**Due in class** : December 10, 2026.

In the following the letter $U$ denotes a domain in $\mathbb{C}$, i.e., an open and connected subset of $\mathbb{C}$.

**Problem 1.** Suppose $U$ is bounded. Let $(f_n)$ be a sequence of continuous functions on $\overline{U}$ which are holomorphic on $U$. Prove that if $(f_n)$ converges uniformly on $\partial U$, then $(f_n)$ converges uniformly on the closure of $U$.

**Problem 2.** Suppose that $U$ is symmetric with respect to the real axis $\mathbb{R}$. Set $U^+ := \{z \in U : \mathrm{Im}\,(z) > 0\}$ and $U^- := \{z \in U : \mathrm{Im}\,(z) < 0\}$. Show that if $f$ is holomorphic on $U^+$, continuous on $U^+ \cup (U \cap \mathbb{R})$, and if $\mathrm{Im}\,(f) \equiv 0$ on $U \cap \mathbb{R}$, then the function defined by

$$F(z) = \begin{cases} f(z) & \text{if } z \in U \setminus U^- \\ \overline{f(\overline{z})} & \text{if } z \in U^- \end{cases}$$

is holomorphic in $U$.
(*This is usually referred to as the Schwarz reflection principle*).

**Problem 3.** Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) \neq 0$. Show that $f$ does not vanish on $\mathbb{D}(0, |f(0)|)$.

**Problem 4.** Let $f_n : U \to \mathbb{C}$ be holomorphic functions and suppose that $(f_n)$ converges locally uniformly in $U$ to a function $f : U \to \mathbb{C}$. Let $U' \subset \mathbb{C}$ be a domain with $f_n(U) \subset U'$ for all $n \in \mathbb{N}$. Prove that either $f$ is constant or $f(U) \subset U'$.

**Problem 5.** Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism. Extend $f$ to a bijection from $\widehat{\mathbb{C}}$ onto itself by setting $f(\infty) := \infty$. Show that the resulting extension is a homeomorphism from $\widehat{\mathbb{C}}$ onto itself.

**Problem 6.** Let $\gamma_N$ be the square with vertices $(N + 1/2)(\pm 1 \pm i)$.

**a.** By considering the horizontal and vertical segments of $\gamma_N$ separately, show that

$$|e^{2\pi i z} - 1| > 1 - e^{-\pi} \qquad (z \in \gamma_N).$$

**b.** Deduce that the function $g(z) := \pi \cot \pi z$ is uniformly bounded on $\gamma_N$ independently of $N$, i.e., there exists $M < \infty$ such that

$$|g(z)| \leq M \qquad (z \in \gamma_N, N \in \mathbb{N}).$$

(*Hint*: Write $g(z) = \pi i T(e^{2\pi i z})$, where $T$ is the Möbius transformation $T(w) := (w+1)/(w-1)$.)

**Problem 7.** A *finite Blaschke product* is a rational function of the form

$$B(z) := \alpha z^m \prod_{k=1}^{n} \left( \frac{z - p_k}{1 - \overline{p_k} z} \right),$$

where $|\alpha| = 1$, $m \in \mathbb{Z}$ and the $p_k$ are non-zero complex numbers with $|p_k| \neq 1$.

**a.** Show that any such $B$ satisfies $B(1/\overline{z}) = 1/\overline{B(z)}$ for all $z \in \mathbb{C}$.

**b.** Deduce that $|B(z)| = 1$ whenever $|z| = 1$.

**c.** Show that if all the zeros of $B$ belong to $\mathbb{D}$, then $B(\mathbb{D}) \subset \mathbb{D}$ and $B(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

**Problem 8.** Suppose $f$ is a rational function such that $|f(z)| = 1$ whenever $|z| = 1$. Show that $f$ is a finite Blaschke product.

**Problem 9.** Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic such that $f(p_k) = 0$ for $k = 1, \ldots, n$, where $p_1, \ldots, p_n \in \mathbb{D}$. Prove that

$$|f(z)| \leq \prod_{k=1}^{n} \left| \frac{z - p_k}{1 - \overline{p_k} z} \right| \qquad (z \in \mathbb{D}).$$

**Problem 10.** Suppose $f : \mathbb{D}(0, r) \to \mathbb{C}$ is holomorphic and $|f(z)| \leq M$ for all $z \in \mathbb{D}(0, r)$. Show that if $f$ vanishes at $p_1, \ldots, p_n \in \mathbb{D}(0, r) \setminus \{0\}$, then

$$\frac{r^n |f(0)|}{|p_1 \cdots p_n|} \leq M.$$

**Problem 11.** Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic with $f(0) = 0$ and

$$|f(z) + z f'(z)| < 1$$

for all $z \in \mathbb{D}$. Show that

$$|f(z)| \leq \frac{|z|}{2} \qquad (z \in \mathbb{D}).$$

**Problem 12.** Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. Show that there exists a sequence $(z_n) \subset \mathbb{D}$ with $|z_n| \to 1$ as $n \to \infty$ such that the sequence $(f(z_n))$ is bounded.
(*Hint*: First show that the conclusion holds if $f$ has infinitely many zeros. If $f$ has finitely many zeros and if $|f(z)| \to \infty$ whenever $|z| \to 1$, then consider $B/f$ for a suitable finite Blaschke product.)

**Problem 13.** Let $f_n : \mathbb{C} \to \mathbb{C}$ be holomorphic functions, and suppose that $(f_n)$ converges uniformly on $\mathbb{C}$ as $n \to \infty$. What can you conclude about

the functions $f_n$ for large $n$?

**Problem 14.** Let $f_n : U \to \mathbb{C}$ be holomorphic functions. Suppose that the sequence $(f_n)$ is uniformly bounded on each compact subset of $U$. Show that if $\lim_{n\to\infty} f_n(z)$ exists for all $z$ belonging to a subset $E \subset U$ which has an accumulation point in $U$, then $(f_n)$ converges uniformly on compact subsets of $U$ to a holomorphic function $f : U \to \mathbb{C}$.

**Problem 15.** Show that if $U \subset \mathbb{C}$ is simply connected, then $\widehat{\mathbb{C}} \setminus U$ is connected.
(*Hint*: We may assume $\widehat{\mathbb{C}} \setminus U$ contains more than one point. In this case use a conformal map $f : \mathbb{D} \to U$ to write $\widehat{\mathbb{C}} \setminus U$ as a decreasing intersection of compact connected sets.)

**Problem 16.** Let $f : \mathbb{D} \to U$ be a conformal map with $f(0) = 0$. If $g : \mathbb{D} \to U$ is any holomorphic function with $g(0) = 0$, show that $g(\mathbb{D}(0, r)) \subset f(\mathbb{D}(0, r))$ for all $0 < r < 1$.

**Problem 17.** Let $f : U \to \mathbb{D}$ be a conformal map and let $p \in U$ with $f(p) = 0$. If $g : U \to \mathbb{D}$ is any holomorphic function with $g(p) = 0$, show that $|g'(p)| \leq |f'(p)|$, with equality if and only if $g = \alpha f$ for some constant $\alpha$ with $|\alpha| = 1$.

**Problem 18.** Let $U \subsetneq \mathbb{C}$ be a simply connected domain and let $f : U \to U$ be a holomorphic function.

**a.** Show that $f$ has at most one fixed point in $U$ unless it is the identity map.

**b.** If $p = f(p)$, show that $|f'(p)| \leq 1$.

**c.** If $p = f(p)$ and $|f'(p)| = 1$, show that $f \in \mathrm{Aut}(U)$.

**d.** If $p = f(p)$ and $f'(p) = 1$, show that $f$ is the identity map.

**Problem 19.** Suppose that $U \subset \mathbb{C}$ is a convex domain. Let $f : U \to \mathbb{C}$ be holomorphic with $\mathrm{Re}\, f'(z) > 0$ for all $z \in U$. Show that $f : U \to f(U)$ is conformal.

**Problem 20.** Does there exist a conformal map $f : \mathbb{C} \to \mathbb{D}$? What about a homeomorphism $f : \mathbb{C} \to \mathbb{D}$?