PEANO CURVES IN COMPLEX ANALYSIS

MALIK YOUNSI

Abstract. A Peano curve is a continuous function from the unit interval into the plane whose image contains a nonempty open set. In this note, we show how such space-filling curves arise naturally from Cauchy transforms in complex analysis.

1. Introduction.

A *Peano curve* (or *space-filling* curve) is a continuous function $f : [0,1] \to \mathbb{C}$, where $\mathbb{C}$ denotes the complex plane, such that $f([0,1])$ contains a nonempty open set.

The first example of such a curve was constructed by Peano [6] in 1890, motivated by Cantor's proof of the fact that the unit interval and the unit square have the same cardinality. Indeed, Peano's construction has the property that f maps $[0,1]$ continuously onto $[0,1] \times [0,1]$. Note, however, that topological considerations prevent such a function f from being injective.

One year later, in 1891, Hilbert [3] constructed another example of a space-filling curve, as a limit of piecewise-linear curves. Hilbert's elegant geometric construction has now become quite classical and is usually taught at the undergraduate level.

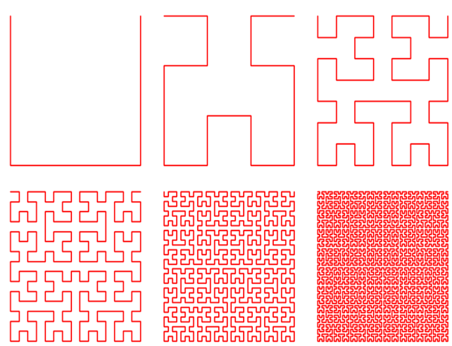

Figure 1. The first six steps of Hilbert's iterative construction of a Peano curve.

However, much less known is the fact that Peano curves can be obtained by the use of complex-analytic methods, more precisely, from the boundary values of certain power series defined on the unit disk. This was observed by Salem and Zygmund in 1945 in the following theorem:

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Theorem 1.1 (Salem–Zygmund [9]). *Let $f(z) = \sum_k a_k z^{n_k}$ be a lacunary power series, meaning that there is a constant $\lambda > 1$ such that*

$$\frac{n_{k+1}}{n_k} \geq \lambda \qquad (k \geq 1).$$

Suppose moreover that $\sum_k |a_k| < \infty$, so that f defines a continuous function on the closed unit disk $\overline{\mathbb{D}}$ that is analytic on $\mathbb{D}$.

Then there is an absolute constant λ_0 such that if $\lambda \geq \lambda_0$ and if $\sum_k |a_k|$ converges slowly enough (in some precise sense), then $f(\partial\mathbb{D})$ contains a nonempty open set.

Note that if f is as in Theorem 1.1, then $t \mapsto f(e^{2\pi i t})$ defines a Peano curve, by definition.

A few years later, in 1952, Piranian, Titus and Young [8] gave a particularly simple example showing that one can construct f such that $f(\partial\mathbb{D}) = f(\overline{\mathbb{D}}) = [0,1] \times [0,1]$. This was later extended to a whole class of series by Schaeffer [10]. See also [4] for other results on Peano curves and power series, as well as [2] and [7] for Peano curves arising from function algebras.

The purpose of this note is to show that Peano curves can also be constructed using Cauchy integrals. The proof relies on a surprisingly little-known folklore theorem from complex analysis as well as on a classical lemma in geometric measure theory due to Frostman.

2. A folklore theorem.

In the following, we denote the Riemann sphere by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Theorem 2.1. *Let $E \subset \mathbb{C}$ be a nonempty compact set, and let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a bounded continuous function analytic on $\widehat{\mathbb{C}} \setminus E$. Then*

$$f(E) = f(\widehat{\mathbb{C}}).$$

In other words, every value taken by f in the sphere is also taken by f in E.

Theorem 2.1 appears in Browder's textbook on function algebras [1, Lemma 3.5.4] in the case where E has empty interior, with some details left to the reader. We supply all the details in the general case for the sake of convenience.

Proof. Clearly $f(E) \subset f(\widehat{\mathbb{C}})$. For the other inclusion, let $w \in \widehat{\mathbb{C}}$. We have to show that if $w \in f(\widehat{\mathbb{C}})$, then there exists $z \in E$ with $f(z) = w$. Replacing f by $f - w$ if necessary, we may assume that $w = 0$.

Suppose, in order to obtain a contradiction, that f has a zero in $\widehat{\mathbb{C}}$ but no zero in E. First, note that f cannot have zeros tending to ∞. Indeed, if this were the case, then f would have a nonisolated zero at ∞, in which case we would have $f \equiv 0$ on $\widehat{\mathbb{C}} \setminus E$ and hence $f \equiv 0$ on $\partial E \subset E$ by continuity, contradicting our assumption. It follows that f can have only finitely many zeros in the whole sphere, since otherwise a sequence of zeros would accumulate at a point of E and f would vanish at that point, again by continuity. Let $z_1, \dots, z_n$ denote the zeros of f, listed with multiplicities, and define

$$g(z) := \frac{f(z)}{(z - z_1)\cdots(z - z_n)} \qquad (z \in \widehat{\mathbb{C}}).$$

We do not include any z_j equal to ∞ in the above formula for g. In particular, we may have $g = f$, if f has only one zero, at ∞.

Now, note that g is a continuous and nonvanishing function in the plane, and therefore has a continuous logarithm $h : \mathbb{C} \to \mathbb{C}$. Moreover, the function h is necessarily analytic outside E, since g is analytic there. We claim that this contradicts the fact that $g(\infty) = 0$. Indeed, in order to see this, we consider the type of isolated singularity that h has at ∞ (i.e., the singularity of $h(1/z)$ at $z = 0$). If ∞ is a removable singularity of h, then the limit $\lim_{z\to\infty} h(z)$ exists, in which case $\lim_{z\to\infty} g(z) = \lim_{z\to\infty} e^{h(z)}$ would be a nonzero complex number, a contradiction. If h has an essential singularity at ∞, then by the Casorati–Weierstrass theorem, the set $h(\{|z| > R\})$ for $R > 0$ large enough is dense in $\mathbb{C}$, again contradicting the fact that $\lim_{z\to\infty} e^{h(z)} = 0$. The only remaining possibility is that ∞ is a pole of h. In this case, there exists some integer $n \geq 1$ and some nonzero complex number α such that

$$\lim_{|z|\to\infty} \frac{h(z)}{z^n} = \alpha.$$

Write $\alpha = |\alpha| e^{i\theta}$, where θ is the argument of the complex number α. Then we have

$$\lim_{|z|\to\infty} \frac{h(|z|e^{-i\theta/n})}{(|z|e^{-i\theta/n})^n} = |\alpha| e^{i\theta},$$

so that in particular there exists $M > 0$ such that

$$\mathrm{Re}(h(|z|e^{-i\theta/n})) \geq \frac{1}{2}|\alpha||z|^n \qquad (|z| > M).$$

Taking the exponential and noting that $|g| = |e^h| = e^{\mathrm{Re}\, h}$ gives

$$|g(|z|e^{-i\theta/n})| \geq e^{\frac{1}{2}|\alpha||z|^n} \qquad (|z| > M).$$

This contradicts the fact that the left-hand side tends to 0 as $|z| \to \infty$.

Since all possible cases lead to a contradiction, we get that $f(E) = f(\widehat{\mathbb{C}})$, as required.

$\square$

Remark. Theorem 2.1 is clearly interesting only if f is not constant. In this case, the set $f(\widehat{\mathbb{C}})$ is open, by the open mapping theorem. In particular, the set $f(E)$ has non-empty interior, even though E may not!

The above remark raises the following question: For which compact sets E does there exist a nonconstant bounded continuous function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that is analytic outside E? Can we find such sets with empty interior?

As we shall see in the next section, the answer is affirmative.

3. Peano curves from Cauchy integrals.

Theorem 3.1. *Let $E \subset \mathbb{C}$ be compact. Suppose that E has empty interior and that its Hausdorff dimension satisfies $\dim_{\mathcal{H}}(E) > 1$. Then there exists a bounded continuous function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, analytic on $\widehat{\mathbb{C}} \setminus E$, that is not constant.*

Remark. In other words, compact sets of dimension bigger than one are *nonremovable* for bounded continuous functions analytic outside the set. On the other hand, a well-known result generally attributed to Painlevé states that compact sets of Hausdorff dimension less than one are removable [11, Corollary 2.8]. This case is not interesting from the point of view of Theorem 2.1, since for such sets only constant functions satisfy the assumptions.

For example, in Theorem 3.1, one could take E to be a fractal curve Γ with Hausdorff dimension strictly between one and two, such as the Koch snowflake for instance. Combining Theorem 3.1 with Theorem 2.1 then yields examples of Peano curves.

Corollary 3.2. *Let Γ be any curve with $1 < \dim_{\mathcal{H}}(\Gamma) < 2$, and let f be as in Theorem 3.1. Then $f(\Gamma)$ is a Peano curve.*

4. Proof of Theorem 3.1.

For the proof of Theorem 3.1, we construct the function f as a Cauchy-type integral.

Suppose that $E \subset \mathbb{C}$ is a compact set with empty interior, and let μ be a nontrivial Radon measure supported on E. The function

$$\mathcal{C}\mu(z) := \int_E \frac{d\mu(\zeta)}{\zeta - z} \qquad (z \in \widehat{\mathbb{C}} \setminus E) \tag{1}$$

is called the *Cauchy transform* of the measure μ. By differentiating under the integral sign, one easily sees that $\mathcal{C}\mu$ defines an analytic function outside E. Moreover, that function is not constant, since

$$\lim_{z \to \infty} \mathcal{C}\mu(z) = 0,$$

whereas

$$\lim_{z \to \infty} z\mathcal{C}\mu(z) = -\mu(E) \neq 0.$$

Cauchy transforms are therefore good candidates for the function f in Theorem 3.1. The problem, however, is that in general $\mathcal{C}\mu$ may not bounded, let alone continuous on the sphere. For this to hold, we need additional assumptions on the measure μ.

Lemma 4.1. *Let $E \subset \mathbb{C}$ be a compact set with empty interior, and let μ be a nontrivial Radon measure supported on E. Suppose moreover that μ satisfies the growth condition*

$$\mu(\mathbb{D}(z_0, r)) \leq r^s \qquad (z_0 \in \mathbb{C}, r > 0),$$

for some $1 < s < 2$. Then the Cauchy transform $\mathcal{C}\mu$ defined by (1) is a nonconstant analytic function on $\widehat{\mathbb{C}} \setminus E$ that extends to a bounded continuous function on $\widehat{\mathbb{C}}$.

Proof. We already mentioned that $\mathcal{C}\mu$ is analytic outside E and not constant.

We show that the growth property of μ implies that $\mathcal{C}\mu$ is Hölder continuous outside E, so that in particular it extends to a bounded continuous function on the whole sphere, by uniform continuity. The argument is quite standard, see e.g., [11, Theorem 2.10]. Fix $z, w \in \mathbb{C} \setminus E$, $z \neq w$, and write $\delta := |z - w|$. Then

$$|\mathcal{C}\mu(z) - \mathcal{C}\mu(w)| \leq \delta \int \frac{d\mu(\zeta)}{|\zeta - z||\zeta - w|}.$$

We split the integral over the four disjoint sets

$$\begin{aligned}
A_1 &:= \{\zeta \in E : |\zeta - z| < \delta/2\},\\
A_2 &:= \{\zeta \in E : |\zeta - w| < \delta/2\},\\
A_3 &:= \{\zeta \in E : |\zeta - z| \leq |\zeta - w|, |\zeta - z| \geq \delta/2\},\\
A_4 &:= \{\zeta \in E : |\zeta - z| > |\zeta - w|, |\zeta - w| \geq \delta/2\}.
\end{aligned}$$

On A_1, $|\zeta - w| > \delta/2$, so we have

$$\begin{aligned}
\delta \int_{A_1} \frac{d\mu(\zeta)}{|\zeta - z||\zeta - w|} &\leq 2 \int_{A_1} \int_{|\zeta - z|}^{\infty} t^{-2} dt\, d\mu(\zeta)\\
&= 2 \int_{A_1} \int_{|\zeta - z|}^{\delta/2} t^{-2} dt\, d\mu(\zeta) + 2 \int_{A_1} \int_{\delta/2}^{\infty} t^{-2} dt\, d\mu(\zeta)\\
&\leq 2 \int_0^{\delta/2} \mu(\mathbb{D}(z,t)) t^{-2} dt + 4\delta^{-1} \mu(A_1)\\
&\leq 2 \int_0^{\delta/2} t^{s-2} dt + 4\delta^{-1} \delta^s 2^{-s}\\
&= C\delta^{s-1},
\end{aligned}$$

where C is independent of δ. Similarly for the integral over A_2. For the integral over A_3, we have

$$\begin{aligned}
\delta \int_{A_3} \frac{d\mu(\zeta)}{|\zeta - z||\zeta - w|} &\leq \delta \int_{\{|\zeta - z| \geq \delta/2\}} \frac{d\mu(\zeta)}{|\zeta - z|^2}\\
&= 2\delta \int_{\delta/2}^{\infty} \mu(\{\delta/2 \leq |\zeta - z| < t\}) t^{-3} dt\\
&\leq 2\delta \int_{\delta/2}^{\infty} t^{s-3} dt\\
&= C' \delta^{s-1}.
\end{aligned}$$

Here we used the fact that $s < 2$. Similarly for the integral over A_4.

This completes the proof of the lemma. □

The final ingredient for the proof of Theorem 3.1 is the following consequence of a classical result of Frostman.

Lemma 4.2 (Frostman's lemma)**.** *Let $E \subset \mathbb{C}$ be a compact set such that $\dim_{\mathcal{H}}(E) > 1$. Then for any $1 < s < \dim_{\mathcal{H}}(E)$, there exists a nontrivial Radon measure μ supported on E with growth*

$$\mu(\mathbb{D}(z_0, r)) \leq r^s \qquad (z_0 \in \mathbb{C}, r > 0).$$

Proof. See e.g., [5, Theorem 8.8].

□

Theorem 3.1 now follows directly from Lemma 4.1 and Lemma 4.2.

Proof. Let μ be as in Lemma 4.2. Then by Lemma 4.1, the function $f = \mathcal{C}\mu$ is a bounded continuous function on the whole sphere which is analytic on $\widehat{\mathbb{C}} \setminus E$, but not constant. $\square$

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Department of Mathematics, University of Hawaii Manoa, Honolulu, HI 96822, USA.
E-mail address: `malik.younsi@gmail.com`