

# HOLOMORPHIC MOTIONS, DIMENSION, AREA AND QUASICONFORMAL MAPPINGS

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**ABSTRACT.** We describe the variation of the Minkowski, packing and Hausdorff dimensions of a set moving under a holomorphic motion, as well as the variation of its area. Our method provides a new, unified approach to various celebrated theorems about quasiconformal mappings, including the work of Astala on the distortion of area and dimension under quasiconformal mappings and the work of Smirnov on the dimension of quasicircles.

## 1. INTRODUCTION

In what follows, we write  $\mathbb{D}, \mathbb{C}$  and  $\widehat{\mathbb{C}}$  for the open unit disk, the complex plane and the Riemann sphere, respectively.

**Definition 1.1.** Let  $A$  be a subset of  $\widehat{\mathbb{C}}$ . A *holomorphic motion* of  $A$  is a map  $f : \mathbb{D} \times A \rightarrow \widehat{\mathbb{C}}$  such that:

- (i) for each fixed  $z \in A$ , the map  $\lambda \mapsto f(\lambda, z)$  is holomorphic on  $\mathbb{D}$ ;
- (ii) for each fixed  $\lambda \in \mathbb{D}$ , the map  $z \mapsto f(\lambda, z)$  is injective on  $A$ ;
- (iii) for all  $z \in A$ , we have  $f(0, z) = z$ .

We write  $f_\lambda(z) := f(\lambda, z)$  and  $A_\lambda := f_\lambda(A)$ . We sometimes abuse terminology and call the set-valued map  $\lambda \mapsto A_\lambda$  a holomorphic motion of  $A$ .

Holomorphic motions were introduced by Mañé, Sad and Sullivan [15]. They established the so-called  $\lambda$ -lemma, which says that every holomorphic motion  $f : \mathbb{D} \times A \rightarrow \widehat{\mathbb{C}}$  has an extension to a holomorphic motion  $F : \mathbb{D} \times \overline{A} \rightarrow \widehat{\mathbb{C}}$ , and that  $F$  is jointly continuous in  $(\lambda, z)$ . They exploited this result to describe the variation of Julia sets of holomorphic families of hyperbolic rational maps. Holomorphic motions have since been applied in various other areas of dynamical systems, notably in describing the variation of limit sets of Kleinian groups, see e.g. [2, §12.2.1].

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Consider the following problem. Let  $\lambda \mapsto A_\lambda$  be a holomorphic motion such that  $A_\lambda \subset \mathbb{C}$  for all  $\lambda \in \mathbb{D}$ . What sort of functions are  $\lambda \mapsto \dim(A_\lambda)$  and  $\lambda \mapsto |A_\lambda|$ ? Here  $|\cdot|$  denotes the area measure (two-dimensional Lebesgue measure) and  $\dim(\cdot)$  can denote any reasonable notion of dimension. Various aspects of this problem have been treated in the literature, see for example [1, 3, 4, 7, 8, 11, 14, 17, 18, 19, 21, 23]. We shall discuss some of these contributions in more detail later.

In this article, we shall be mainly interested in three notions of dimension, namely the Minkowski, packing and Hausdorff dimensions. To state our results, it is convenient to introduce another definition.

**Definition 1.2.** Let  $D$  be a domain in  $\mathbb{C}$ . A positive function  $u : D \rightarrow [0, \infty)$  is called *inf-harmonic* if there exists a family  $\mathcal{H}$  of harmonic functions on  $D$  such that  $u(\lambda) = \inf_{h \in \mathcal{H}} h(\lambda)$  for all  $\lambda \in D$ .

In Theorems 1.3–1.6, we consider a holomorphic motion  $f : \mathbb{D} \times A \rightarrow \mathbb{C}$  of a subset  $A$  of  $\mathbb{C}$ , and write  $A_\lambda := f_\lambda(A)$ .

Our first result describes the variation of the Minkowski dimension, or more precisely the upper Minkowski dimension  $\overline{\dim}_M$ , of a bounded set moving under a holomorphic motion.

**Theorem 1.3.** *Let  $\lambda \mapsto A_\lambda$  be a holomorphic motion of a bounded subset  $A$  of  $\mathbb{C}$ . Then  $A_\lambda$  is bounded for all  $\lambda \in \mathbb{D}$ , and either  $\overline{\dim}_M(A_\lambda) = 0$  for all  $\lambda \in \mathbb{D}$ , or  $\lambda \mapsto 1/\overline{\dim}_M(A_\lambda)$  is an inf-harmonic function on  $\mathbb{D}$ .*

From this theorem, we deduce an analogous result for the packing dimension  $\dim_P$ .

**Theorem 1.4.** *Let  $\lambda \mapsto A_\lambda$  be a holomorphic motion of a subset  $A$  of  $\mathbb{C}$ . Then either  $\dim_P(A_\lambda) = 0$  for all  $\lambda \in \mathbb{D}$ , or  $\lambda \mapsto 1/\dim_P(A_\lambda)$  is an inf-harmonic function on  $\mathbb{D}$ .*

From these theorems, we obtain the following corollary.

**Corollary 1.5.** *Under the respective assumptions of Theorems 1.3 and 1.4,  $\overline{\dim}_M(A_\lambda)$  and  $\dim_P(A_\lambda)$  are continuous, logarithmically subharmonic functions of  $\lambda \in \mathbb{D}$  (and hence also subharmonic on  $\mathbb{D}$ ). In particular, if either these functions attains a maximum on  $\mathbb{D}$ , then it is constant.*

*Proof.* As we shall see, an inf-harmonic function is a continuous superharmonic function. Using Jensen's inequality, it is easy to see that, if  $1/v$  is a positive superharmonic function, then  $\log v$  is a subharmonic function, and hence also  $v$ . The last part of the corollary is a consequence of the maximum principle for subharmonic functions.  $\square$

For the Hausdorff dimension  $\dim_H$ , there is a result similar to Theorems 1.3 and 1.4, but with a weaker conclusion.

**Theorem 1.6.** *Let  $\lambda \mapsto A_\lambda$  be a holomorphic motion of a subset  $A$  of  $\mathbb{C}$ . Then either  $\dim_H(A_\lambda) = 0$  for all  $\lambda \in \mathbb{D}$ , or  $\dim_H(A_\lambda) > 0$  for all  $\lambda \in \mathbb{D}$ .*

In the latter case,  $\lambda \mapsto (1/\dim_H(A_\lambda) - 1/2)$  is the supremum of a family of inf-harmonic functions on  $\mathbb{D}$ .

The nature of the conclusion in Theorem 1.6 does not permit us to deduce that  $\log \dim_H(A_\lambda)$  or  $\dim_H(A_\lambda)$  is a subharmonic function of  $\lambda \in \mathbb{D}$ . We shall return to this problem at the end of the article.

Our next theorem is a sort of converse result.

**Theorem 1.7.** *Let  $d : \mathbb{D} \rightarrow (0, 2]$  be a function such that  $1/d$  is inf-harmonic on  $\mathbb{D}$ . Then there exists a holomorphic motion  $f : \mathbb{D} \times A \rightarrow \mathbb{C}$  of a compact subset  $A$  of  $\mathbb{C}$  such that, setting  $A_\lambda := f_\lambda(A)$ , we have  $\dim_P(A_\lambda) = \dim_H(A_\lambda) = d(\lambda)$  for all  $\lambda \in \mathbb{D}$ .*

We remark that Theorems 1.4 and 1.7 together yield a complete characterization of the variation of the packing dimension of a set moving under a holomorphic motion.

The holomorphic motions that arise from Julia sets of holomorphic families of hyperbolic rational maps (as considered in [15]) have the additional property that their Hausdorff and packing dimensions vary as real-analytic functions of  $\lambda$ . This is a special case of a result of Ruelle [21]. (Ruelle stated his theorem for Hausdorff dimension, but it coincides with packing dimension in this case.) For general holomorphic motions, it is known that the Hausdorff and packing dimensions need not be real-analytic (see e.g. [3]). The following corollary of Theorem 1.7 shows that in fact they may have the same lack of smoothness as an arbitrary concave function.

**Corollary 1.8.** *Given a concave function  $\psi : \mathbb{D} \rightarrow [0, \infty)$ , there exists a holomorphic motion  $f : \mathbb{D} \times A \rightarrow \mathbb{C}$  of a compact subset  $A$  of  $\mathbb{C}$  such that, setting  $A_\lambda := f_\lambda(A)$ , we have*

$$\dim_H(A_\lambda) = \dim_P(A_\lambda) = \frac{2}{1 + \psi(\lambda)} \quad (\lambda \in \mathbb{D}).$$

*Proof.* Every positive concave function on  $\mathbb{D}$  is inf-harmonic, since it is the lower envelope of a family of affine functions  $\lambda \mapsto a \operatorname{Re}(\lambda) + b \operatorname{Im}(\lambda) + c$ , each of which is harmonic on  $D$ . Thus the map  $\lambda \mapsto \frac{1}{2}(1 + \psi(\lambda))$  is inf-harmonic on  $\mathbb{D}$ . Also, it is clearly bounded below by  $1/2$ , so its reciprocal takes values in  $(0, 2]$ . The result therefore follows from Theorem 1.7.  $\square$

We now turn to the discussion of the variation of the area of a set  $A \subset \mathbb{C}$  moving under a holomorphic motion  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$ . As before, for  $\lambda \in \mathbb{D}$ , we write  $f_\lambda(z) := f(\lambda, z)$  and  $A_\lambda := f_\lambda(A)$ . Then each  $f_\lambda : \mathbb{C} \rightarrow \mathbb{C}$  is quasiconformal and we denote its complex dilatation by  $\mu_{f_\lambda}$  (see §3 for the definitions). Our next result gives a partial description of the function  $\lambda \mapsto |A_\lambda|$ , where  $|\cdot|$  denotes area measure.

**Theorem 1.9.** *Suppose that there exists a compact subset  $\Delta$  of  $\mathbb{C}$  such that, for each  $\lambda \in \mathbb{D}$ , the map  $f_\lambda$  is conformal on  $\mathbb{C} \setminus \Delta$  and  $f_\lambda(z) = z + O(1)$  near  $\infty$ . Let  $A$  be a Borel subset of  $\Delta$  such that  $|A| > 0$ .*

- (i) If  $\mu_{f_\lambda} = 0$  a.e. on  $A$ , then  $\lambda \mapsto \log(\pi c(\Delta)^2/|A_\lambda|)$  is an inf-harmonic function on  $\mathbb{D}$ , where  $c(\Delta)$  denotes the logarithmic capacity of  $\Delta$ .
- (ii) If  $\mu_{f_\lambda} = 0$  a.e. on  $\mathbb{C} \setminus A$ , then  $\lambda \mapsto |A_\lambda|$  is an inf-harmonic function on  $\mathbb{D}$ .

Note that if  $|A| = 0$ , then  $|A_\lambda| = 0$  for all  $\lambda \in \mathbb{D}$ , because quasiconformal mappings preserve zero area.

Our approach based on inf-harmonic functions also permits us to present a unified treatment of several celebrated theorems about the distortion of area and dimension under quasiconformal maps.

We emphasize here that prior works on the distortion of dimension under quasiconformal mappings relied on some of their more involved analytic properties, such as higher order integrability of the Jacobian. Our approach, on the other hand, only requires the fact that quasiconformal mappings satisfy a “weak” quasisymmetry property, as stated in Corollary 3.7.

For instance, a simple application of the Harnack inequality allows us to obtain the following two results. In Theorem 1.10,  $\dim$  denotes any one of  $\dim_P$ ,  $\dim_H$  or  $\overline{\dim}_M$ . (In the case of  $\overline{\dim}_M$ , we also suppose that  $A$  is bounded.)

**Theorem 1.10.** *Let  $F : \mathbb{C} \rightarrow \mathbb{C}$  be a  $k$ -quasiconformal homeomorphism, and let  $A$  be a subset of  $\mathbb{C}$  such that  $\dim(A) > 0$ . Then*

$$\frac{1}{\dim A} \left( \frac{1}{\dim A} - \frac{1}{2} \right) \leq \left( \frac{1}{\dim F(A)} - \frac{1}{2} \right) \leq K \left( \frac{1}{\dim A} - \frac{1}{2} \right),$$

where  $K := (1+k)/(1-k)$ .

For the Hausdorff dimension, the above estimate was first suggested by Gehring and Väisälä [11] and finally proved by Astala [1, Theorem 1.4]. For packing dimension it is a special case of a result of Kaufmann [14, Theorem 4].

**Theorem 1.11.** *Let  $F : \mathbb{C} \rightarrow \mathbb{C}$  be a  $k$ -quasiconformal homeomorphism which is conformal on  $\mathbb{C} \setminus \Delta$ , where  $\Delta$  is a compact set of logarithmic capacity at most 1, and such that  $F(z) = z + o(1)$  near  $\infty$ . Let  $A$  be a Borel subset of  $\Delta$ .*

- (i) If  $\mu_F = 0$  a.e. on  $A$ , then

$$|F(A)| \leq \pi^{1-1/K} |A|^{1/K}.$$

- (ii) If  $\mu_F = 0$  a.e. on  $\mathbb{C} \setminus A$ , then

$$|F(A)| \leq K|A|.$$

- (iii) Hence, in general,

$$|F(A)| \leq K\pi^{1-1/K} |A|^{1/K}.$$

Here again  $K = (1+k)/(1-k)$ .

Theorem 1.11 is a sharpened form of a result of Astala [1, Theorem 1] due to Eremenko and Hamilton [8, Theorem 1].

We also show how the proof of Theorem 1.6 can be adapted to obtain the following upper bound for the Hausdorff dimension of quasicircles due to Smirnov [23].

**Theorem 1.12.** *If  $\Gamma$  is a  $k$ -quasicircle, then  $\dim_H(\Gamma) \leq 1 + k^2$ .*

Finally, we obtain a result on the distortion of dimension under quasi-symmetric maps. For the Hausdorff dimension  $\dim_H$ , it was proved by Prause and Smirnov, see the main result of [18] and also [17, Theorem 3.1]. In the theorem below,  $\dim$  denotes one of  $\overline{\dim}_M$  or  $\dim_P$ . In the case of  $\overline{\dim}_M$ , we also assume that  $A$  is bounded.

**Theorem 1.13.** *Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a  $k$ -quasisymmetric map, where  $k \in [0, 1)$ . Then, given a set  $A \subset \mathbb{R}$  with  $\dim(A) = \delta$ ,  $0 < \delta \leq 1$ , we have*

$$\Delta(\delta, k) \leq \dim(g(A)) \leq \Delta^*(\delta, k).$$

Here

$$\Delta(\delta, k) := 1 - \left( \frac{k + l}{1 + kl} \right)^2$$

where  $l := \sqrt{1 - \delta}$ , and  $\Delta^*(\delta, k)$  is the inverse

$$\Delta^*(\delta, k) := \Delta(\delta, -\min(k, \sqrt{1 - \delta})).$$

In particular, if  $\dim A = \delta = 1$ , then  $l = 0$  and  $\Delta(\delta, k) = 1 - k^2$ , whence

$$\dim(g(A)) \geq 1 - k^2.$$

The remainder of the paper is organized as follows. We review the notions of Hausdorff, packing and Minkowski dimensions in §2. In §3 we discuss holomorphic motions in more detail, in particular their relation to quasiconformal maps. The basic properties of inf-harmonic functions that we need are developed in §4. Our main results, Theorems 1.3, 1.4, 1.6, 1.7 and 1.9, are proved in §§5–9. The applications to quasiconformal mappings, namely Theorems 1.10, 1.11, 1.12 and 1.13, are treated in §10. We conclude in §11 with an open problem.

## 2. NOTIONS OF DIMENSION

In this section we present a very brief review of some basic notions of dimension, introducing the notation, and concentrating on the aspects that will be useful to us later. Our account is based on the books of Bishop–Peres [6] and Falconer [9].

**2.1. Hausdorff dimension.** We begin with the definition. Let  $A \subset \mathbb{C}$ . For  $s \geq 0$  and  $\delta > 0$ , define

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{j=1}^{\infty} \text{diam}(A_j)^s \right\},$$

where the infimum is taken over all countable covers  $\{A_j\}$  of  $A$  by sets of diameter at most  $\delta$ . Since  $\mathcal{H}_\delta^s(A)$  increases as  $\delta$  decreases, the limit

$$\mathcal{H}^s(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A)$$

exists, possibly 0 or  $\infty$ . The set function  $\mathcal{H}^s(\cdot)$  is an outer measure on  $\mathbb{C}$ , called the *s-dimensional Hausdorff measure*. The *Hausdorff dimension* of  $A$  is defined as the unique real number  $\dim_H(A) \in [0, 2]$  such that

$$\mathcal{H}^s(A) = \begin{cases} \infty, & s < \dim_H(A), \\ 0, & s > \dim_H(A). \end{cases}$$

We shall need a slight variant of this construction. A *dyadic square* is a subset of  $\mathbb{C}$  of the form  $Q = [m2^{-k}, (m+1)2^{-k}) \times [n2^{-k}, (n+1)2^{-k})$ , where  $k, m, n$  are integers (possibly negative). Define

$$\tilde{\mathcal{H}}_\delta^s(A) := \inf \left\{ \sum_{j=1}^{\infty} \text{diam}(Q_j)^s \right\},$$

where now the infimum is taken merely over countable covers  $\{Q_j\}$  of  $A$  by dyadic squares of diameter at most  $\delta$ . As before, we also set

$$\tilde{\mathcal{H}}^s(A) := \lim_{\delta \rightarrow 0} \tilde{\mathcal{H}}_\delta^s(A).$$

Clearly we have  $\tilde{\mathcal{H}}_\delta^s(A) \geq \mathcal{H}_\delta^s(A)$  for all  $\delta$ , and hence  $\tilde{\mathcal{H}}^s(A) \geq \mathcal{H}^s(A)$ . Also, it is not hard to see that any bounded subset of  $\mathbb{C}$  can be covered by 9 dyadic squares of smaller diameter, from which it follows that  $\tilde{\mathcal{H}}_\delta^s(A) \leq 9\mathcal{H}_\delta^s(A)$  for all  $\delta$ , and hence  $\tilde{\mathcal{H}}^s(A) \leq 9\mathcal{H}^s(A)$ . In particular, we deduce the following result.

**Proposition 2.1.** *With the above notation, we have*

$$\tilde{\mathcal{H}}^s(A) = \begin{cases} \infty, & s < \dim_H(A), \\ 0, & s > \dim_H(A). \end{cases}$$

Dyadic squares have the property that any two of them are either nested or disjoint. Thus the sets  $Q_j$  in the definition of  $\tilde{\mathcal{H}}_\delta^s(A)$  may be taken to be disjoint. This will be useful for us later. For more on this, see [6, §1.3, p.11].

We conclude by noting that Hausdorff dimension is *countably stable*, i.e., for any sequence of sets  $(A_j)$  we have  $\dim_H(\cup_{j \geq 1} A_j) = \sup_{j \geq 1} \dim_H(A_j)$  (see e.g. [9, p.49]).

**2.2. Packing dimension.** The notion of packing dimension is in some sense dual to that of Hausdorff dimension. It was introduced by Tricot in [25].

Once again, we begin with the definition. Let  $A \subset \mathbb{C}$ . For  $s \geq 0$  and  $\delta > 0$ , define

$$\mathcal{P}_\delta^s(A) := \sup \left\{ \sum_{j=1}^n \text{diam}(D_j)^s \right\},$$

where the supremum is taken over all finite sets of disjoint disks  $\{D_j\}$  with centres in  $A$  and of diameters at most  $\delta$ . Since  $\mathcal{P}_\delta^s(A)$  decreases as  $\delta$  decreases, the limit

$$\mathcal{P}_0^s(A) := \lim_{\delta \rightarrow 0} \mathcal{P}_\delta^s(A)$$

exists, possibly 0 or  $\infty$ . This is not yet an outer measure, because it is not countably subadditive. It is sometimes called the *s-dimensional pre-packing measure* of  $A$ . We modify it to make it an outer measure, defining the *s-dimensional packing measure* of  $A$  by

$$\mathcal{P}^s(A) := \inf \left\{ \sum_{j \geq 1} \mathcal{P}_0^s(A_j) : A = \cup_{j \geq 1} A_j \right\},$$

where the infimum is taken over all countable covers of  $A$  by subsets  $(A_j)_{j \geq 1}$ . The *packing dimension* of  $A$  is then defined as the unique real number  $\dim_P(A) \in [0, 2]$  such that

$$\mathcal{P}^s(A) = \begin{cases} \infty, & s < \dim_P(A), \\ 0, & s > \dim_P(A). \end{cases}$$

As in the case of Hausdorff dimension, the packing dimension is countably stable:  $\dim_P(\cup_{j \geq 1} A_j) = \sup_{j \geq 1} \dim_P(A_j)$ . Also, we always have

$$\dim_H(A) \leq \dim_P(A),$$

and the inequality may be strict.

**2.3. Minkowski dimension.** Let  $A$  be a bounded subset of  $\mathbb{C}$ . Given  $\delta > 0$ , we denote by  $N_\delta(A)$  the smallest number of sets of diameter at most  $\delta$  needed to cover  $A$ . The *upper* and *lower Minkowski dimensions* of  $A$  are respectively defined by

$$\overline{\dim}_M(A) := \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(A)}{\log(1/\delta)} \quad \text{and} \quad \underline{\dim}_M(A) := \liminf_{\delta \rightarrow 0} \frac{\log N_\delta(A)}{\log(1/\delta)}.$$

Of course we always have  $\underline{\dim}_M(A) \leq \overline{\dim}_M(A)$ . The inequality may be strict. If equality holds, then we speak simply of the Minkowski dimension of  $A$ , denoted  $\dim_M(A)$ . It is also called the *box-counting dimension* of  $A$ .

The Minkowski dimension has the virtue of simplicity, but it also suffers from the drawback that, unlike the Hausdorff and packing dimensions, it is not countably stable, i.e., it can happen that  $\dim_M(\cup_j A_j) > \sup_j \dim_M(A_j)$ .

There is a useful relationship between upper Minkowski dimension and the pre-packing measure  $\mathcal{P}_0^s$  introduced in the previous subsection. The following result is due to Tricot [25, Corollary 2].

**Proposition 2.2.** *If  $A$  is a bounded subset of  $\mathbb{C}$ , then*

$$\mathcal{P}_0^s(A) = \begin{cases} \infty, & s < \overline{\dim}_M(A), \\ 0, & s > \overline{\dim}_M(A). \end{cases}$$

Using this result, we can express the packing dimension in terms of the upper Minkowski dimension. The following theorem is again due to Tricot [25, Proposition 2], see also [6, Theorem 2.7.1].

**Proposition 2.3.** *If  $A$  is a subset of  $\mathbb{C}$ , then*

$$\dim_P(A) = \inf \left\{ \sup_{j \geq 1} \overline{\dim}_M(A_j) : A = \bigcup_{j \geq 1} A_j \right\},$$

where the infimum is taken over all countable covers of  $A$  by bounded subsets  $(A_j)$ .

From this result, it is obvious that, for every bounded set  $A$ , we have

$$\dim_P(A) \leq \overline{\dim}_M(A).$$

In general the inequality can be strict. The books [6] and [9] both contain a discussion of conditions under which equality holds.

**2.4. Similarity dimension.** There is one further notion of dimension that will prove useful in what follows. It applies to a specific example.

Consider a finite system of contractive similarities

$$\gamma_j(z) = a_j z + b_j \quad (j = 1, \dots, n),$$

where  $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}$  and  $|a_j| < 1$  for all  $j$ . In this situation, there is a unique compact subset  $L$  of  $\mathbb{C}$  such that  $L = \bigcup_{j=1}^n \gamma_j(L)$ , called the *limit set* of the iterated function system  $\{\gamma_1, \dots, \gamma_n\}$ .

The system  $\{\gamma_1, \dots, \gamma_n\}$  is said to satisfy the *open set condition* if there exists a non-empty open subset  $U$  of  $\mathbb{C}$  such that  $\gamma_j(U) \subset U$  for all  $j$  and  $\gamma_i(U) \cap \gamma_j(U) = \emptyset$  whenever  $i \neq j$ . The following result is due to Hutchinson [12], generalizing an earlier result of Moran [16], see also [9, Theorem 9.3] or [6, Theorem 2.2.2].

**Theorem 2.4.** *If the system  $\{\gamma_1, \dots, \gamma_n\}$  satisfies the open set condition, then the Hausdorff and packing dimensions of its limit set  $L$  are given by  $\dim_H(L) = \dim_P(L) = s$ , where  $s$  is the unique solution of the equation*

$$\sum_{j=1}^n |a_j|^s = 1.$$

The number  $s$  (with or without the open set condition) is called the *similarity dimension* of the system.



## 3. HOLOMORPHIC MOTIONS AND QUASICONFORMAL MAPS

Holomorphic motions were defined in Definition 1.1. As was mentioned in the introduction, they were introduced in [15] by Mañé, Sad and Sullivan, who also established the  $\lambda$ -lemma. Their result was later improved by Ślodkowski in [22], confirming a conjecture of Sullivan and Thurston [24]. Ślodkowski's result is often called the extended  $\lambda$ -lemma. There are now several proofs; another one can be found in [2, §12].

**Theorem 3.1** (Extended  $\lambda$ -lemma). *A holomorphic motion  $f : \mathbb{D} \times A \rightarrow \mathbb{C}$  has an extension to a holomorphic motion  $F : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$ . The function  $F$  is jointly continuous on  $\mathbb{D} \times \mathbb{C}$ .*

As was already remarked in [15], holomorphic motions are closely related to quasiconformal maps. We now define this term and state some results that will be needed in the sequel. Our treatment follows that in [2].

**Definition 3.2.** Let  $\Omega, \Omega'$  be plane domains. A homeomorphism  $f : \Omega \rightarrow \Omega'$  is called *quasiconformal* if:

- (i)  $f$  is orientation-preserving;
- (ii) its distributional Wirtinger derivatives  $\partial f / \partial z$  and  $\partial f / \partial \bar{z}$  both belong to  $L^2_{\text{loc}}(\Omega)$ , and
- (iii)  $f$  satisfies the *Beltrami equation*:

$$\frac{\partial f}{\partial \bar{z}} = \mu_f \frac{\partial f}{\partial z} \quad \text{a.e. on } \Omega,$$

where  $\mu_f$  is a measurable function on  $\Omega$  such that  $\|\mu_f\|_\infty < 1$ .

The function  $\mu_f$  is called the *Beltrami coefficient* or *complex dilatation* of  $f$ . We shall say that the mapping  $f$  is *k-quasiconformal* if  $\|\mu_f\|_\infty \leq k$ .

*Remark.* Many authors (including those of [2]) use the term *K-quasiconformal* to mean *k-quasiconformal* in our sense with  $K = (1 + k)/(1 - k)$ .

We shall need the following fundamental result on the existence and uniqueness of solutions to the Beltrami equation [2, Theorem 5.3.4].

**Theorem 3.3** (Measurable Riemann mapping theorem). *Let  $\mu$  be a measurable function on  $\mathbb{C}$  with  $\|\mu\|_\infty < 1$ . Then there exists a unique quasiconformal mapping  $f : \mathbb{C} \rightarrow \mathbb{C}$  fixing 0 and 1 with  $\mu_f = \mu$  a.e. on  $\mathbb{C}$ .*

It is well known that solutions of the Beltrami equation depend holomorphically on the parameter  $\mu$  [2, Corollary 5.7.5]. Combined with [2, Theorem 12.3.2], this is the key to the following characterization of holomorphic motions.

**Theorem 3.4.** *Let  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  be a function. The following statements are equivalent:*

- (i) *The map  $f$  is a holomorphic motion.*

- (ii) For each  $\lambda \in \mathbb{D}$ , the map  $f_\lambda : \mathbb{C} \rightarrow \mathbb{C}$  is quasiconformal with Beltrami coefficient  $\mu_\lambda$  satisfying  $\|\mu_\lambda\|_\infty \leq |\lambda|$ . Moreover, the map  $f_0$  is the identity, and the  $L^\infty(\mathbb{C})$ -valued map  $\lambda \mapsto \mu_\lambda$  is holomorphic on  $\mathbb{D}$ .

These results can be used to show that every quasiconformal homeomorphism of  $\mathbb{C}$  can be embedded as part of a holomorphic motion [2, Theorem 12.5.3].

**Theorem 3.5.** *If  $F : \mathbb{C} \rightarrow \mathbb{C}$  is a  $k$ -quasiconformal homeomorphism, then there exists a holomorphic motion  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  such that  $f_k = F$ .*

Quasiconformal maps exhibit numerous interesting properties. An important one for us is the fact that quasiconformal homeomorphisms of  $\mathbb{C}$  are quasismetric in the sense described in the next theorem [2, Theorem 3.5.3].

**Theorem 3.6.** *Given  $k \in [0, 1)$ , there exists an increasing homeomorphism  $\eta : [0, \infty) \rightarrow [0, \infty)$  such that every  $k$ -quasiconformal map  $f : \mathbb{C} \rightarrow \mathbb{C}$  satisfies*

$$(3.1) \quad \frac{|f(z_0) - f(z_1)|}{|f(z_0) - f(z_2)|} \leq \eta\left(\frac{|z_0 - z_1|}{|z_0 - z_2|}\right) \quad (z_0, z_1, z_2 \in \mathbb{C}).$$

We shall exploit this result via the following simple corollary.

**Corollary 3.7.** *Given  $k \in [0, 1)$ , there exist constants  $\delta, \delta' > 0$  such that every  $k$ -quasiconformal homeomorphism  $f : \mathbb{C} \rightarrow \mathbb{C}$  has the following properties:*

- (i) *If  $z_0 \in \mathbb{C}$  and  $D$  is an open disk with centre  $z_0$ , then  $f(D)$  contains the open disk with centre  $f(z_0)$  and radius  $\delta \operatorname{diam} f(D)$ .*
- (ii) *If  $z_0 \in \mathbb{C}$  and  $Q$  is an open square with centre  $z_0$ , then  $f(Q)$  contains the open disk with centre  $f(z_0)$  and radius  $\delta' \operatorname{diam} f(Q)$ .*

*Proof.* Let  $\eta$  be the function associated to  $k$  by Theorem 3.6. In the case of the square, if  $z_1, z_2 \in \partial Q$ , then  $|z_0 - z_1|/|z_0 - z_2| \leq \sqrt{2}$ , so by (3.1) we have  $|f(z_0) - f(z_1)|/|f(z_0) - f(z_2)| \leq \eta(\sqrt{2})$ . It follows that (ii) holds with  $\delta' = 1/(2\eta(\sqrt{2}))$ . The proof of (i) is similar, now with  $\delta = 1/(2\eta(1))$ .  $\square$

#### 4. INF-HARMONIC FUNCTIONS

Recall from Definition 1.2 that a function  $u : D \rightarrow [0, \infty)$  defined on a plane domain  $D$  is inf-harmonic if it is the lower envelope of a family of harmonic functions. It is inherent in the definition that  $u$  is positive, so the harmonic functions are positive too. This has the consequence that inf-harmonic functions inherit several of the good properties of positive harmonic functions.

We begin by showing that inf-harmonic functions satisfy Harnack's inequality. Recall that, given  $\lambda_1, \lambda_2 \in D$ , there exists  $\tau > 0$  such that, for all

positive harmonic functions  $h$  on  $D$

$$\frac{1}{\tau} \leq \frac{h(\lambda_1)}{h(\lambda_2)} \leq \tau.$$

The smallest such  $\tau$  is called the *Harnack distance* between  $\lambda_1, \lambda_2$ , denoted  $\tau_D(\lambda_1, \lambda_2)$ . For example,  $\tau_{\mathbb{D}}(0, \lambda) = (1 + |\lambda|)/(1 - |\lambda|)$  for  $\lambda \in \mathbb{D}$ .

**Proposition 4.1.** *Let  $u$  be an inf-harmonic function on a domain  $D$ , and suppose that  $u \not\equiv 0$ . Then  $u(\lambda) > 0$  for all  $\lambda \in D$  and*

$$(4.1) \quad \frac{1}{\tau_D(\lambda_1, \lambda_2)} \leq \frac{u(\lambda_1)}{u(\lambda_2)} \leq \tau_D(\lambda_1, \lambda_2) \quad (\lambda_1, \lambda_2 \in D).$$

*Proof.* For each positive harmonic function  $h$  on  $D$ , we have

$$\frac{1}{\tau_D(\lambda_1, \lambda_2)} h(\lambda_2) \leq h(\lambda_1) \leq \tau_D(\lambda_1, \lambda_2) h(\lambda_2) \quad (\lambda_1, \lambda_2 \in D).$$

Taking the infimum over all  $h$  such that  $h \geq u$ , we obtain

$$\frac{1}{\tau_D(\lambda_1, \lambda_2)} u(\lambda_2) \leq u(\lambda_1) \leq \tau_D(\lambda_1, \lambda_2) u(\lambda_2) \quad (\lambda_1, \lambda_2 \in D).$$

Since  $u \not\equiv 0$ , this shows that  $u(\lambda) > 0$  for all  $\lambda \in D$ , and (4.1) now follows immediately.  $\square$

**Corollary 4.2.** *If  $u$  is an inf-harmonic function on  $D$ , then it is a continuous superharmonic function on  $D$ .*

*Proof.* The continuity of  $u$  follows from Proposition 4.1, since  $\tau_D$  is continuous on  $D \times D$ . As  $u$  is the infimum of harmonic functions, it clearly satisfies the super-mean value property, so it is superharmonic on  $D$ .  $\square$

The next result is a normal-family property.

**Proposition 4.3.** *Let  $(D_n)_{n \geq 1}$  be an increasing sequence of domains, and let  $D := \cup_{n \geq 1} D_n$ . For each  $n$ , let  $u_n$  be an inf-harmonic function on  $D_n$ . Then either  $u_n \rightarrow \infty$  locally uniformly on  $D$ , or else some subsequence  $u_{n_j} \rightarrow u$  locally uniformly on  $D$ , where  $u$  is inf-harmonic on  $D$ .*

*Proof.* If there exists a point  $\lambda_0 \in D$  such that  $u_n(\lambda_0) \rightarrow \infty$ , then by Proposition 4.1 the sequence  $u_n \rightarrow \infty$  locally uniformly in each  $D_m$  and hence also on  $D$ . Likewise if  $u_n(\lambda_0) \rightarrow 0$ , then  $u_n \rightarrow 0$  locally uniformly in  $D$ . Thus, replacing  $(u_n)$  by a subsequence if necessary, we may assume that there exists  $\lambda_0 \in D_1$  and  $M > 1$  such that  $1/M \leq u_n(\lambda_0) \leq M$  for all  $n$ . In this case, by Proposition 4.1 once more, the sequence  $(u_n)$  is equicontinuous on each  $D_m$ , and by the Arzelà–Ascoli theorem, a subsequence  $(u_{n_j})$  converges locally uniformly on  $D$  to a finite-valued function  $u$ .

It remains to show that  $u$  is itself inf-harmonic on  $D$ . Relabelling, if necessary, we can suppose that the whole sequence  $u_n$  converges to  $u$  locally uniformly on  $D$ . Let  $\lambda_0 \in D$ . Choose  $n_0$  so that  $\lambda_0 \in D_{n_0}$ . For each  $n \geq n_0$ , the function  $u_n$  is inf-harmonic on  $D_n$ , so there exists a (positive) harmonic function  $h_n$  on  $D_n$  such that  $h_n \geq u_n$  on  $D_n$  and  $h_n(\lambda_0) \leq u(\lambda_0) + 1/n$ . By

a standard normal-family argument, a subsequence  $(h_{n_j})$  converges locally uniformly on  $D$  to a function  $h$  that is harmonic on  $D$ . Clearly  $h \geq u$  on  $D$  and  $h(\lambda_0) = u(\lambda_0)$ . Such an  $h$  exists for each choice of  $\lambda_0 \in D$ , so we conclude that  $u$  is indeed inf-harmonic on  $D$ .  $\square$

The following result lists some closure properties of the family of inf-harmonic functions.

- Proposition 4.4.** (i) *If  $u$  and  $v$  are inf-harmonic on  $D$  and if  $\alpha, \beta \geq 0$ , then  $\alpha u + \beta v$  is inf-harmonic on  $D$ .*  
(ii) *If  $u$  is inf-harmonic on  $D$  and  $h$  is harmonic on  $D$ , and if  $u \geq h$ , then  $u - h$  is inf-harmonic on  $D$ .*  
(iii) *If  $(u_n)_{n \geq 1}$  are inf-harmonic functions on  $D$ , and if  $u_n \rightarrow u$  pointwise on  $D$ , then either  $u$  is inf-harmonic on  $D$  or  $u \equiv \infty$ .*  
(iv) *If  $(D_n)$  is an increasing sequence of domains with  $\cup_{n \geq 1} D_n = D$ , and if  $u$  is a function on  $D$  such that  $u|_{D_n}$  is inf-harmonic on  $D_n$  for each  $n$ , then  $u$  is inf-harmonic on  $D$ .*  
(v) *If  $\mathcal{V}$  is a family of inf-harmonic functions on  $D$  and  $u := \inf_{v \in \mathcal{V}} v$ , then  $u$  is inf-harmonic on  $D$ .*  
(vi) *If  $\mathcal{V}$  is an upward-directed family of inf-harmonic functions on  $D$  (i.e., given  $v_1, v_2 \in \mathcal{V}$ , there exists  $v_3 \in \mathcal{V}$  with  $v_3 \geq \max\{v_1, v_2\}$ ), and if  $u := \sup_{v \in \mathcal{V}} v$ , then either  $u$  is inf-harmonic on  $D$  or  $u \equiv \infty$ .*

*Proof.* (i),(ii) These are both obvious.

(iii) Assume that  $u \not\equiv \infty$ . Then, by Proposition 4.3, a subsequence of the  $(u_n)$  converges locally uniformly on  $D$  to an inf-harmonic function  $v$ . Since the same subsequence converges pointwise to  $u$ , we must have  $v = u$ . Hence  $u$  is inf-harmonic.

(iv) This follows by applying Proposition 4.3 with  $u_n := u|_{D_n}$ .

(v) Again, this is obvious.

(vi) Assume that  $u \not\equiv \infty$ . Then, by Proposition 4.1,  $u$  is finite-valued and continuous on  $D$ . Let  $\Lambda = (\lambda_j)$  be a sequence that is dense in  $D$ . Using the fact that  $\mathcal{V}$  is upward-directed, we may construct an increasing sequence of functions  $v_n \in \mathcal{V}$  such that  $v_n(\lambda_j) \geq u(\lambda_j) - 1/n$  for all  $j \in \{1, 2, \dots, n\}$  and all  $n \geq 1$ . Then  $v_n$  converges pointwise to a function  $v$  such that  $v \leq u$  and  $v = u$  on  $\Lambda$ . By part (iii) above,  $v$  is inf-harmonic on  $D$ . As  $v = u$  on the dense subset  $\Lambda$  and both  $u, v$  are continuous, we have  $v = u$  on  $D$ . Thus  $u$  is inf-harmonic on  $D$ , as asserted.  $\square$

We conclude this section with an implicit function theorem for inf-harmonic functions.

**Theorem 4.5.** *Let  $D$  be a plane domain, and let  $a_j : D \rightarrow (0, 1)$  be a finite or infinite sequence of functions such that  $\log(1/a_j)$  is inf-harmonic on  $D$  for each  $j$ . Let  $c > 0$ , and define  $s : D \rightarrow [0, \infty]$  by*

$$s(\lambda) := \inf \left\{ \alpha > 0 : \sum_j a_j(\lambda)^\alpha \leq c \right\} \quad (\lambda \in D),$$

where we interpret  $\inf \emptyset = \infty$ . Then either  $s \equiv 0$  or  $1/s$  is an inf-harmonic function on  $D$ .

It is perhaps worth emphasizing the case where there are only finitely many functions  $a_j$ . It then becomes a result closely linked to the notion of similarity dimension defined in §2. It generalizes a result of Baribeau and Roy [4, Theorem 1].

**Corollary 4.6.** *Let  $a_1, \dots, a_n : D \rightarrow (0, 1)$  be functions such that  $\log(1/a_j)$  is inf-harmonic on  $D$  for each  $j$ . Let  $c \in (0, n)$  and, for each  $\lambda \in D$ , let  $s(\lambda)$  be the unique solution of the equation*

$$\sum_{j=1}^n a_j(\lambda)^{s(\lambda)} = c.$$

*Then  $1/s$  is an inf-harmonic function on  $D$ .*

We shall deduce Theorem 4.5 from a more general abstract result. To formulate this result, it is convenient to introduce some terminology.

Let  $X$  be a set and let  $\mathcal{U}$  be a family of functions  $u : X \rightarrow [0, \infty)$ . We call  $\mathcal{U}$  an *inf-cone* on  $X$  if it satisfies the following closure properties:

- if  $u, v \in \mathcal{U}$  and  $\alpha, \beta \geq 0$ , then  $\alpha u + \beta v \in \mathcal{U}$ ;
- if  $\emptyset \neq \mathcal{V} \subset \mathcal{U}$  and  $u := \inf_{v \in \mathcal{V}} v$ , then  $u \in \mathcal{U}$ .

By Proposition 4.4 parts (i) and (v), the set of inf-harmonic functions on a domain  $D$  is an inf-cone on  $D$ .

The following result may be viewed as an abstract implicit function theorem for inf-cones.

**Lemma 4.7.** *Let  $\mathcal{U}$  be an inf-cone on  $X$ , let  $(u_j)_{j \geq 1}$  be a sequence in  $\mathcal{U}$ , and for each  $j$  let  $\phi_j : [0, \infty) \rightarrow [0, \infty)$  be a continuous, decreasing, convex function. Define  $v : X \rightarrow [0, \infty]$  by*

$$v(x) := \sup \left\{ t > 0 : \sum_{j \geq 1} \phi_j(u_j(x)/t) \leq 1 \right\} \quad (x \in X),$$

*where we interpret  $\sup \emptyset = 0$ . Then  $v \in \mathcal{U}$  or  $v \equiv \infty$ .*

*Proof.* For each  $j$ , let  $\mathcal{L}_j$  be the family of functions of the form  $L(y) := b_L - a_L y$ , such that  $a_L \geq 0, b_L \in \mathbb{R}$  and  $L \leq \phi_j$ . As  $\phi_j$  is a continuous decreasing convex function, we have  $\phi_j = \sup_{L \in \mathcal{L}_j} L$ . Consequently, if  $x \in X$

and  $t > 0$ , then

$$\begin{aligned}
& \sum_{j \geq 1} \phi_j(u_j(x)/t) \leq 1 \\
& \iff \sum_{j=1}^n \phi_j(u_j(x)/t) \leq 1 \quad (n \geq 1) \\
& \iff \sum_{j=1}^n (b_{L_j} - a_{L_j} u_j(x)/t) \leq 1 \quad (n \geq 1, L_1 \in \mathcal{L}_1, \dots, L_n \in \mathcal{L}_n) \\
& \iff t \left( \sum_{j=1}^n b_{L_j} - 1 \right) \leq \sum_{j=1}^n a_{L_j} u_j(x) \quad (n \geq 1, L_1 \in \mathcal{L}_1, \dots, L_n \in \mathcal{L}_n).
\end{aligned}$$

There are now two possibilities. If  $\sum_{j=1}^n b_{L_j} \leq 1$  for all  $n$  and all choices of  $(L_1, \dots, L_n) \in \mathcal{L}_1 \times \dots \times \mathcal{L}_n$ , then the above conditions are satisfied for all  $t > 0$  and all  $x \in X$ . In this case  $v \equiv \infty$ . In the other case, we have

$$v(x) = \inf \left\{ \frac{\sum_{j=1}^n a_{L_j} u_j(x)}{\sum_{j=1}^n b_{L_j} - 1} \right\} \quad (x \in X),$$

where the infimum is taken over all  $n \geq 1$  and all  $(L_1, \dots, L_n) \in \mathcal{L}_1 \times \dots \times \mathcal{L}_n$  such that  $\sum_{j=1}^n b_{L_j} > 1$ . Hence  $v \in \mathcal{U}$  in this case.  $\square$

*Proof of Theorem 4.5.* This result follows from Lemma 4.7 upon taking  $\mathcal{U}$  to be the set of inf-harmonic functions on  $D$ , and  $\psi_j(y) := (1/c) \exp(-y)$  for each  $j$ .  $\square$

## 5. PROOF OF THEOREM 1.3

We have  $A_\lambda = f_\lambda(A) = f(\lambda, A)$ , where  $f : \mathbb{D} \times A \rightarrow \mathbb{C}$  is a holomorphic motion. By Theorem 3.1, we may extend  $f$  to a holomorphic motion  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$ . We shall assume that  $f$  has been so extended. Since  $A$  is bounded and  $f$  is continuous, it follows that  $A_\lambda$  is bounded for all  $\lambda \in \mathbb{D}$ .

The following lemma establishes the link with inf-harmonic functions. We recall that  $D(a, r)$  denotes the open disk with centre  $a$  and radius  $r$ , and that  $\text{diam}(S)$  denotes the euclidean diameter of  $S$ .

**Lemma 5.1.** *Let  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic motion. Let  $B$  be a bounded subset of  $\mathbb{C}$  and let  $\rho \in (0, 1)$ . Then  $M := \text{diam } f(D(0, \rho) \times B) < \infty$ . If  $S$  is a subset of  $B$ , then the map  $\lambda \mapsto \log(M / \text{diam } f_\lambda(S))$  is an inf-harmonic function on  $D(0, \rho)$ . Consequently, we have*

$$(5.1) \quad \frac{\rho - |\lambda|}{\rho + |\lambda|} \leq \frac{\log(M / \text{diam } f_\lambda(S))}{\log(M / \text{diam } S)} \leq \frac{\rho + |\lambda|}{\rho - |\lambda|} \quad (\lambda \in D(0, \rho)).$$

*Proof.* As  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  is a continuous map and  $\overline{D}(0, \rho) \times \overline{B}$  is a compact subset of  $\mathbb{D} \times \mathbb{C}$ , it follows that  $f(\overline{D}(0, \rho) \times \overline{B})$  is a compact subset of  $\mathbb{C}$ . In particular it has finite diameter, so  $M < \infty$ .

Given  $S \subset B$ , we have

$$\log\left(\frac{M}{\text{diam } f_\lambda(S)}\right) = \inf\left\{\log\left(\frac{M}{|f_\lambda(z) - f_\lambda(w)|}\right) : z, w \in S, z \neq w\right\}.$$

For each pair  $z, w \in S$  with  $z \neq w$ , the function  $\lambda \mapsto \log(M/|f_\lambda(z) - f_\lambda(w)|)$  is positive and harmonic on  $D(0, \rho)$ . Therefore  $\lambda \mapsto \log(M/\text{diam } f_\lambda(S))$  is inf-harmonic on  $D(0, \rho)$ .

Finally, the inequality (5.1) is a direct consequence of Harnack's inequality for inf-harmonic functions, Proposition 4.1.  $\square$

The next lemma contains the heart of the proof of Theorem 1.3. We recall that the upper Minkowski dimension  $\overline{\dim}_M$  can be characterized using Proposition 2.2.

**Lemma 5.2.** *If  $\overline{\dim}_M(A) > 0$ , then there exists an inf-harmonic function  $u$  on  $\mathbb{D}$  such that*

$$u(0) = 1/\overline{\dim}_M(A) \quad \text{and} \quad u(\lambda) \geq 1/\overline{\dim}_M(A_\lambda) \quad (\lambda \in \mathbb{D}).$$

*Proof.* Let  $\rho \in (0, 1)$ . We shall carry out the proof on the disk  $D(0, \rho)$ , and then let  $\rho \rightarrow 1$  at the very end.

Let  $(d_n)$  be a sequence such that  $0 < d_n < \overline{\dim}_M(A)$  and  $d_n \rightarrow \overline{\dim}_M(A)$ . By Proposition 2.2, for each  $n$  there exists a finite set  $\mathcal{D}_n$  of disjoint disks with centres in  $A$  such that, as  $n \rightarrow \infty$ ,

$$(5.2) \quad \max_{D \in \mathcal{D}_n} \text{diam}(D) \rightarrow 0 \quad \text{and} \quad \sum_{D \in \mathcal{D}_n} \text{diam}(D)^{d_n} \rightarrow \infty.$$

Let  $B$  be the union of all the disks in  $\cup_{n \geq 1} \mathcal{D}_n$ . This is a bounded set, so, by Lemma 5.1,  $M := \text{diam } f(D(0, \rho) \times B) < \infty$ , and  $\lambda \mapsto \log(M/\text{diam } f_\lambda(D))$  is inf-harmonic on  $D(0, \rho)$  for each  $D \in \cup_n \mathcal{D}_n$ .

For each  $\lambda \in D(0, \rho)$ , let  $s_n(\lambda)$  be the unique solution of the equation

$$\sum_{D \in \mathcal{D}_n} (\text{diam } f_\lambda(D)/M)^{s_n(\lambda)} = \sum_{D \in \mathcal{D}_n} (\text{diam } D/M)^{d_n}.$$

Clearly  $s_n(0) = d_n$ . Also, by the implicit function theorem, Corollary 4.6, the function  $1/s_n$  is inf-harmonic on  $D(0, \rho)$ . By Proposition 4.3, a subsequence of  $1/s_n$  (which, by relabelling, we may suppose to be the whole sequence) converges locally uniformly to an inf-harmonic function  $u$  on  $D(0, \rho)$ . Clearly we have  $u(0) = \lim_n (1/d_n) = 1/\overline{\dim}_M(A)$ . We shall show that  $u(\lambda) \geq 1/\overline{\dim}_M(A_\lambda)$  for all  $\lambda \in D(0, \rho)$ .

Fix  $\lambda \in D(0, \rho)$ , and let  $c \in (0, 1/u(\lambda))$ . Then  $s_n(\lambda) > c$  for all large enough  $n$ , and so, for these  $n$ , we have

$$\begin{aligned} \sum_{D \in \mathcal{D}_n} (\text{diam } f_\lambda(D)/M)^c &\geq \sum_{D \in \mathcal{D}_n} (\text{diam } f_\lambda(D)/M)^{s_n(\lambda)} \\ &= \sum_{D \in \mathcal{D}_n} (\text{diam } D/M)^{d_n}, \end{aligned}$$

whence

$$\sum_{D \in \mathcal{D}_n} (\text{diam } f_\lambda(D))^c \geq M^{c-d_n} \sum_{D \in \mathcal{D}_n} (\text{diam } D)^{d_n}.$$

For a given value of  $n$ , the sets  $\{f_\lambda(D) : D \in \mathcal{D}_n\}$  are disjoint, but they are not disks. However, we can circumvent this difficulty by invoking the theory of quasiconformal mappings. By Theorem 3.4, the map  $f_\lambda$  is a  $\rho$ -quasiconformal self-homeomorphism of  $\mathbb{C}$ . Consequently, by Corollary 3.7(i), there exists a  $\delta > 0$  such that, for each  $w \in \mathbb{C}$  and each open disk  $D$  with centre  $w$ , the set  $f_\lambda(D)$  contains the open disk with centre  $f_\lambda(w)$  and radius  $\delta \text{diam } f_\lambda(D)$ . In particular, for each  $D \in \mathcal{D}_n$ , the set  $f_\lambda(D)$  contains a disk with centre in  $f_\lambda(A)$  and diameter at least  $\delta \text{diam } f_\lambda(D)$ . Denoting by  $\mathcal{D}'_n$  the set of such disks, we obtain a finite set of disjoint disks  $D'$  with centres in  $A_\lambda$  and such that

$$\sum_{D' \in \mathcal{D}'_n} (\text{diam } D')^c \geq \sum_{D \in \mathcal{D}_n} (\delta \text{diam } f_\lambda(D))^c \geq \delta^c M^{c-d_n} \sum_{D \in \mathcal{D}_n} (\text{diam } D)^{d_n}.$$

From (5.2), we have  $\sum_{D \in \mathcal{D}_n} (\text{diam } D)^{d_n} \rightarrow \infty$ , whence it follows that

$$(5.3) \quad \sum_{D' \in \mathcal{D}'_n} (\text{diam } D')^c \rightarrow \infty \quad (n \rightarrow \infty).$$

Also from (5.2), we have  $\max_{D \in \mathcal{D}_n} \text{diam}(D) \rightarrow 0$ , which, together with the inequality (5.1), implies that

$$(5.4) \quad \max_{D' \in \mathcal{D}'_n} \text{diam}(D') \rightarrow 0 \quad (n \rightarrow \infty).$$

Taken together, the limits (5.3) and (5.4) show that  $\overline{\dim}_M(A_\lambda) \geq c$ . As this holds for each  $c \in (0, 1/u(\lambda))$ , we deduce that  $\overline{\dim}_M(A_\lambda) \geq 1/u(\lambda)$ , in other words, that  $u(\lambda) \geq 1/\overline{\dim}_M(A_\lambda)$ , as desired.

The proof of the lemma is nearly complete, save for the fact that  $u$  is defined only on  $D(0, \rho)$ , not on  $\mathbb{D}$ . To fix this, let us choose an increasing sequence  $(\rho_m)$  in  $(0, 1)$  such that  $\rho_m \rightarrow 1$ . For each  $m$ , the argument above furnishes an inf-harmonic function  $u_m$  defined on  $D(0, \rho_m)$  such that  $u_m(0) = 1/\overline{\dim}_M(A)$  and  $u_m(\lambda) \geq 1/\overline{\dim}_M(A_\lambda)$  for all  $\lambda \in D(0, \rho_m)$ . By Proposition 4.3, a subsequence of  $(u_m)$  converges locally uniformly to an inf-harmonic function  $u$  on  $\mathbb{D}$ . Clearly we have  $u(0) = 1/\overline{\dim}_M(A)$  and  $u(\lambda) \geq 1/\overline{\dim}_M(A_\lambda)$  for all  $\lambda \in \mathbb{D}$ . The proof is now complete.  $\square$

From here, it is a small step to establish the main result.

*Proof of Theorem 1.3.* It is enough to show that, for each  $\lambda_0 \in \mathbb{D}$  such that  $\overline{\dim}_M(A_{\lambda_0}) > 0$ , there exists an inf-harmonic function  $u$  on  $\mathbb{D}$  such that

$$(5.5) \quad u(\lambda_0) = 1/\overline{\dim}_M(A_{\lambda_0}) \quad \text{and} \quad u(\lambda) \geq 1/\overline{\dim}_M(A_\lambda) \quad (\lambda \in \mathbb{D}).$$

The special case  $\lambda_0 = 0$  has already been proved in Lemma 5.2. The general case can be deduced from this as follows.



Fix a Möbius automorphism  $\phi$  of  $\mathbb{D}$  such that  $\phi(0) = \lambda_0$ . Then  $\tilde{f}_\lambda := f_{\phi(\lambda)} \circ f_{\lambda_0}^{-1}$  is a holomorphic motion mapping  $\mathbb{D} \times \mathbb{C}$  into  $\mathbb{C}$ . Also  $\tilde{A} := A_{\lambda_0}$  is a bounded subset of  $\mathbb{C}$ , such that  $\tilde{f}_\lambda(\tilde{A}) = A_{\phi(\lambda)}$  for all  $\lambda \in \mathbb{D}$ . Thus, applying Lemma 5.2 with  $A, f$  replaced by the pair  $\tilde{A}, \tilde{f}$ , we deduce that there exists an inf-harmonic function  $v$  on  $\mathbb{D}$  such that

$$v(0) = 1/\overline{\dim}_M(A_{\phi(0)}) \quad \text{and} \quad v(\lambda) \geq 1/\overline{\dim}_M(A_{\phi(\lambda)}) \quad (\lambda \in \mathbb{D}).$$

Then  $u := v \circ \phi^{-1}$  is an inf-harmonic function on  $\mathbb{D}$  satisfying (5.5). This completes the proof of the theorem.  $\square$

## 6. PROOF OF THEOREM 1.4

As in the previous section, we may suppose that  $A_\lambda = f_\lambda(A) = f(\lambda, A)$ , where  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  is a holomorphic motion.

We shall deduce Theorem 1.4 from Theorem 1.3, using the characterization of packing dimension in terms of Minkowski dimension given in Proposition 2.3. From that result, we have

$$\dim_P(A) = \inf \left\{ \sup_{j \geq 1} \overline{\dim}_M(A_j) : A = \cup_{j \geq 1} A_j \right\},$$

where the infimum is taken over all countable covers of  $A$  by bounded subsets  $(A_j)$ . Since  $f_\lambda$  is a bijection of  $A$  onto  $A_\lambda$ , it follows that,

$$\dim_P(A_\lambda) = \inf \left\{ \sup_{j \geq 1} \overline{\dim}_M(f_\lambda(A_j)) : A = \cup_{j \geq 1} A_j \right\} \quad (\lambda \in \mathbb{D}),$$

and hence

$$(6.1) \quad \frac{1}{\dim_P(A_\lambda)} = \sup \left\{ \inf_{j \geq 1} \frac{1}{\overline{\dim}_M(f_\lambda(A_j))} : A = \cup_{j \geq 1} A_j \right\} \quad (\lambda \in \mathbb{D}).$$

Let  $A = \cup_{j \geq 1} A_j$  be a countable cover of  $A$  by bounded subsets of  $A$ . By Theorem 1.3, for each  $j$ , either  $\overline{\dim}_M(f_\lambda(A_j)) \equiv 0$  or  $\lambda \mapsto 1/\overline{\dim}_M(f_\lambda(A_j))$  is an inf-harmonic function on  $\mathbb{D}$ . It follows that either  $\overline{\dim}_M(f_\lambda(A_j)) \equiv 0$  for all  $j \geq 1$  or else  $\lambda \mapsto \inf_{j \geq 1} 1/\overline{\dim}_M(f_\lambda(A_j))$  is an inf-harmonic function on  $\mathbb{D}$ . In the first case, (6.1) implies that  $\dim_P(A_\lambda) \equiv 0$ . In the second case, the relation (6.1) expresses  $1/\dim_P(A_\lambda)$  as the supremum of a family of inf-harmonic functions.

Ordinarily, the supremum of a family of inf-harmonic functions is no longer inf-harmonic. However, this particular family is an upward-directed set, in the sense of Proposition 4.4 (vi). Indeed, given any two countable covers  $A = \cup_i A_i = \cup_j B_j$  of  $A$  by bounded sets, there is a third such cover, namely  $A = \cup_{i,j} (A_i \cap B_j)$ , with the property that

$$\sup_{i,j} \overline{\dim}_M(A_i \cap B_j) \leq \min \left\{ \sup_i \overline{\dim}_M(A_i), \sup_j \overline{\dim}_M(B_j) \right\},$$

which implies upward-directedness in (6.1). By Proposition 4.4 (vi), it follows that either  $\dim_P(A_\lambda) \equiv 0$  or  $\lambda \mapsto 1/\dim_P(A_\lambda)$  is inf-harmonic on  $\mathbb{D}$ . This completes the proof of Theorem 1.4.  $\square$

## 7. PROOF OF THEOREM 1.6

The proof of Theorem 1.6 follows a similar pattern to that of Theorem 1.3, presented in §5, except that, because Hausdorff dimension is defined in terms of coverings rather than packings, some of the inequalities go in the other direction. Unfortunately, this leads ultimately to a weaker result.

We have  $A_\lambda = f_\lambda(A) = f(\lambda, A)$ , where  $f : \mathbb{D} \times A \rightarrow \mathbb{C}$  is a holomorphic motion. As before, we may extend  $f$  to a holomorphic motion  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$ , and we shall assume that  $f$  has been so extended.

The core of the proof is contained in the following lemma.

**Lemma 7.1.** (i) *If  $\dim_H(A) = 0$ , then  $\dim_H(A_\lambda) = 0$  for all  $\lambda \in \mathbb{D}$ .*  
(ii) *If  $\dim_H(A) > 0$ , then there exists an inf-harmonic function  $u$  on  $\mathbb{D}$  such that*

$$u(0) = 1/\dim_H(A) \quad \text{and} \quad 1/2 \leq u(\lambda) \leq 1/\dim_H(A_\lambda) \quad (\lambda \in \mathbb{D}).$$

*Proof.* If  $\dim_H(A) = 2$ , then we may simply take  $u \equiv 1/2$ . Henceforth, we suppose that  $0 \leq \dim_H(A) < 2$ .

Let  $\rho \in (0, 1)$ . We shall carry out the proof on the disk  $D(0, \rho)$ , and then let  $\rho \rightarrow 1$  at the very end.

Let  $(d_n)$  be a sequence such that  $\dim_H(A) < d_n < 2$  and  $d_n \rightarrow \dim_H(A)$ . By Proposition 2.1, for each  $n$  there exists a (countable) cover  $\mathcal{Q}_n$  of  $A$  by disjoint dyadic squares such that, as  $n \rightarrow \infty$ ,

$$(7.1) \quad \sup_{Q \in \mathcal{Q}_n} \text{diam}(Q) \rightarrow 0 \quad \text{and} \quad \sum_{Q \in \mathcal{Q}_n} \text{diam}(Q)^{d_n} \rightarrow 0.$$

We can suppose that all the squares in  $\cup_n \mathcal{Q}_n$  meet  $A$ . Thus, if  $B$  is the union of all the squares in  $\cup_n \mathcal{Q}_n$ , then  $B$  is a bounded set. By Lemma 5.1,  $M := \text{diam } f(D(0, \rho) \times B) < \infty$ , and  $\lambda \mapsto \log(M/\text{diam } f_\lambda(Q))$  is inf-harmonic on  $D(0, \rho)$  for each  $Q \in \cup_n \mathcal{Q}_n$ .

Fix a constant  $C$ , to be chosen later (it will depend only on  $\rho$ ), and, for each  $\lambda \in D(0, \rho)$ , set

$$s_n(\lambda) := \inf \left\{ \alpha > 0 : \sum_{Q \in \mathcal{Q}_n} \left( \frac{\text{diam } f_\lambda(Q)}{M} \right)^\alpha \leq C \right\}.$$

By the implicit function theorem, Theorem 4.5, either  $s_n \equiv 0$  or  $1/s_n$  is inf-harmonic on  $D(0, \rho)$ . By Proposition 4.3, a subsequence of  $(s_n)$  (which, by relabelling, we may suppose to be the whole sequence) converges locally uniformly to  $s$  on  $D(0, \rho)$ , where either  $s \equiv 0$  or  $1/s$  is inf-harmonic on  $D(0, \rho)$ .

From (7.1) we have we have  $s_n(0) \leq d_n$  for all sufficiently large  $n$ , so

$$(7.2) \quad s(0) = \lim_{n \rightarrow \infty} s_n(0) \leq \lim_{n \rightarrow \infty} d_n = \dim_H(A).$$

If  $\alpha > s(\lambda)$  for some  $\lambda \in D(0, \rho)$ , then  $\alpha > s_n(\lambda)$  for all large enough  $n$ , so, for these  $n$ ,

$$\sum_{Q \in \mathcal{Q}_n} \text{diam } f_\lambda(Q)^\alpha \leq CM^\alpha.$$

For each  $n$ , the family  $\{f_\lambda(Q) : Q \in \mathcal{Q}_n\}$  is a cover of  $A_\lambda$ . Also, from (7.1) and (5.1), we have  $\sup_{Q \in \mathcal{Q}_n} \text{diam } f_\lambda(Q) \rightarrow 0$  as  $n \rightarrow \infty$ . It follows from the definition of Hausdorff dimension that  $\dim_H(A_\lambda) \leq \alpha$ . As this holds for each  $\alpha > s(\lambda)$ , we conclude that

$$(7.3) \quad s(\lambda) \geq \dim_H(A_\lambda) \quad (\lambda \in D(0, \rho)).$$

Next we show that, if the constant  $C$  is chosen sufficiently large, then we also have  $s(\lambda) \leq 2$  for all  $\lambda \in D(0, \rho)$ . To achieve this, we once again invoke the theory of quasiconformal mappings. By Theorem 3.4, the map  $f_\lambda$  is a  $\rho$ -quasiconformal self-homeomorphism of  $\mathbb{C}$ . Consequently, by Corollary 3.7 (ii), there exists a constant  $\delta' > 0$ , depending only on  $\rho$ , such that, for each open square  $Q$  in  $\mathbb{C}$ , the set  $f_\lambda(Q)$  contains an open disk of radius  $\delta' \text{diam } f_\lambda(Q)$ . In particular, for each  $n$ , the disjoint sets  $\{f_\lambda(Q) : Q \in \mathcal{Q}_n\}$  contain disjoint disks of radii  $\delta' \text{diam } f_\lambda(Q)$ . As these disks are all contained within the set  $f(D(0, \rho) \times B)$ , which has diameter  $M$ , consideration of their areas leads to the inequality

$$\sum_{Q \in \mathcal{Q}_n} \pi(\delta' \text{diam } f_\lambda(Q))^2 \leq \pi M^2,$$

in other words,

$$\sum_{Q \in \mathcal{Q}_n} \left( \frac{\text{diam}(f_\lambda(Q))}{M} \right)^2 \leq 1/\delta'^2.$$

This shows that, if  $C \geq 1/\delta'^2$ , then  $s_n(\lambda) \leq 2$  for all  $n$ , and consequently  $s(\lambda) \leq 2$ .

To summarize, we have shown that, if  $\dim_H(A) = 0$ , then  $\dim_H(A_\lambda) = 0$  for all  $\lambda \in D(0, \rho)$  (combine (7.2) and (7.3)), and, if  $\dim_H(A) > 0$ , then  $u := 1/s$  is an inf-harmonic function on  $D(0, \rho)$  such that

$$u(0) = 1/\dim_H(A) \quad \text{and} \quad 1/2 \leq u(\lambda) \leq 1/\dim_H(A_\lambda) \quad (\lambda \in D(0, \rho)).$$

The proof of the lemma is nearly complete, except that  $u$  is defined only on  $D(0, \rho)$ , not on  $\mathbb{D}$ . We fix this in exactly the same way as at the end of the proof of Lemma 5.2.  $\square$

*Remark.* Part (i) of Lemma 7.1 could also have been proved using the well-known fact that the quasiconformal image of a set of Hausdorff dimension zero also has Hausdorff dimension zero.

*Proof of Theorem 1.6.* We claim that, for each  $\zeta \in \mathbb{D}$ , if  $\dim_H(A_\zeta) = 0$ , then  $\dim_H(A_\lambda) = 0$  for all  $\lambda \in \mathbb{D}$ , and, if  $\dim_H(A_\zeta) > 0$ , then there exists an inf-harmonic function  $u_\zeta$  on  $\mathbb{D}$  such that

$$u_\zeta(\zeta) = 1/\dim_H(A_\zeta) \quad \text{and} \quad 1/2 \leq u_\zeta(\lambda) \leq 1/\dim_H(A_\lambda) \quad (\lambda \in \mathbb{D}).$$

The special case  $\zeta = 0$  has been proved in Lemma 7.1, and the general case is deduced from this just as in the proof of Theorem 1.3 at the end of §5.

Thus, either  $\dim_H(A_\lambda) = 0$  for all  $\lambda \in \mathbb{D}$ , or  $\dim_H(A_\lambda) > 0$  for all  $\lambda \in \mathbb{D}$ . In the latter case, we have

$$\frac{1}{\dim_H(A_\lambda)} - \frac{1}{2} = \sup_{\zeta \in \mathbb{D}} (u_\zeta(\lambda) - 1/2) \quad (\lambda \in \mathbb{D}),$$

where the right-hand side is the supremum of a family of functions that are inf-harmonic on  $\mathbb{D}$ .  $\square$

## 8. PROOF OF THEOREM 1.7

The essential idea of the proof is contained in the following lemma, which is based on a construction in Astala's paper [1].

**Lemma 8.1.** *Let  $h : \mathbb{D} \rightarrow (0, \infty)$  be a positive harmonic function, and let  $n \geq 10$ . Then there exists a holomorphic motion  $\lambda \mapsto E_\lambda$  such that  $E_\lambda$  is a compact subset of  $\mathbb{D}$  for all  $\lambda \in \mathbb{D}$  and*

$$\frac{1}{\dim_H(E_\lambda)} = \frac{1}{\dim_P(E_\lambda)} = h(\lambda) + \frac{1}{2} + \frac{\log 2}{2 \log n} \quad (\lambda \in \mathbb{D}).$$

*Proof.* As  $h$  is a positive harmonic function on  $\mathbb{D}$ , there exists a holomorphic function  $a : \mathbb{D} \rightarrow \mathbb{D} \setminus \{0\}$  such that

$$\log |a(\lambda)| = -h(\lambda) \log n \quad (\lambda \in \mathbb{D}).$$

Let  $\overline{D}(w_1, r), \dots, \overline{D}(w_n, r)$  be disjoint closed disks inside  $\mathbb{D}$ , where  $r = 1/\sqrt{2n}$ . Such disks may be found if  $n \geq 10$ . For  $j = 1, \dots, n$  and  $\lambda \in \mathbb{D}$ , define

$$\gamma_{j,\lambda}(z) := ra(\lambda)z + w_j \quad (z \in \mathbb{C}).$$

Note that  $\gamma_{j,\lambda}(\mathbb{D}) \subset D(w_j, r)$  for each  $j = 1, \dots, n$  and each  $\lambda \in \mathbb{D}$ . Thus, for each  $\lambda \in \mathbb{D}$ , the family  $\{\gamma_{j,\lambda} : j = 1, \dots, n\}$  generates an iterated function system satisfying the open set condition. If we denote by  $E_\lambda$  its limit set, then  $\lambda \mapsto E_\lambda$  is a compact-valued holomorphic motion (see e.g. [4, Theorem 4]) such that  $E_\lambda \subset \cup_{j=1}^n \overline{D}(w_j, r) \subset \mathbb{D}$  for all  $\lambda \in \mathbb{D}$ . Moreover, by a special case of the Hutchinson–Moran formula Theorem 2.4, the Hausdorff and packing dimensions of  $E_\lambda$  are given by  $\dim_H E_\lambda = \dim_P E_\lambda = s(\lambda)$ , where  $s(\lambda)$  is the solution of the equation

$$n(r|a(\lambda)|)^{s(\lambda)} = 1.$$

Solving this equation, we obtain

$$\frac{1}{s(\lambda)} = -\frac{\log(r|a(\lambda)|)}{\log n} = \frac{\log(\sqrt{2n}) + h(\lambda) \log n}{\log n} = \frac{\log 2}{2 \log n} + \frac{1}{2} + h(\lambda).$$

This completes the proof.  $\square$

**Lemma 8.2.** *Let  $D$  be a domain and let  $u$  be an inf-harmonic function on  $D$ . Then there exists a sequence  $(h_n)_{n \geq 1}$  of positive harmonic functions on  $D$  such that, for every  $m \geq 1$ , we have  $u = \inf_{n \geq m} h_n$  on  $D$ .*

*Proof.* Let  $S$  be a countable dense subset of  $D$ , and let  $(\lambda_n)_{n \geq 1}$  be a sequence in  $S$  that visits every point of  $S$  infinitely often. Since  $u$  is inf-harmonic on  $D$ , for each  $n \geq 1$  there exists a positive harmonic function  $h_n$  on  $D$  such that  $h_n \geq u$  and  $h_n(\lambda_n) < u(\lambda_n) + 1/n$ . Then, for each  $m \geq 1$ , we have  $u = \inf_{n \geq m} h_n$  on  $S$ . Since  $S$  is dense in  $D$  and inf-harmonic functions are automatically continuous, it follows that  $u = \inf_{n \geq m} h_n$  on  $D$ .  $\square$

*Proof of Theorem 1.7.* Set  $u(\lambda) := 1/d(\lambda) - 1/2$ . Since  $1/d$  is inf-harmonic and  $1/d \geq 1/2$ , it follows that  $u$  is inf-harmonic as well. By Lemma 8.2, there exists a sequence  $(h_n)_{n \geq 1}$  of positive harmonic functions on  $\mathbb{D}$  such that  $u = \inf_{n \geq m} h_n$  for every  $m \geq 1$ .

By Lemma 8.1, for each  $n \geq 10$ , there exists a compact-valued holomorphic motion  $\lambda \mapsto E_\lambda^{(n)}$  in  $\mathbb{D}$  such that

$$\frac{1}{\dim_H(E_\lambda^{(n)})} = \frac{1}{\dim_P(E_\lambda^{(n)})} = h_n(\lambda) + \frac{1}{2} + \frac{\log 2}{2 \log n} \quad (\lambda \in \mathbb{D}).$$

Fix a sequence of disjoint closed disks  $\overline{D}(\zeta_n, s_n)$  in  $\mathbb{C}$  such that  $\zeta_n \rightarrow 0$  and  $s_n \rightarrow 0$ , and define

$$A_\lambda := \bigcup_{n \geq 10} (s_n E_\lambda^{(n)} + \zeta_n) \cup \{0\} \quad (\lambda \in \mathbb{D}).$$

Then  $\lambda \mapsto A_\lambda$  is a union of holomorphic motions taking place in disjoint disks, so it is itself a holomorphic motion. Moreover  $A_\lambda$  is a compact set for each  $\lambda \in \mathbb{D}$ . Finally, since both Hausdorff dimension and packing dimension are countably stable, and these dimensions are unchanged under similarities, we have

$$\begin{aligned} \frac{1}{\dim_H(A_\lambda)} &= \frac{1}{\dim_P(A_\lambda)} = \inf_{n \geq 10} \left( \frac{1}{\dim_P(E_\lambda^{(n)})} \right) \\ &= \inf_{n \geq 10} \left( h_n(\lambda) + \frac{1}{2} + \frac{\log 2}{2 \log n} \right) \\ &= u(\lambda) + \frac{1}{2} = \frac{1}{d(\lambda)}. \end{aligned}$$

In other words,  $\dim_H(A_\lambda) = \dim_P(A_\lambda) = d(\lambda)$  for all  $\lambda \in \mathbb{D}$ . This completes the proof.  $\square$

## 9. PROOF OF THEOREM 1.9

In this section, we prove Theorem 1.9 on the variation of the area of a set moving under a holomorphic motion. The proof of part (i) follows closely the ideas of [8], as elaborated in [2, §13.1]. We first need the following lemmas.

**Lemma 9.1.** *Let  $(\Omega, \nu)$  be a measure space and let  $a : \Omega \rightarrow (0, \infty)$  be a measurable function such that  $\int_\Omega a d\nu < \infty$ . Then, for every measurable*

function  $p : \Omega \rightarrow (0, \infty)$  such that  $\int_{\Omega} p \, d\nu = 1$ , we have

$$\log\left(\int_{\Omega} a \, d\nu\right) \geq \int_{\Omega} p \log\left(\frac{a}{p}\right) d\nu,$$

with equality if  $p = a/(\int_{\Omega} a \, d\nu)$ .

*Proof.* The inequality follows from Jensen's inequality applied to the concave function  $\log x$  and the probability space  $(\Omega, p \, d\nu)$ . The case of equality is obvious.  $\square$

**Lemma 9.2.** *Let  $D$  be a plane domain, let  $(\Omega, \nu)$  be a finite measure space, and let  $h : D \times \Omega \rightarrow \mathbb{R}$  be a measurable function such that:*

- $\lambda \mapsto h(\lambda, \omega)$  is harmonic on  $D$ , for each  $\omega \in \Omega$ ;
- $\sup_{K \times \Omega} |h(\lambda, \omega)| < \infty$  for each compact  $K \subset D$ .

*Then the function  $H(\lambda) := \int_{\Omega} h(\lambda, \omega) \, d\nu(\omega)$  is harmonic on  $D$ .*

*Proof.* The function  $H$  is continuous on  $D$ , by the dominated convergence theorem. Also it satisfies the mean-value property on  $D$ , by the harmonicity of  $h(\cdot, \omega)$  and Fubini's theorem. Therefore  $H$  is harmonic on  $D$ .  $\square$

**Lemma 9.3.** *Let  $k \in (0, 1)$  and let  $R > 0$ . Let  $g, g_n : \mathbb{C} \rightarrow \mathbb{C}$  be  $k$ -quasiconformal homeomorphisms such that*

- $\mu_{g_n} \rightarrow \mu_g$  a.e. on  $\mathbb{C}$ ,
- $\text{supp } \mu_{g_n} \subset D(0, R)$  for each  $n \geq 1$ ,
- $g_n(z) = z + o(1) = g(z)$  as  $|z| \rightarrow \infty$  for each  $n \geq 1$ .

*Then*

$$\|\partial_z g_n - \partial_z g\|_{L^2(\mathbb{C})} \rightarrow 0 \quad \text{and} \quad \|\partial_{\bar{z}} g_n - \partial_{\bar{z}} g\|_{L^2(\mathbb{C})} \rightarrow 0.$$

*Proof.* The second limit holds by [2, Lemma 5.3.1]. The first limit is an automatic consequence, since  $\|\partial_z g_n - \partial_z g\|_{L^2(\mathbb{C})} = \|\partial_{\bar{z}} g_n - \partial_{\bar{z}} g\|_{L^2(\mathbb{C})}$ . This is because the Beurling transform, which takes  $\partial_{\bar{z}} f$  to  $\partial_z f$ , is a unitary operator on  $L^2(\mathbb{C})$  (see the discussion on [2, p.95]).  $\square$

*Proof of Theorem 1.9.* Let  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic motion. Suppose that there exists a compact subset  $\Delta$  of  $\mathbb{C}$  such that, for each  $\lambda \in \mathbb{D}$ , the map  $f_{\lambda}$  is conformal on  $\mathbb{C} \setminus \Delta$  and  $f_{\lambda}(z) = z + O(1)$  near  $\infty$ . Let  $A$  be a Borel subset of  $\Delta$  such that  $|A| > 0$ . We begin with some preliminary remarks.

The first remark is that, in the normalization  $f_{\lambda}(z) = z + O(1)$  near  $\infty$ , we may as well suppose that in fact  $f_{\lambda}(z) = z + o(1)$  near  $\infty$ . Indeed, it suffices to consider the translated holomorphic motion  $f(\lambda, z) - a_0(\lambda)$ , where  $a_0(\lambda)$  is the constant coefficient in the Laurent expansion of  $f_{\lambda}(z)$  near infinity. Note that  $a_0(\lambda)$  is holomorphic in  $\mathbb{D}$ , as can be seen from the formula

$$a_0(\lambda) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f_{\lambda}(z)}{z} dz,$$

valid for all  $R$  large enough so that  $\Delta \subset D(0, R)$ .

Next, we claim that there is a simple *a priori* bound on  $|A_\lambda|$ , namely

$$(9.1) \quad |A_\lambda| \leq \pi c(\Delta)^2 \quad (\lambda \in \mathbb{D}).$$

Here  $c(\Delta)$  is the logarithmic capacity of  $\Delta$ , see e.g. [20, Chapter 5] for the definition. Indeed, since  $f_\lambda$  is a conformal homeomorphism of  $\mathbb{C} \setminus \Delta$  onto  $\mathbb{C} \setminus f_\lambda(\Delta)$  satisfying  $f_\lambda(z) = z + o(1)$  at infinity, the sets  $\Delta$  and  $f_\lambda(\Delta)$  have the same logarithmic capacity:

$$c(f_\lambda(\Delta)) = c(\Delta) \quad (\lambda \in \mathbb{D}),$$

by [20, Theorem 5.2.3]. From the isoperimetric inequality for logarithmic capacity ([20, Theorem 5.3.5]) we have  $|f_\lambda(\Delta)| \leq \pi c(f_\lambda(\Delta))^2$ , and it follows that

$$|A_\lambda| = |f_\lambda(A)| \leq |f_\lambda(\Delta)| \leq \pi c(f_\lambda(\Delta))^2 = \pi c(\Delta)^2,$$

as claimed.

We now turn to the proof of part (i) of the theorem. Suppose first that  $A$  is compact and that there exists an open neighbourhood  $U$  of  $A$  such that  $\mu_{f_\lambda} \equiv 0$  on  $U$  for all  $\lambda \in \mathbb{D}$ . Then each  $f_\lambda$  is a conformal mapping on  $U$ , so  $f'_\lambda(z) \neq 0$  for all  $z \in U$ . By the standard Jacobian formula for area, we have

$$|A_\lambda| = |f_\lambda(A)| = \int_A |f'_\lambda(z)|^2 dm(z),$$

where  $dm$  denotes area measure on  $\mathbb{C}$ . Using Lemma 9.1, we can write  $\log |A_\lambda|$  as

$$\log |A_\lambda| = \sup_p \left\{ \int_A p(z) \log \left( \frac{|f'_\lambda(z)|^2}{p(z)} \right) dm(z) \right\},$$

where the supremum is taken over all continuous functions  $p : A \rightarrow (0, \infty)$  such that  $\int_A p dm = 1$ . By Lemma 9.2, each of the integrals is a harmonic function of  $\lambda \in \mathbb{D}$ . Therefore  $\log(C/|A_\lambda|)$  is an inf-harmonic function on  $\mathbb{D}$  for each  $C \geq \sup_{\lambda \in \mathbb{D}} |A_\lambda|$ , in particular for  $C = \pi c(\Delta)^2$ , by (9.1).

Suppose now that  $A$  is merely Borel, but still that  $\mu_{f_\lambda} \equiv 0$  on  $U$  for all  $\lambda \in \mathbb{D}$ . We have

$$\log \left( \frac{\pi c(\Delta)^2}{|A_\lambda|} \right) = \inf_F \log \left( \frac{\pi c(\Delta)^2}{|F_\lambda|} \right) \quad (\lambda \in \mathbb{D}),$$

where the infimum is taken over all compact subsets  $F$  of  $A$ . Each function on the right-hand side is inf-harmonic on  $\mathbb{D}$ , by what we have already proved. Therefore the left-hand side is inf-harmonic on  $\mathbb{D}$  as well.

Finally, suppose merely that  $\mu_{f_\lambda} = 0$  a.e. on  $A$  for each  $\lambda \in \mathbb{D}$ . Let  $U_n$  be a decreasing sequence of bounded open sets such that  $|U_n \setminus A| \rightarrow 0$ . By Theorem 3.4, for each  $n$  there exists a holomorphic motion  $f_n : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  such that, for each  $\lambda \in \mathbb{D}$ , we have  $\mu_{f_{n,\lambda}} = 1_{\mathbb{C} \setminus U_n} \mu_{f_\lambda}$  a.e. on  $\mathbb{C}$  and  $f_{n,\lambda}(z) = z + o(1)$  near  $\infty$ . By Lemma 9.3, it follows that

$$\|\partial_z f_{n,\lambda} - \partial_z f_\lambda\|_{L^2(\mathbb{C})} \rightarrow 0 \quad \text{and} \quad \|\partial_{\bar{z}} f_{n,\lambda} - \partial_{\bar{z}} f_\lambda\|_{L^2(\mathbb{C})} \rightarrow 0.$$

Therefore

$$\int_A |\partial_z f_\lambda|^2 dm = \lim_{n \rightarrow \infty} \int_A |\partial_z f_{n,\lambda}|^2 dm = \lim_{n \rightarrow \infty} |f_{n,\lambda}(A)|$$

and

$$\int_A |\partial_{\bar{z}} f_\lambda|^2 dm = \lim_{n \rightarrow \infty} \int_A |\partial_{\bar{z}} f_{n,\lambda}|^2 dm = 0.$$

Hence, using [2, formula (2.24)], we obtain

$$|f_\lambda(A)| = \int_A (|\partial_z f_\lambda|^2 - |\partial_{\bar{z}} f_\lambda|^2) dm = \lim_{n \rightarrow \infty} |f_{n,\lambda}(A)|.$$

Thus

$$\log\left(\frac{\pi c(\Delta)^2}{|A_\lambda|}\right) = \lim_{n \rightarrow \infty} \log\left(\frac{\pi c(\Delta)^2}{|f_{n,\lambda}(A)|}\right) \quad (\lambda \in \mathbb{D}).$$

By what we have already proved, the right-hand sides are inf-harmonic functions of  $\lambda$ . It follows from Proposition 4.4 part (iii) that the left-hand side is inf-harmonic on  $\mathbb{D}$  as well. This completes the proof of part (i) of the theorem.

We now turn to the proof of part (ii). Set  $R_0 := \sup_{z \in \Delta} |z|$  and, for  $R > R_0$ , set  $\Delta_R := \overline{D}(0, R)$  (so  $c(\Delta_R) = R$ ). By hypothesis  $\mu_{f_\lambda} = 0$  a.e. on  $\Delta_R \setminus A$ . So, applying what we have proved in part (i) (with  $\Delta$  replaced by  $\Delta_R$  and  $A$  replaced by  $\Delta_R \setminus A$ ), we see that

$$\lambda \mapsto \log\left(\frac{\pi R^2}{|f_\lambda(\Delta_R \setminus A)|}\right)$$

is an inf-harmonic function on  $\mathbb{D}$ . Now, fix  $\lambda \in \mathbb{D}$ . Then  $|f_\lambda(\Delta_R \setminus A)| = |f_\lambda(\Delta_R)| - |A_\lambda|$ , and by the area theorem from univalent function theory,

$$|f_\lambda(\Delta_R)| = \pi R^2 - \pi \sum_{n \geq 1} n |a_n(\lambda)|^2 R^{-2n},$$

where  $f_\lambda(z) = z + \sum_{n \geq 1} a_n(\lambda) z^{-n}$  is the Laurent expansion of  $f_\lambda$  near infinity. In particular,  $|f_\lambda(\Delta_R)| = \pi R^2 + O(R^{-2})$  as  $R \rightarrow \infty$ . Hence

$$\log\left(\frac{\pi R^2}{|f_\lambda(\Delta_R \setminus A)|}\right) = \log\left(\frac{\pi R^2}{\pi R^2 - |A_\lambda| + O(R^{-2})}\right) = \frac{|A_\lambda|}{\pi R^2} + O(R^{-4}) \quad (R \rightarrow \infty).$$

It follows that

$$|A_\lambda| = \lim_{R \rightarrow \infty} \pi R^2 \log\left(\frac{\pi R^2}{|f_\lambda(\Delta_R \setminus A)|}\right) \quad (\lambda \in \mathbb{D}).$$

By what we have shown earlier, the right-hand sides are inf-harmonic functions of  $\lambda$ . It follows from Proposition 4.4 part (iii) that the left-hand side is inf-harmonic on  $\mathbb{D}$  as well. This completes the proof of part (ii) of the theorem.  $\square$



## 10. APPLICATIONS TO QUASICONFORMAL MAPS

In this section we show how our results lead to a unified approach to the four theorems on quasiconformal distortion of area and dimension that were stated at the end of the introduction.

**10.1. Distortion of dimension by quasiconformal maps.** In this subsection we establish Theorem 1.10, to the effect that, if  $F : \mathbb{C} \rightarrow \mathbb{C}$  is a  $k$ -quasiconformal homeomorphism and  $\dim A > 0$ , then

$$(10.1) \quad \frac{1}{K} \left( \frac{1}{\dim A} - \frac{1}{2} \right) \leq \left( \frac{1}{\dim F(A)} - \frac{1}{2} \right) \leq K \left( \frac{1}{\dim A} - \frac{1}{2} \right),$$

where  $K = (1+k)/(1-k)$ . Here  $\dim$  denotes any one of  $\dim_P, \dim_H$  or  $\overline{\dim}_M$ . In the case of  $\overline{\dim}_M$ , we also suppose that the set  $A$  is bounded.

*Proof of Theorem 1.10.* By Theorem 3.5, there exists a holomorphic motion  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  such that  $f_k = F$ . For  $\lambda \in \mathbb{D}$ , set  $A_\lambda := f_\lambda(A)$ . By Theorems 1.3, 1.4 and 1.6, either  $\lambda \mapsto (1/\dim(A_\lambda) - 1/2)$  is an inf-harmonic function on  $\mathbb{D}$ , or, at the very least, it is a supremum of inf-harmonic functions. Either way, it satisfies Harnack's inequality, so, for all  $\lambda \in \mathbb{D}$ , we have

$$\frac{1-|\lambda|}{1+|\lambda|} \left( \frac{1}{\dim(A_0)} - \frac{1}{2} \right) \leq \left( \frac{1}{\dim(A_\lambda)} - \frac{1}{2} \right) \leq \frac{1+|\lambda|}{1-|\lambda|} \left( \frac{1}{\dim(A_0)} - \frac{1}{2} \right).$$

In particular, taking  $\lambda = k$ , we obtain (10.1).  $\square$

*Remark.* One consequence of Theorem 1.10 is that, if  $f : \mathbb{D} \times A \rightarrow \mathbb{C}$  is a holomorphic motion and  $A_\lambda = f_\lambda(A)$ , then the map  $\lambda \mapsto \dim(A_\lambda)$  is a continuous function. For the Minkowski and packing dimensions, this was also proved in Corollary 1.5. For all three notions of dimension, it can also be seen more directly as follows.

As  $\lambda \rightarrow \lambda_0 \in \mathbb{D}$ , the transition map  $f_\lambda \circ f_{\lambda_0}^{-1}$  is  $k$ -quasiconformal with  $k$  tending to 0, hence also Hölder-continuous with Hölder exponent tending to 1 (see [2, Theorem 12.2.3 and Corollary 3.10.3]). Thus  $\dim(A_\lambda) = \dim(f_\lambda \circ f_{\lambda_0}^{-1})(A_{\lambda_0}) \rightarrow \dim(A_{\lambda_0})$  as  $\lambda \rightarrow \lambda_0$ .

**10.2. Distortion of area by quasiconformal maps.**

*Proof of Theorem 1.11.* Let  $F : \mathbb{C} \rightarrow \mathbb{C}$  be a  $k$ -quasiconformal homeomorphism which is conformal on  $\mathbb{C} \setminus \Delta$ , where  $\Delta$  is a compact set of logarithmic capacity at most 1, and such that  $F(z) = z + o(1)$  near  $\infty$ . Let  $A$  be a Borel subset of  $\Delta$ .

Let  $k := (K-1)/(K+1)$ . By Theorem 3.4, there is a holomorphic motion  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  with  $f_k = F$  and  $\mu_{f_\lambda} = (\lambda/k)\mu_F$  for each  $\lambda \in \mathbb{D}$ . We may also require that  $f_\lambda(z) = z + o(1)$  near  $\infty$ .

Suppose first that  $\mu_F = 0$  a.e. on  $A$ . By Theorem 1.9(i), the function  $\lambda \mapsto \log(\pi/|A_\lambda|)$  is inf-harmonic on  $\mathbb{D}$ . In particular, it satisfies Harnack's

inequality there:

$$\log\left(\frac{\pi}{|A_\lambda|}\right) \geq \frac{1-|\lambda|}{1+|\lambda|} \log\left(\frac{\pi}{|A|}\right) \quad (\lambda \in \mathbb{D}).$$

Setting  $\lambda = k$ , we obtain

$$\log\left(\frac{\pi}{|F(A)|}\right) \geq \frac{1}{K} \log\left(\frac{\pi}{|A|}\right).$$

This proves (i).

Suppose instead that  $\mu_F = 0$  a.e. on  $\mathbb{C} \setminus A$ . By Theorem 1.9(ii), the function  $\lambda \mapsto |A_\lambda|$  is inf-harmonic on  $\mathbb{D}$ . In particular, it satisfies Harnack's inequality there:

$$|A_\lambda| \leq \frac{1+|\lambda|}{1-|\lambda|} |A| \quad (\lambda \in \mathbb{D}).$$

Setting  $\lambda = k$ , we obtain

$$|F(A)| \leq K|A|.$$

This proves (ii).

Finally, the general case (iii) is deduced from (i) and (ii) via a standard factorization process, see e.g. [2, Theorem 13.1.4].  $\square$

*Remark.* As mentioned in §9, our proof of part (i) of Theorem 1.11 is quite similar to the original proof of Eremenko and Hamilton [8], as presented in [2, §13.1]. On the other hand, our proof of part (ii) is completely different from (and rather simpler than) the methods used in [8] and [2].

**10.3. Symmetric holomorphic motions and inf-sym-harmonic functions.** In preparation for the proofs of Theorems 1.12 and 1.13, we study what can be said about the function  $1/\dim(A_\lambda)$  when  $A$  is a subset of  $\mathbb{R}$  and  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  is a holomorphic motion that is symmetric in the sense defined below.

**Definition 10.1.** We say that a holomorphic motion  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  is *symmetric* if

$$f_\lambda(z) = \overline{f_{\bar{\lambda}}(\bar{z})} \quad (\lambda \in \mathbb{D}, z \in \mathbb{C}).$$

**Definition 10.2.** We say that a harmonic function  $h : \mathbb{D} \rightarrow \mathbb{R}$  is *symmetric* if  $h(\bar{\lambda}) = h(\lambda)$  for all  $\lambda \in \mathbb{D}$ . A function  $u : \mathbb{D} \rightarrow [0, \infty)$  is *inf-sym-harmonic* if there is a family  $\mathcal{H}$  of symmetric harmonic functions on  $\mathbb{D}$  such that

$$u(\lambda) = \inf_{h \in \mathcal{H}} h(\lambda) \quad (\lambda \in \mathbb{D}).$$

We now state symmetric versions of Lemmas 5.2 and 7.1.

**Lemma 10.3.** *Let  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  be a symmetric holomorphic motion and let  $A$  be a bounded subset of  $\mathbb{R}$  with  $\overline{\dim}_M(A) > 0$ . Set  $A_\lambda := f_\lambda(A)$ . Then there exists an inf-sym-harmonic function  $u$  on  $\mathbb{D}$  such that*

$$u(0) = 1/\overline{\dim}_M(A) \quad \text{and} \quad u(\lambda) \geq 1/\overline{\dim}_M(A_\lambda) \quad (\lambda \in \mathbb{D}).$$

**Lemma 10.4.** *Let  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  be a symmetric holomorphic motion and let  $A$  be a subset of  $\mathbb{R}$  with  $\dim_H(A) > 0$ . Set  $A_\lambda := f_\lambda(A)$ . Then there exists an inf-sym-harmonic function  $u$  on  $\mathbb{D}$  such that*

$$u(0) = 1/\dim_H(A) \quad \text{and} \quad 1/2 \leq u(\lambda) \leq 1/\dim_H(A_\lambda) \quad (\lambda \in \mathbb{D}).$$

*Proof.* The proofs follow closely those of Lemmas 5.2 and 7.1, with the following differences:

- If  $S \subset \mathbb{R}$ , then the function  $\log(M/\text{diam } f_\lambda(S))$  defined in Lemma 5.1 is inf-sym-harmonic on  $D(0, \rho)$ . This can be seen directly from the formula

$$\log\left(\frac{M}{\text{diam } f_\lambda(S)}\right) = \inf\left\{\log\left(\frac{M}{|f_\lambda(z) - f_\lambda(w)|}\right) : z, w \in S, z \neq w\right\},$$

using the symmetry relation  $f_\lambda(z) = \overline{f_\lambda(\bar{z})}$ .

- Consequently, if we replace the occurrences of  $f_\lambda(D)$  and  $f_\lambda(Q)$  in the proofs of Lemmas 5.2 and 7.1 by  $f_\lambda(D \cap \mathbb{R})$  and  $f_\lambda(Q \cap \mathbb{R})$  respectively, then all the functions that were previously inf-harmonic are now inf-sym-harmonic. Intersecting with  $\mathbb{R}$  leads to no loss of information about  $A$ , since  $A \subset \mathbb{R}$ .
- When applying the implicit function theorem or its corollary (Theorem 4.5 and Corollary 4.6), it is now assumed that the functions  $\log(1/a_j)$  are inf-sym-harmonic, and the conclusion is now that  $1/s$  is inf-sym-harmonic (or  $s \equiv 0$ ). This follows by applying Lemma 4.7, taking  $\mathcal{U}$  to be the inf-cone of inf-sym-harmonic functions.  $\square$

**10.4. Dimension of quasicircles.** In this subsection, we establish Theorem 1.12. More precisely, we use Lemma 10.4 to show that the Hausdorff dimension of a  $k$ -quasicircle is at most  $1 + k^2$ .

**Definition 10.5.** Let  $k \in [0, 1)$ . A curve  $\Gamma$  in  $\mathbb{C}$  is a  $k$ -quasicircle if  $\Gamma = g(\mathbb{R})$ , where  $g : \mathbb{C} \rightarrow \mathbb{C}$  is a normalized  $k$ -quasiconformal homeomorphism. By normalized, we mean simply that  $g$  fixes 0 and 1.

Quasicircles have been studied extensively over the years because of the desirable function-theoretic properties of the domains that they bound, see e.g. [10]. In particular, the problem of finding upper bounds for the Hausdorff dimension of a  $k$ -quasicircle in terms of  $k$  has attracted much interest. Theorem 1.10 implies that if  $\Gamma$  is a  $k$ -quasicircle, then

$$\dim_H(\Gamma) \leq 1 + k.$$

Motivated by examples of Becker and Pommerenke [5], Astala asked in [1] whether the upper bound can be replaced by  $1 + k^2$ . This was answered in the affirmative by Smirnov in [23]. As we will now see, Astala's question can also be answered using inf-harmonic functions.

We first need a result on symmetrization of Beltrami coefficients due to Smirnov [23, Theorem 4]. See also [2, §13.3.1].

**Lemma 10.6.** *The function  $g$  in Definition 10.5 may be chosen so that, in addition, its Beltrami coefficient satisfies the antisymmetry relation*

$$(10.2) \quad \overline{\mu_g(\bar{z})} = -\mu_g(z) \quad \text{a.e. in } \mathbb{C}.$$

We will also need the following Harnack-type inequality for inf-sym-harmonic functions, reminiscent of [23, Lemma 7]. See also [2, Lemma 13.3.8].

**Lemma 10.7.** *Let  $v : \mathbb{D} \rightarrow [0, \infty)$  be an inf-sym-harmonic function. Then*

$$\frac{1-y^2}{1+y^2}v(0) \leq v(iy) \leq \frac{1+y^2}{1-y^2}v(0) \quad (y \in (-1, 1)).$$

*Proof.* Write

$$v(\lambda) = \inf_{h \in \mathcal{H}} h(\lambda) \quad (\lambda \in \mathbb{D}),$$

where each  $h \in \mathcal{H}$  is a positive and symmetric harmonic function on  $\mathbb{D}$ . Fix  $h \in \mathcal{H}$ , and set  $k(\lambda) := (h(\lambda) + h(-\lambda))/2$ . Clearly  $k$  is an even positive harmonic function on  $\mathbb{D}$ . Thus it can be written as  $k(\lambda) = l(\lambda^2)$ , where  $l$  is a positive harmonic function on  $\mathbb{D}$ . Applying the standard Harnack inequality to  $l$ , we get

$$\frac{1-|\lambda|^2}{1+|\lambda|^2}k(0) \leq k(\lambda) \leq \frac{1+|\lambda|^2}{1-|\lambda|^2}k(0) \quad (\lambda \in \mathbb{D}).$$

As  $h$  is symmetric, we have

$$h(iy) = \frac{h(iy) + h(-iy)}{2} = k(iy) \quad (y \in (-1, 1)).$$

Hence

$$\frac{1-y^2}{1+y^2}h(0) \leq h(iy) \leq \frac{1+y^2}{1-y^2}h(0) \quad (\lambda \in \mathbb{D}).$$

Taking the infimum over all  $h \in \mathcal{H}$  gives the result.  $\square$

We can now prove the main result of this subsection.

*Proof of Theorem 1.12.* Let  $\Gamma$  be a  $k$ -quasicircle. By Lemma 10.6, we can write  $\Gamma = g(\mathbb{R})$  for some normalized  $k$ -quasiconformal mapping  $g : \mathbb{C} \rightarrow \mathbb{C}$  whose Beltrami coefficient  $\mu_g$  satisfies the antisymmetry relation (10.2). For  $\lambda \in \mathbb{D}$ , define a Beltrami coefficient  $\mu_\lambda$  by

$$\mu_\lambda := \frac{\lambda}{ik} \mu_g,$$

and denote by  $f_\lambda : \mathbb{C} \rightarrow \mathbb{C}$  the unique normalized quasiconformal mapping whose Beltrami coefficient is  $\mu_\lambda$ , as given by Theorem 3.3. Note that  $f_0$  is the identity and  $f_{ik} = g$ . It follows from Theorem 3.4 that the maps  $f_\lambda$  define a holomorphic motion of  $\mathbb{C}$ . Moreover, we have

$$\overline{\mu_\lambda(\bar{z})} = \frac{\lambda}{-ik} \overline{\mu_g(\bar{z})} = \mu_\lambda(z) \quad \text{a.e. in } \mathbb{C}.$$

It easily follows that the maps  $f_\lambda$  inherit the same symmetry:

$$f_\lambda(z) = \overline{f_{\bar{\lambda}}(\bar{z})} \quad (\lambda \in \mathbb{D}, z \in \mathbb{C}),$$

see e.g. [2, Section 13.3.1]. In other words, the holomorphic motion  $f$  is symmetric in the sense of Definition 10.1.

Now, let  $A := \mathbb{R}$ . By Lemma 10.4, there is an inf-sym-harmonic function  $u$  on  $\mathbb{D}$  such that

$$u(0) = 1/\dim_H(A) = 1 \quad \text{and} \quad 1/2 \leq u(\lambda) \leq 1/\dim_H(A_\lambda) \quad (\lambda \in \mathbb{D}).$$

In particular, the function  $v := u - 1/2$  is also inf-sym-harmonic, and Lemma 10.7 yields

$$v(ik) \geq \frac{1 - k^2}{1 + k^2} v(0) = \frac{1}{2} \frac{1 - k^2}{1 + k^2}.$$

But also

$$v(ik) \leq \frac{1}{\dim_H(f_{ik}(A))} - \frac{1}{2} = \frac{1}{\dim_H(\Gamma)} - \frac{1}{2},$$

and hence we obtain

$$\dim_H(\Gamma) \leq 1 + k^2,$$

as required.  $\square$

*Remark.* In fact, the upper bound in Theorem 1.12 is not sharp, as recently proved by Oleg Ivrii [13].

**10.5. Quasisymmetric distortion spectrum.** In this subsection, we prove Theorem 1.13. More precisely, we use Lemma 10.3 to estimate the Minkowski and packing dimensions of the image of a subset of the real line under a quasisymmetric map.

**Definition 10.8.** Let  $k \in [0, 1)$ . A homeomorphism  $g : \mathbb{R} \rightarrow \mathbb{R}$  is called *k-quasisymmetric* if it extends to a normalized  $k$ -quasiconformal map  $g : \mathbb{C} \rightarrow \mathbb{C}$  such that  $g(z) = \overline{g(\bar{z})}$  for all  $z \in \mathbb{C}$ .

For the proof of Theorem 1.13, we need the following Schwarz–Pick type inequality, see [18, Lemma 2.2].

**Lemma 10.9.** *Let  $\phi : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic function. Suppose that  $\phi(\lambda) = \overline{\phi(\bar{\lambda})}$  for all  $\lambda \in \mathbb{D}$  and that  $\phi(\lambda) \geq 0$  for all  $\lambda \in (-1, 1)$ . Then*

$$\phi(k) \leq \left( \frac{k + \sqrt{\phi(0)}}{1 + k\sqrt{\phi(0)}} \right)^2 \quad (0 \leq k < 1).$$

*Proof of Theorem 1.13.* It is enough to prove the result for the Minkowski dimension. The case of the packing dimension then follows easily by applying Proposition 2.3.

Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a  $k$ -quasisymmetric map, and let  $A \subset \mathbb{R}$  be a bounded set with  $\overline{\dim}_M(A) = \delta$ , where  $0 < \delta \leq 1$ . It suffices to show that  $\overline{\dim}_M(g(A)) \geq \Delta(\delta, k)$ , since the upper bound follows from the lower bound, replacing  $g$  by  $g^{-1}$  and using the definition of  $\Delta^*(\delta, k)$ .

Extend  $g$  to a normalized  $k$ -quasiconformal mapping  $g : \mathbb{C} \rightarrow \mathbb{C}$  such that  $g(z) = \overline{g(\bar{z})}$  for all  $z \in \mathbb{C}$ . The Beltrami coefficient  $\mu_g$  satisfies

$$\mu_g(z) = \overline{\mu_g(\bar{z})} \quad (z \in \mathbb{C}).$$

Therefore, by a similar construction to that in the proof of Theorem 1.12, there is a symmetric holomorphic motion  $f : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$  with  $f_k = g$ . By Lemma 10.3, there exists an inf-sym-harmonic function  $u$  on  $\mathbb{D}$  such that

$$u(0) = 1/\overline{\dim}_M(A) = 1/\delta \quad \text{and} \quad u(\lambda) \geq 1/\overline{\dim}_M(A_\lambda) \quad (\lambda \in \mathbb{D}).$$

The function  $v := u - 1/2$  is also inf-sym-harmonic, and we can write

$$v(\lambda) = \inf_{h \in \mathcal{H}} h(\lambda),$$

where each  $h \in \mathcal{H}$  is a positive, symmetric harmonic function on  $\mathbb{D}$ .

Fix  $h \in \mathcal{H}$ . Since  $h$  is harmonic and  $h(\bar{\lambda}) = h(\lambda)$  for all  $\lambda \in \mathbb{D}$ , there is a holomorphic function  $H$  on  $\mathbb{D}$  with  $\operatorname{Re} H = h$  and  $\overline{H(\bar{\lambda})} = H(\lambda)$  for all  $\lambda \in \mathbb{D}$ . Then  $H$  maps  $\mathbb{D}$  into the right half-plane. Also, for  $\lambda \in (-1, 1)$ , we have

$$H(\lambda) = h(\lambda) \geq v(\lambda) \geq \frac{1}{\overline{\dim}_M(A_\lambda)} - \frac{1}{2} \geq \frac{1}{2},$$

since  $A_\lambda \subset \mathbb{R}$  by the symmetry of the holomorphic motion. It follows that the function

$$\phi := \frac{2H - 1}{2H + 1}$$

satisfies the assumptions of Lemma 10.9, and we get

$$\frac{2h(k) - 1}{2h(k) + 1} = \phi(k) \leq \left( \frac{k + l'}{1 + kl'} \right)^2,$$

where  $l' = \sqrt{\phi(0)}$ . Using the fact that the functions  $x \mapsto (2x - 1)/(2x + 1)$  and  $x \mapsto (k + x)/(1 + kx)$  are increasing, we obtain, after taking the infimum over all  $h \in \mathcal{H}$ ,

$$\frac{2v(k) - 1}{2v(k) + 1} \leq \left( \frac{k + l}{1 + kl} \right)^2,$$

where

$$l = \left( \frac{2v(0) - 1}{2v(0) + 1} \right)^{1/2} = \left( \frac{2(1/\delta - 1/2) - 1}{2(1/\delta - 1/2) + 1} \right)^{1/2} = \sqrt{1 - \delta}.$$

Note that

$$\frac{2v(k) - 1}{2v(k) + 1} = \frac{2u(k) - 2}{2u(k)} = 1 - \frac{1}{u(k)} \geq 1 - \overline{\dim}_M(g(A)).$$

This gives the desired inequality, namely

$$\overline{\dim}_M(g(A)) \geq 1 - \left( \frac{k + l}{1 + kl} \right)^2 = \Delta(\delta, k). \quad \square$$

## 11. AN OPEN PROBLEM

As remarked in the introduction, Theorems 1.4 and 1.7 between them provide a complete characterization of the variation of the packing dimension of a set moving under a holomorphic motion. Such a characterization for the Hausdorff dimension is currently lacking, due to the fact that the conclusion in Theorem 1.6 is weaker than that in Theorem 1.4. This naturally raises the following question.

**Question 11.1.** *Let  $A$  be a subset of  $\mathbb{C}$  such that  $\dim_H A > 0$ , and let  $f : \mathbb{D} \times A \rightarrow \mathbb{C}$  be a holomorphic motion. Set  $A_\lambda := f_\lambda(A)$ . Then must  $\lambda \mapsto 1/\dim_H(A_\lambda)$  be an inf-harmonic function on  $\mathbb{D}$ ?*

The same question was posed 30 years ago in [19]. As far as we know, it is still an open problem.

It was shown in [19] that the answer to Question 11.1 is affirmative in the following special case. Let  $(R_\lambda)_{\lambda \in \mathbb{D}}$  be a holomorphic family of hyperbolic rational maps. Then the holomorphic motion  $\lambda \mapsto J(R_\lambda)$  defined by their Julia sets has the property that  $1/\dim_H J(R_\lambda)$  is an inf-harmonic function on  $\mathbb{D}$ . The proof relies on an explicit formula for the Hausdorff dimension, namely the Bowen–Ruelle–Manning formula.

Another special case was established by Baribeau and Roy [4]. They showed that, if  $L_\lambda$  is the limit set of an iterated function system of contractive similarities depending holomorphically on a parameter  $\lambda \in \mathbb{D}$ , then, subject to a technical condition, the map  $\lambda \mapsto L_\lambda$  is a holomorphic motion for which  $1/\dim_H(L_\lambda)$  is an inf-harmonic function on  $\mathbb{D}$ . Their proof also relies on an explicit formula for the Hausdorff dimension, this time the Hutchinson–Moran formula, Theorem 2.4.

In fact, in both these special cases, it turns out that the Hausdorff dimension coincides with the packing dimension, so both results are now consequences of Theorem 1.4, without any recourse to explicit formulas for the dimension.

Finally, we remark that an affirmative answer to Question 11.1 would imply that  $\lambda \mapsto \dim_H(A_\lambda)$  is a subharmonic function (in much the same way that Corollary 1.5 was proved for the packing and Minkowski dimensions). Even this apparently weaker statement is also still an open problem. As an interesting test case, we pose the following question.

**Question 11.2.** *Does the Hausdorff dimension of a holomorphic motion  $\lambda \mapsto A_\lambda$  always satisfy the inequality*

$$\dim_H(A_0) \leq \max_{|\lambda|=1/2} \dim_H(A_\lambda)?$$

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