**Due in class** : February 25, 2025.

In the following, the letter $U$ denotes a domain in $\mathbb{C}$, i.e. a non-empty, connected and open subset of $\mathbb{C}$.

**Problem 1.** Let $(a_n)$ be a sequence of complex numbers satisfying

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

**a.** Show that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in the open unit disk $\mathbb{D}$.

**b.** Compute

$$\lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$$

in terms of the sequence $(a_n)$.

**Problem 2.** Recall the differential operators

$$f_z := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad f_{\overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

acting on smooth functions $f : U \to \mathbb{C}$. If $f := u + iv$, we write $\overline{f} := u - iv$.

**a.** Verify the product rules

$$(fg)_z = f_z g + f g_z \quad (fg)_{\overline{z}} = f_{\overline{z}} g + f g_{\overline{z}}.$$

**b.** Show that

$$\overline{f}_{\overline{z}} = \overline{f_z} \quad \overline{f}_z = \overline{f_{\overline{z}}}.$$

**c.** Show that

$$\Delta f = 4 f_{z\overline{z}} = 4 f_{\overline{z}z},$$

where $\Delta$ is the Laplacian operator $\Delta f := f_{xx} + f_{yy}$. Deduce that $\Delta f = 0$ in $U$ if $f : U \to \mathbb{C}$ is holomorphic.

**d.** Show that if $f : U \to \mathbb{C}$ is holomorphic, then

$$\Delta |f|^2 = 4|f'|^2$$

in $U$.

**Problem 3.** Let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be holomorphic. Suppose that

$$|f(z)|^2 + |g(z)|^2 = 1$$

for all $z \in U$. Show that $f$ and $g$ are constant.

**Problem 4.** Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Show that if

$$\int_{\mathbb{C}} |f(x+iy)|\, dx\, dy < \infty,$$

then $f(z) = 0$ for all $z \in \mathbb{C}$.

(*Hint*: Use polar coordinates and Cauchy's formula for derivatives to show that $f^{(n)}(0) = 0$ for all $n$)

**Problem 5.** Verify that the function

$$f(z) := \frac{1}{2\pi} \int_0^{2\pi} e^{2z\cos t}\, dt \qquad (z \in \mathbb{C})$$

is entire, and that its power series representation is

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z^n}{n!}\right)^2 \qquad (z \in \mathbb{C}).$$

**Problem 6.** A function $f : \mathbb{C} \to \mathbb{C}$ is *doubly periodic* if there are non-zero complex numbers $\alpha, \beta$ with $\alpha/\beta \notin \mathbb{R}$ such that

$$f(z+\alpha) = f(z+\beta) = f(z) \qquad (z \in \mathbb{C}).$$

Show that every doubly periodic entire function is constant.

**Problem 7.** Compute the following integrals. Here $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle, oriented counter-clockwise.

**a.** $\int_{\mathbb{T}} \frac{dz}{4z^2+1}$.

**b.** $\int_{\mathbb{T}} \frac{e^z - e^{-z}}{z^4}$.

**c.** $\int_{\mathbb{T}} z^n e^z\, dz \qquad (n \in \mathbb{Z})$.

**Problem 8.** Let $f : \mathbb{C} \to \mathbb{C}$ be entire and non-constant. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

(*Hint:* If $f(\mathbb{C})$ is not dense in $\mathbb{C}$, then there exists $w \in \mathbb{C}$ and $\epsilon > 0$ such that $|w - f(z)| \geq \epsilon$ for all $z \in \mathbb{C}$. Now consider $g(z) := 1/(w - f(z))$).

**Problem 9.** Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Show that if either $\operatorname{Re} f$ or $\operatorname{Im} f$ is bounded from above or below in the whole plane, then $f$ is constant.

**Problem 10.** Let $f : \mathbb{C} \to \mathbb{C}$ be entire, and suppose that

$$|f(z)| \leq A + B|z|^n \qquad (z \in \mathbb{C}),$$

for some constants $A$ and $B$. Prove that $f$ is a polynomial of degree at most $n$.

**Problem 11.** Let $f : U \to \mathbb{C}$ be holomorphic in $U \setminus \{z_0\}$ for some $z_0 \in U$. Suppose in addition that $f$ is bounded near $z_0$, i.e., there exists $M < \infty$ and $r > 0$ such that

$$|f(z)| \leq M \qquad (z \in \mathbb{D}(z_0, r)).$$

Show that $f$ can be defined at $z_0$ so that the extended function is holomorphic in $U$.

(*Hint:* Define $h : U \to \mathbb{C}$ by $h(z) := (z - z_0)^2 f(z)$ for $z \neq z_0$ and $h(z_0) := 0$. Then $h$ has a power series representation $h(z) = \sum_{n=2}^{\infty} a_n (z - z_0)^n$ (why?). Now let $f(z_0) := a_2$ and conclude.)

**Problem 12.** Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be entire. Suppose that

$$|f(z)| \leq |g(z)| \qquad (z \in \mathbb{C}).$$

Prove that there exists a constant $c \in \mathbb{C}$ such that

$$f(z) = cg(z) \qquad (z \in \mathbb{C}).$$

**Problem 13.** Let $f : U \to \mathbb{C}$ be holomorphic. Suppose that for each $z \in U$ there exists an integer $n$ such that $f^{(n)}(z) = 0$. Prove that $f$ is a polynomial.

**Problem 14.** Let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be holomorphic. Suppose that

$$\mathrm{Re}(f(z)) = \mathrm{Re}(g(z)) \qquad (z \in U).$$

Show that there exists a constant $c \in \mathbb{R}$ such that

$$f(z) = g(z) + ic \qquad (z \in U).$$

**Problem 15.** Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and non-constant. Suppose that

$$f(f(z)) = f(z) \qquad (z \in \mathbb{C}).$$

Show that $f(z) = z$ for all $z \in \mathbb{C}$.

**Problem 16.** Let $f$ be holomorphic in a neighborhood of $\overline{\mathbb{D}}$. Suppose that $f(0) = 1$ and $|f(z)| > 2$ whenever $|z| = 1$. Must $f$ have a zero in $\mathbb{D}$?

**Problem 17.** Let $f$ be holomorphic and non-constant in a neighborhood of $\overline{\mathbb{D}}$. Prove that if $|f|$ is constant on $\mathbb{T}$, then $f$ has at least one zero in $\mathbb{D}$.

**Problem 18.** Does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$$

for all $n \in \mathbb{N}$?

**Problem 19.** Does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f\left(\frac{1}{n^2}\right) = \frac{1}{n}$$

for all $n \in \mathbb{N}$?

**Problem 20.** Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Suppose that

$$|f(1/n)| \leq e^{-n}$$

for all $n \in \mathbb{N}$. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.