := |R_n(\lambda)|$. For a bounded set $E \subset \mathbb{C}$ and $\delta > 0$, let $N_\delta(E)$ be the smallest number of sets of diameter at most δ needed to cover E. Recall that the lower and upper Minkowski dimensions are defined by

$$\underline{\dim}_{\mathrm{M}}(E) := \liminf_{\delta\downarrow 0} \frac{\log N_\delta(E)}{\log(1/\delta)}, \qquad \overline{\dim}_{\mathrm{M}}(E) := \limsup_{\delta\downarrow 0} \frac{\log N_\delta(E)}{\log(1/\delta)}.$$

For any sufficiently small $\delta > 0$, choose n such that

$$\frac{\ell_n}{3} \le \delta < \frac{\ell_{n-1}}{3}. \tag{4.3}$$

The cylinders of length n provide a cover of A_λ, whereas a set of diameter at most δ cannot meet two cylinders of length $n-1$, by Proposition 3.3. Consequently,

$$2^{n-1} \le N_\delta(A_\lambda) \le 2^n. \tag{4.4}$$

The bounds $1/16 \le \ell_n/\ell_{n-1} \le 1/4$ from (2.4) imply that, for the choice of n in (4.3),

$$-\log \delta = -\log \ell_n + O(1), \qquad \log N_\delta(A_\lambda) = n \log 2 + O(1).$$

The constants are independent of n and δ. It follows that

$$\underline{\dim}_{\mathrm{M}}(A_\lambda) = \liminf_{n\to\infty} \frac{n \log 2}{-\log \ell_n}$$

and

$$\overline{\dim}_{\mathrm{M}}(A_\lambda) = \limsup_{n\to\infty} \frac{n \log 2}{-\log \ell_n},$$

as required. □

5. Packing dimension

In this section we prove the desired formula for the packing dimension of the sets A_λ from Theorem 1.4, namely

$$\dim_{\mathrm{P}}(A_\lambda) = \frac{1}{3 - M(\lambda)} \qquad (\lambda \in \mathbb{D}), \tag{5.1}$$

where $M(\lambda) := \max\{|\operatorname{Re}\lambda|, |\operatorname{Im}\lambda|\}$.

Lemma 5.1. *We have*

$$\dim_{\mathrm{P}}(A_\lambda) = \overline{\dim}_{\mathrm{M}}(A_\lambda) \qquad (\lambda \in \mathbb{D}).$$

Equation (5.1) follows directly from Lemma 5.1 and (4.2).

Proof of Lemma 5.1. We use the following characterization of packing dimension due to Tricot:

$$\dim_{\mathrm{P}}(E) = \inf\left\{ \sup_{j\geq 1} \overline{\dim}_{\mathrm{M}}(E_j) : E = \bigcup_{j\geq 1} E_j \right\}, \tag{5.2}$$

where the infimum is over countable covers by bounded subsets E_j of E; see [2, Proposition 2.3].

Fix $\lambda \in \mathbb{D}$, write $\ell_n := |R_n(\lambda)|$, and put

$$p := \overline{\dim}_{\mathrm{M}}(A_\lambda) = \frac{1}{3 - M(\lambda)}.$$

Taking the cover consisting of A_λ alone in (5.2) gives $\dim_{\mathrm{P}}(A_\lambda) \leq p$.

For the reverse inequality, let μ be the probability measure on A_λ which assigns mass 2^{-n} to every cylinder of length n. Write μ^* for the associated outer measure.

Suppose that $E \subset A_\lambda$ satisfies $\mu^*(E) > 0$. Set $a := \mu^*(E)$. Let m_n be the number of cylinders of length n intersecting E. The union of all such cylinders contains E, and hence

$$a \leq m_n 2^{-n}.$$

By Proposition 3.3, distinct cylinders of length n are separated by at least $2\ell_n/3$. In particular a set of diameter at most $\ell_n/3$ cannot meet two of them. It follows that

$$N_{\ell_n/3}(E) \geq m_n \geq a\, 2^n.$$

Taking a limsup along these scales and using Lemma 4.1 gives

$$\overline{\dim}_{\mathrm{M}}(E) \geq \limsup_{n\to\infty} \frac{n\log 2 + \log a}{-\log \ell_n + \log 3} = p.$$

Monotonicity gives the reverse inequality, because $E \subset A_\lambda$. We have therefore shown that, whenever $E \subset A_\lambda$,

$$\mu^*(E) > 0 \quad \Longrightarrow \quad \overline{\dim}_{\mathrm{M}}(E) = p. \tag{5.3}$$

Now, for any countable cover $A_\lambda = \bigcup_{j\geq 1} E_j$, at least one E_j has positive μ-outer measure. By (5.3), that set has upper Minkowski dimension p. Thus every cover in (5.2) satisfies

$$\sup_{j\geq 1} \overline{\dim}_{\mathrm{M}}(E_j) \geq p,$$

from which it follows that $\dim_{\mathrm{P}}(A_\lambda) \geq p$, as required. □

6. Concluding remarks

Summarizing the results of Sections 3, 4 and 5, we have established Equations (1.1) and (1.2) in Theorem 1.4. The last sentence in the statement of the theorem is straightforward: the Hausdorff and lower Minkowski dimension functions both attain a strict maximum at 0, so, by the maximum principle, neither function can be subharmonic, and hence neither reciprocal is inf-harmonic. Thus Theorem 1.4 is proved.

We conclude by mentioning that, for the holomorphic motion $f : \mathbb{D}\times A \to \mathbb{C}$ of Theorem 1.4, the reciprocals of packing dimension and upper Minkowski dimension are inf-harmonic, which is consistent with Theorem 1.3. Indeed, for $\lambda := x + iy \in \mathbb{D}$, we have

$$\frac{1}{\dim_{\mathrm{P}}(A_\lambda)} = \frac{1}{\overline{\dim}_{\mathrm{M}}(A_\lambda)} = 3 - M(\lambda) = \min\{3+x, 3-x, 3+y, 3-y\},$$

and each function in this minimum is positive and harmonic on $\mathbb{D}$.

The example is also consistent with Theorem 1.6 of [2], which asserts that $1/\dim_{\mathrm{H}}(A_\lambda) - 1/2$ is a supremum of inf-harmonic functions. Indeed, again writing $\lambda = x + iy$, we have

$$\frac{1}{\dim_{\mathrm{H}}(A_\lambda)} - \frac{1}{2} = \max\left\{\frac{5}{2}+x, \frac{5}{2}-x, \frac{5}{2}+y, \frac{5}{2}-y\right\},$$

which is a maximum of four positive harmonic functions on $\mathbb{D}$.

AI tools declaration

In preparing this work, the authors used ChatGPT to develop and explore strategies for the proof of Theorem 1.4. The final mathematical text was written entirely by the authors, who thoroughly corrected, verified, and streamlined the proposed strategy. The authors accept full responsibility for the accuracy and integrity of the final results presented in this paper.

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