University of Hawai'i

MATH 644 (Analytic Function Theory)

Lecture Notes

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Chapter : Preface

Complex analysis is the branch of mathematical analysis that studies holomorphic functions of a complex variables and their properties. As one of the classical areas of mathematics, with roots in the 18th century and prior, complex analysis has proven over the years to be of fundamental importance in a wide varieties of areas of mathematics, such as algebraic geometry, number theory, and applied mathematics, as well as in physics, including hydrodynamics, thermodynamics and quantum mechanics. In modern times, new applications of complex analysis have been discovered and studied extensively, notably in complex dynamics and probability. This makes complex analysis one of the most beautiful areas of mathematics, with a very high ratio of theorems to definitions.

The goal of this course is to study the properties of holomorphic functions of a complex variable. Topics to be covered include:

- Holomorphic functions and their basic properties, complex integration;
- Cauchy's theorem and topological aspects;
- Meromorphic functions: isolated singularities, the Riemann sphere, Laurent series and residues, the argument principle;
- Möbius maps and the Schwarz lemma;
- Convergence in the space of holomorphic functions, normal families;
- Conformal maps: the Riemann mapping theorem and boundary behavior;
- Harmonic functions and their basic properties.

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Chapter A: Motivation: Why study complex analysis?

Complex analysis and, more generally, complex numbers, have a large number of applications in mathematics and related areas. Remarkably, many of these applications have a priori little or nothing to do with complex numbers. In this chapter, we briefly discuss some examples.

A.1 Complex analysis and algebraic equations

In 1545, the Italian thinker Gerolamo Cardano published the now famous formula for solving cubic equations, based upon the work of Scipione del Ferro. Historically this appears to have been the first use of complex numbers to solve mathematical problems. A surprising aspect of Cardano's method is that it sometimes requires to perform operations using complex numbers as an intermediate step, even when the cubic equation has only real roots (casus irreducibilis). It was proved in 1843 by Pierre Wantzel that there cannot exist any solution in real radicals in the casus irreducibilis.

A.2 Complex analysis and analytic combinatorics

One of the most famous asymptotic formulas is *Stirling's formula*, which states that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

as $n \to \infty$. Another well-known formula is the one due to Hardy–Ramanujan for the number $p(n)$ of integer partitions of n:

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}}$$

as $n \to \infty$.

A standard approach to proving these types of formulas uses complex analysis.

A.3 Complex analysis and number theory

Let $\pi(n)$ denote the number of prime numbers less than or equal to n. The famous *prime number theorem* gives an asymptotic expression for this prime-counting function.

Theorem A.1 [Hadamard–de la Vallée Poussin, 1896]

$$\pi(n) \sim \frac{n}{\log n}$$

The usual proof of the prime number theorem uses complex analysis, via the study of the Riemann zeta-function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

first defined for all complex numbers s with $\operatorname{Re} s > 1$, then extended to the whole plane by analytic continuation. The connection between the zeta function and prime numbers was discovered by Euler long before Riemann, who proved the identity

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \qquad (\operatorname{Re} s > 1),$$

where the product is taken over all prime numbers. This is easy to prove using geometric series and the fundamental theorem of arithmetic.

The proof of the prime number theorem relies on the fact that $\zeta(s) \neq 0$ for all s with $\operatorname{Re} s = 1$. In fact, the latter is equivalent to the prime number theorem. The celebrated Riemann hypothesis states that the only zeros of the Riemann zeta function are at the negative even integers or on the vertical line $\{s \in \mathbb{C} : \operatorname{Re} s = 1/2\}$. It is known that the Riemann hypothesis, if true, would give the 'best' error term in the prime number theorem.

Another application of complex analysis in number theory has to do with sums of squares. A classical theorem of Legendre from 1770 states that every positive integer can be represented as a sum of four squares of integers. Given a positive integer n, how many ways can we represent n as a sum of four squares?

Jacobi's four-square theorem from 1834 gives the answer.

Theorem A.2 [Jacobi] If $r_4(n)$ denotes the number of distinct ways in which n can be represented as a sum of four squares, then

$$r_4(n) = 8 \sum_{d|n, 4\nmid d} d.$$

This theorem follows from the theory of elliptic functions and the residue theorem in complex analysis.

A.4 Complex analysis and calculus

Complex analysis offers a set of techniques for evaluating definite integrals that are difficult or impossible to calculate using standard calculus methods. An example is the *Fresnel integral*

$$\int_0^\infty \sin\left(t^2\right) dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

A.5 Complex analysis and partial differential equations

Complex-analytic techniques are very useful for solving several kinds of partial differential equations, particularly those arising in various physics problems in hydrodynamics, heat conduction, electrostatics, and more.

A.6 Complex analysis and quantum physics

The famous Schrödinger equation in quantum mechanics involves the imaginary unit. Schrödinger himself appeared dissatisfied with the idea that his equation uses complex numbers to describe physical reality.

"What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. Ψ is surely fundamentally a real function."

- Erwin Schrödinger, June 6, 1926 letter to Hendrik Lorentz.

A.7 Complex analysis and metric geometry

A famous conjecture of Kepler from 1611 states that the optimal density for packing unit spheres in three dimensions is

$$\frac{\pi}{3\sqrt{2}} \approx 0.74048.$$

Kepler's conjecture was proved in 1998 by Thomas Hales, using complex computer calculations. In 2014 Hales announced the completion of a formal proof using automated proof checking, thereby removing any doubt about the validity of the proof.

In higher dimensions very little is known about optimal sphere packings. In 2016 Viazovska proved that the optimal density for packing unit spheres in dimension 8 is

$$\frac{\pi^4}{384} \approx 0.25367.$$

Viazovska received the Fields medal in 2022 for this breakthrough work. The proof makes use of complex analysis in a fundamental way.

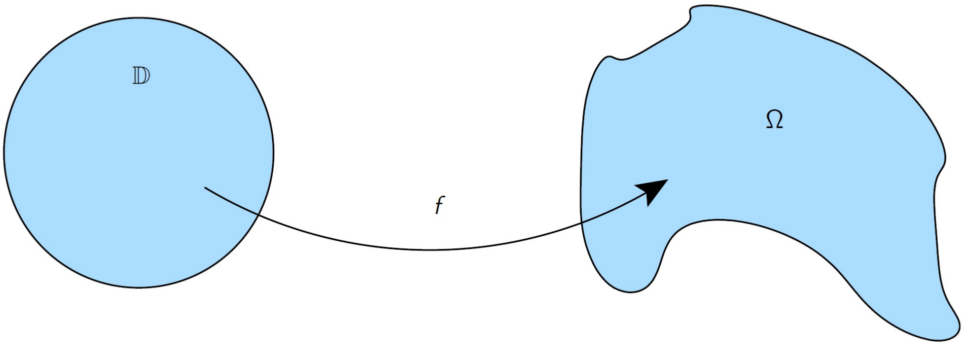
A.8 Complex analysis and plane topology

It is easy to see that for domains in the plane, the property of being simply connected is a topological invariant. More precisely, suppose that Ω is a simply connected domain in the plane. If Ω' is another domain homeomorphic to Ω, then Ω' must be simply connected. But is the converse true?

Namely, is it true that any two simply connected domains in the plane are homeomorphic?

The answer is yes, but the tools of topology alone are not enough to prove it. Using complex analysis one can prove that in fact much more is true.

Theorem A.3 [Riemann mapping theorem] Every simply connected domain $\Omega \subsetneq \mathbb{C}$ is conformally equivalent to the unit disk $\mathbb{D}$.



A conformal map between from unit disk onto simply connected domain Ω

A.9 Complex analysis and dynamical systems

Iteration of analytic functions can be used to generate beautiful fractals with remarkable properties. Even the dynamical behavior of simple quadratic polynomials $P_c(z) := z^2 + c$ is far from being well-understood. For a complex number c, define the *attractive basin of infinity* for P_c by

$$I_c := \{z \in \mathbb{C} : \lim_{n \to \infty} P_c^n(z) = \infty\},$$

where P_c^n denotes the polynomial P_c composed with itself n times. Then the *filled Julia set* of P_c is

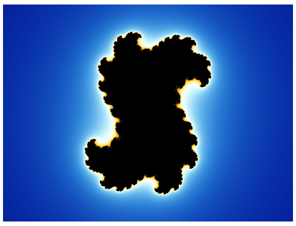
$$K_c := \mathbb{C} \setminus I_c$$

and the *Julia set* of P_c is

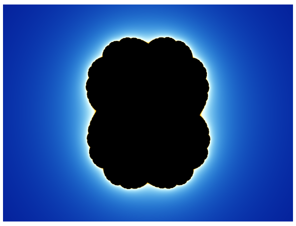
$$J_c := \partial K_c.$$

Example A.1 For $|c| \neq 0$ sufficiently small, the Julia set J_c is a Jordan curve of infinite length. By a classical result of Ruelle from the 1980's, we have

$$\dim_{\mathcal{H}}(J_c) = 1 + \frac{|c|^2}{4 \log 2} + O(|c|^3) \qquad (c \to 0).$$



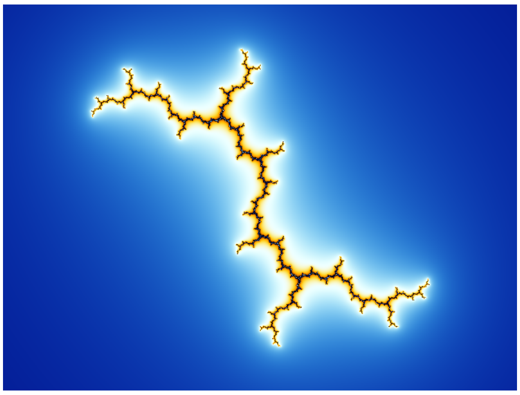
$c = 0.346 - 0.171i$



$c = 0.207 + 0.015i$

Definition A.1 A compact set $E \subset \mathbb{C}$ is a *dendrite* if it is connected, locally connected, has empty interior and connected complement.

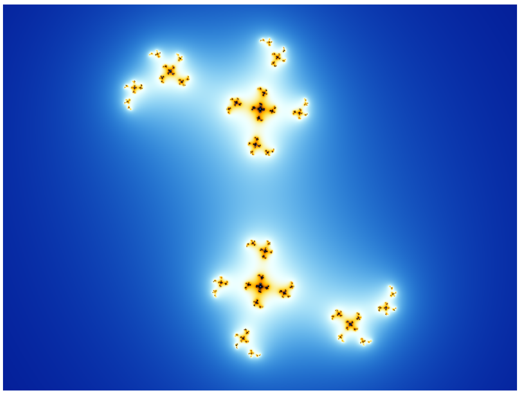
Example A.2 The Julia set of $P_i(z) := z^2 + i$ is a dendrite.



Julia set of $z^2 + i$

DEFINITION A.2 A set $E \subset \mathbb{C}$ is called a *Cantor set* if it is compact, perfect and totally disconnected.

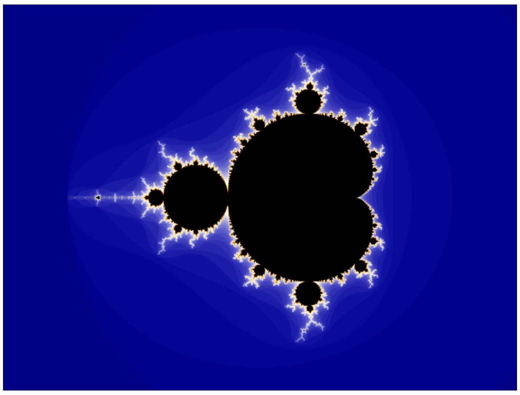
EXAMPLE A.3 For $c \in \mathbb{C}$ with $|c|$ large enough, the Julia set J_c of $P_c(z) = z^2 + c$ is a Cantor set. Moreover, we have $\dim_{\mathcal{H}}(J_c) \to 0$ as $|c| \to \infty$.



Julia set of $z^2 + 0.475 + 0.69i$

DEFINITION A.3 We define the *Mandelbrot set* $\mathcal{M}$ by

$$\mathcal{M} := \{c \in \mathbb{C} : \{P_c^n(0)\} \text{ is bounded}\}.$$



The Mandelbrot set $\mathcal{M}$

REMARK A.4

① $c \in \mathcal{M}$ if and only if K_c is connected.

② If $c \notin \mathcal{M}$, then K_c is a Cantor set.

③ $\mathcal{M}$ is compact and connected.

Whether $\mathcal{M}$ is locally connected or not is a central open problem in the field.

Conjecture A.5 [MLC] The Mandelbrot set $\mathcal{M}$ is locally connected.

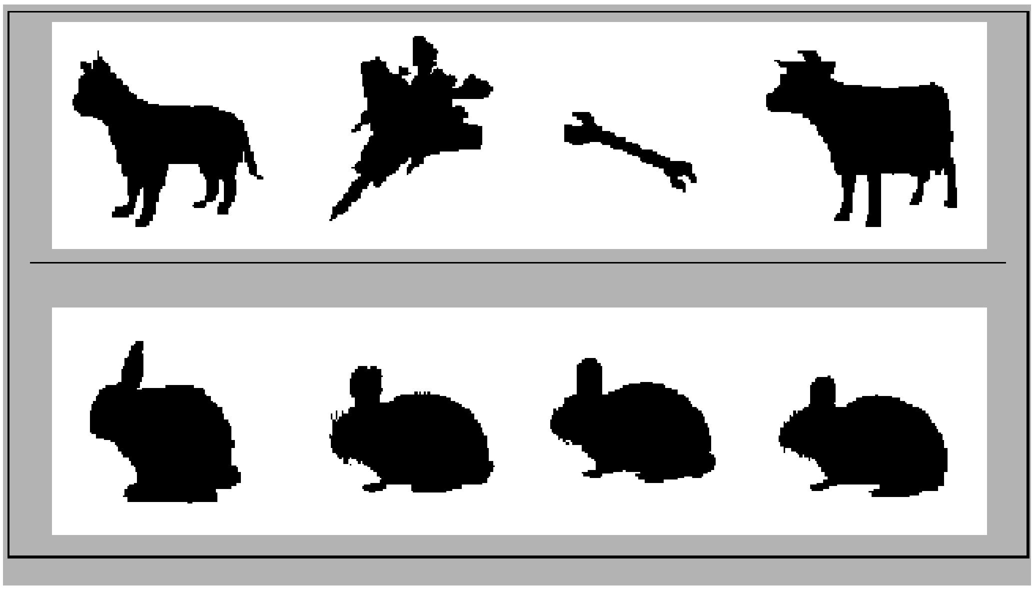
It is known that the boundary of $\mathcal{M}$ has dimension two, but it is still unknown whether it has positive area or not.

Conjecture A.6 The area of the boundary of $\mathcal{M}$ is zero.

Note that there exist Julia sets with positive area, as proved by Buff and Cheritat.

A.10 Complex analysis and numerical vision

How to recognize and classify objects in a large database from their observed shape?



The typical method is

① Set up a pairwise distance between shapes using some 2d spatial comparison (e.g. Hausdorff distance)

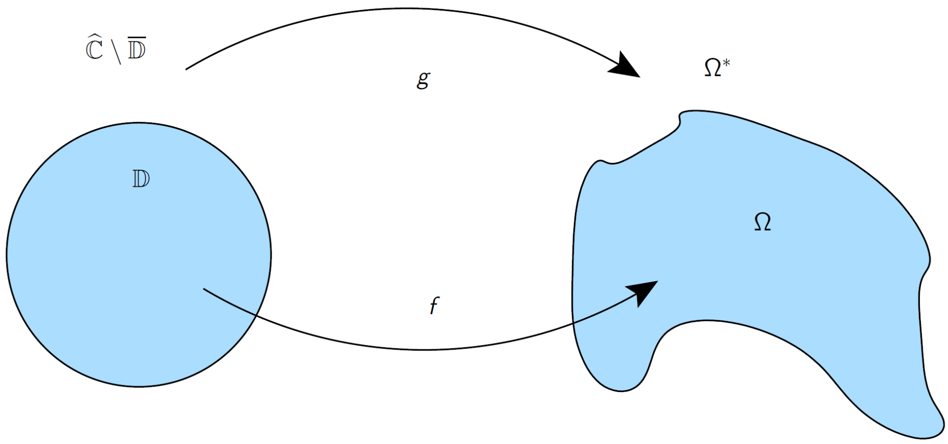
② Apply a clustering algorithm.

But how do we find a meaningful distance and an appropriate clustering algorithm? In 2004, Sharon and Mumford developed a numerical method based on conformal welding. The advantage of their method is that it takes into account scalings and translations of shapes, as we will see.

To describe what conformal welding is, let Γ be a Jordan curve (non self-intersecting and closed curve) in the complex plane $\mathbb{C}$. Denote by Ω and Ω^* the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus \Gamma$ respectively. Let $f : \mathbb{D} \to \Omega$ and $g : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \Omega^*$ be conformal maps given by the Riemann mapping theorem, where $\mathbb{D}$ is the open unit disk. Then f and g extend to the closure of their respective domains, by Carathéodory's extension theorem for conformal maps.

Definition A.4 The *conformal welding* of Γ is the orientation-preserving homeomorphism $h_\Gamma : \partial\mathbb{D} \to \partial\mathbb{D}$ defined by

$$h_\Gamma := g^{-1} \circ f.$$



Conformal welding

Remark A.7

① The conformal welding h_Γ is uniquely determined by Γ up to pre- and post-composition with automorphisms of $\mathbb{D}$.

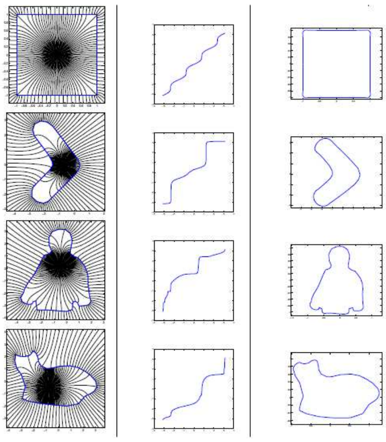
② If Γ' is obtained by Γ by translation or scaling, or more generally by a Möbius transformation, then $h_\Gamma = h'_\Gamma$

③ Conformal welding defines a map $\mathcal{W} : [\Gamma] \mapsto [h_\Gamma]$ from Jordan curves, modulo Möbius transformations, to orientation-preserving homeomorphisms of the circle, modulo automorphisms of $\mathbb{D}$.

④ It is known that the conformal welding map $\mathcal{W}$ is neither injective nor surjective.

⑤ There is no known characterization for the image of $\mathcal{W}$. The question of which orientation-preserving homeomorphisms of the circle are conformal weldings is extremely difficult. It is known that every sufficiently nice homeomorphism is a conformal welding and every sufficiently bad homeomorphism is a conformal welding!

⑥ However restricting the map $\mathcal{W}$ to the set of smooth curves gives a bijection onto the set of smooth diffeomorphisms of the circle. This gives a way to encode smooth curves as smooth diffeomorphisms of the circle, and vice versa.



Sharon and Mumford's method to go back and forth between curves and diffeomorphisms.

A.11 Complex analysis and probability

Over the last few decades, complex analysis has proven to be of fundamental importance in order to understand a wide variety of conformally invariant stochastic objects in the plane. A famous example of such an object is Brownian motion.

Definition A.5 A collection of random variables $\{B_t\}_{t\geq 0}$ is a *standard Brownian motion* if it satisfies the following properties:

(i) $B_0 = 0$;

(ii) Independent increments: for all $0 \leq t_1 < t_2 < \cdots < t_k$, the random variables

$$B_{t_k} - B_{t_{k-1}}, \ldots, B_{t_2} - B_{t_1}$$

are independent;

(iii) Normally distributed: for all $0 \leq t < s$, the random variable $B_s - B_t$ is normally distributed with mean 0 and variance $s - t$;

(iv) Continuous: almost surely $t \mapsto B_t$ is continuous.

If $\{B_t^1\}_{t\geq 0}, \{B_t^2\}_{t\geq 0}$ are two independent standard Brownian motions, then

$$B_t := (B_t^1, B_t^2) \qquad (t \geq 0)$$

is called a *2-dimensional Brownian motion* starting at 0. Similarly we can define d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$.

Remark A.8

① Brownian motion is point-recurrent in dimension $d = 1$, neighborhood-recurrent but not point-recurrent in dimension $d = 2$, and transient in dimension $d \geq 3$.

② 2-dimensional Brownian motion is conformally invariant: If $f : U \to V$ is conformal and if $\{B_t\}$ is a 2-dimension Brownian motion starting at $x \in U$, then $\{f(B_t)\}$ is a Brownian motion in V starting at $f(x) \in V$, up to a time change and stopping time considerations.

Brownian motion can be used to define harmonic measure, an important probability measure that also satisfies a conformal invariance property.

Let Ω be a domain in the plane, and let $z \in \Omega$. Let $\{B_t\}$ be a 2-dimensional Brownian motion starting at z. Consider the stopping time

$$T := \inf\{t \geq 0 : B_t \notin \Omega\}$$

and suppose T is finite almost surely. Then the random variable B_T takes values in $\partial\Omega$.

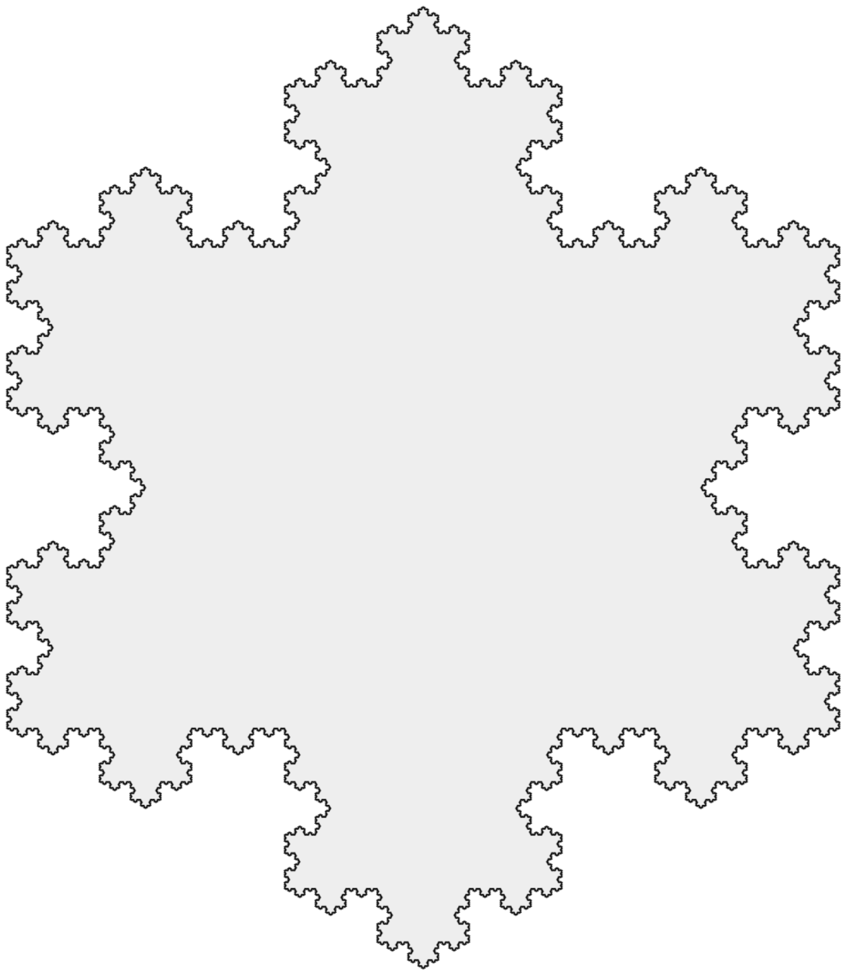
DEFINITION A.6 The harmonic measure $\omega_\Omega^z(\cdot)$ is the distribution measure of the random variable B_T:

$$\omega_\Omega^z(E) := P\{B_T \in E\} \qquad (E \subset \partial\Omega \text{ Borel}).$$

REMARK A.9

① In other words the harmonic measure of a Borel set $E \subset \partial\Omega$ is the probability that a Brownian motion first hits $\partial\Omega$ at a point of E.

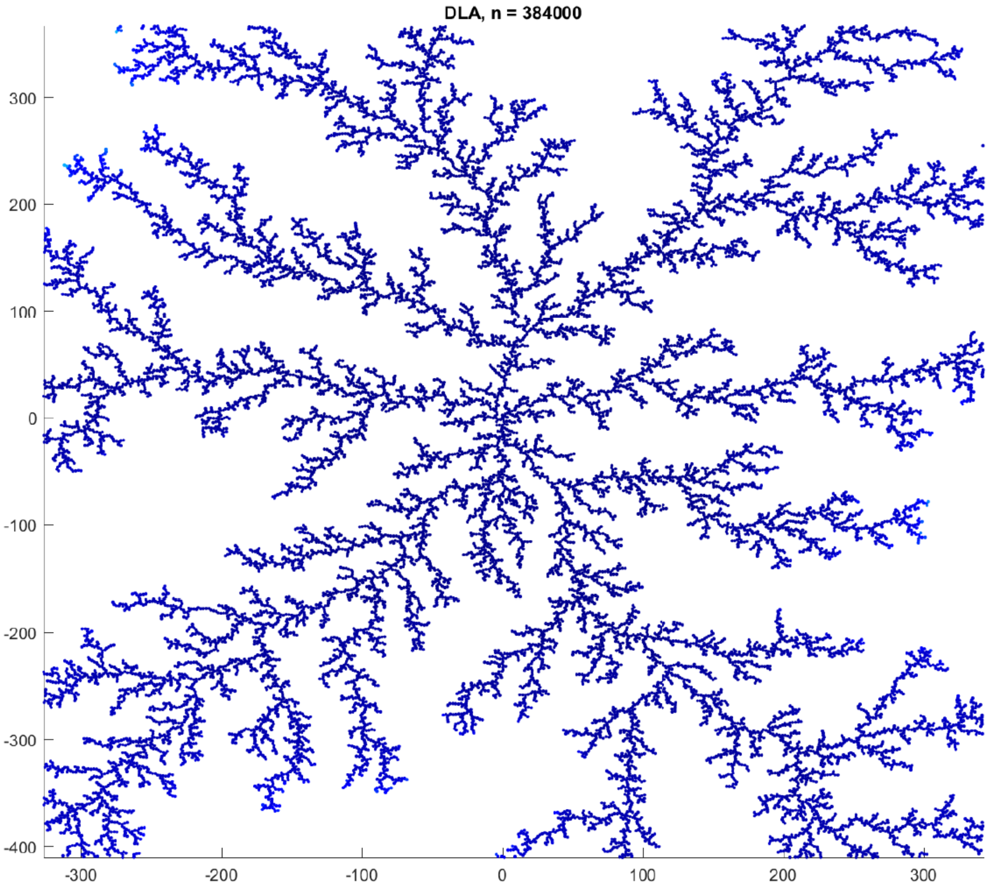
EXAMPLE A.4 If Ω is the unit disk $\mathbb{D}$, then the harmonic measure of a Borel set $E \subset \partial\mathbb{D}$ is just the normalized Lebesgue measure of E. This follows from rotation invariance of Brownian motion.



Harmonic measures from the inside and from the outside are mutually singular

One can use conformal invariance of Brownian motion to show that harmonic measure is conformally invariant. Therefore all the tools of complex analysis can be used to study the behavior of harmonic measure. Complex analysis is also very useful to study random models in the plane such as *diffusion-limited aggregation* (DLA). Roughly, DLA is the process whereby particles undergoing a random walk due to Brownian motion cluster together to form aggregates of such particles. It was introduced by Witten and Sander in 1981.

More precisely, start with a unit disk centered at the origin. Imagine another unit disk, whose center moves as a Brownian motion starting near infinity unit the it hits the first disk and then stops. Now send in another random disk until it hits one of the first two. Continue in this way until n disks have accumulated to form a connected set.



Diffusion-limited aggregation (DLA) with $n = 384000$ disks

How fast does the cluster grow? It is easy to see that the order of growth of the diameter is less than n but greater than $\sqrt{n}$. The following theorem of Kesten from 1990 gives an improved upper bound.

Theorem A.10 [Kesten] Almost surely, the diameter of DLA with n disks is $O(n^{2/3})$.

Remark A.11

① There are no known lower bounds! There have been numerous numerical simulations of DLA and heuristic arguments for estimating its growth and geometry, but after thirty years, Kesten's bound is the only rigorously provable thing we know about DLA.

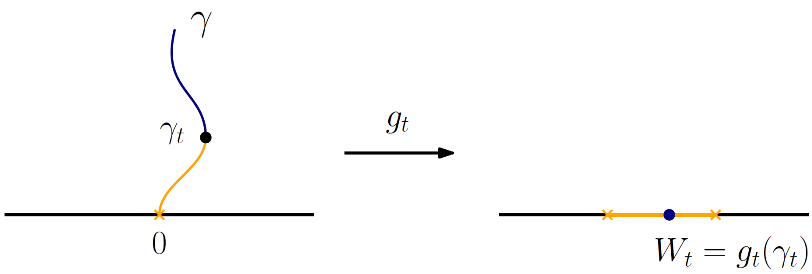
② Fields medalist Stas Smirnov has warned that graduate students and postdocs not be allowed to work on DLA. Apparently they are particularly susceptible to a debilitating condition known as "diffusion limited aggravation".

Brownian motion can also be used to define a family of random curves in the plane called *Schramm-Loewner evolution*.

Let $\mathbb{H}$ denote the upper half-plane, and let $\gamma : [0,\infty) \to \overline{H}$ be a simple curve in $\mathbb{H}$ connecting 0 to ∞, i.e. $\gamma(0) = 0$, $\gamma((0,\infty)) \subset \mathbb{H}$ and $\gamma_t := \gamma(t) \to \infty$ as $t \to \infty$. For each $t \geq 0$, the domain $\mathbb{H} \setminus \gamma([0,t])$ is simply connected, so by the Riemann mapping theorem there is a unique conformal map $g_t : \mathbb{H} \setminus \gamma([0,t]) \to \mathbb{H}$, suitably normalized. Note that $g_0(z) = z$ for all $z \in \mathbb{H}$. One can show that g_t satisfies the *Loewner equation*

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - W_t} \tag{A.1}$$

where $W_t := g_t(\gamma_t)$ is a continuous function called the *driving function* corresponding to the curve γ.



The Loewner equation

The remarkable idea introduced by Oded Schramm in 2000 is to reverse this process starting with a random continuous function $W : \mathbb{R} \to \mathbb{R}$. For instance, suppose that

$$W_t := \sqrt{\kappa} B_t$$

for some parameter $\kappa \geq 0$, where B_t is standard one-dimensional Brownian motion. Then with this choice of W, one can solve the stochastic differential equation $(A.1)$ and obtain a family of random curves in this way, denoted by SLE_κ and called *Schramm-Loewner evolution* with parameter κ.

EXAMPLE A.5 Take $\kappa = 0$. Then the equation (A.1) is a simple deterministic ODE, and the solution is

$$g_t(z) = \sqrt{z^2 + 4t},$$

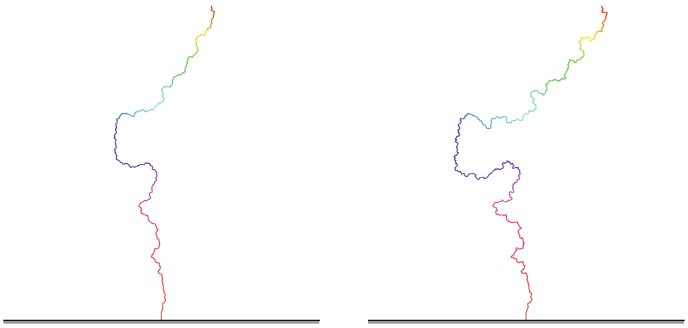
with $\gamma(t) = 2i\sqrt{t}$. Hence SLE_0 is a vertical line starting at 0.

In general the larger κ is the more fractal the curve SLE_κ becomes:

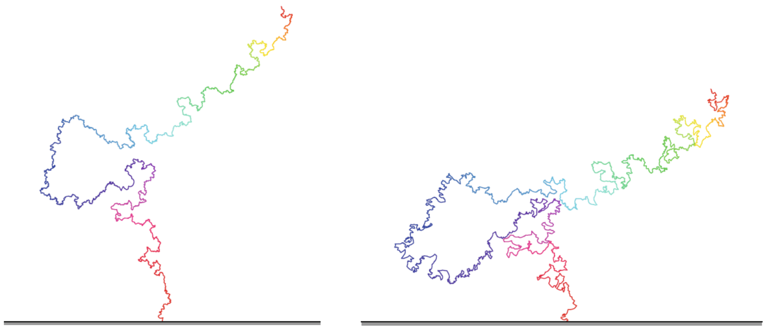
1. For $0 \leq \kappa < 4$ the curve SLE_κ is simple.
2. For $4 \leq \kappa < 8$ the curve SLE_κ intersects itself and every point is contained in a loop but the curve is not space-filling.
3. For $\kappa \geq 8$ the curve SLE_κ is space-filling.

Moreover

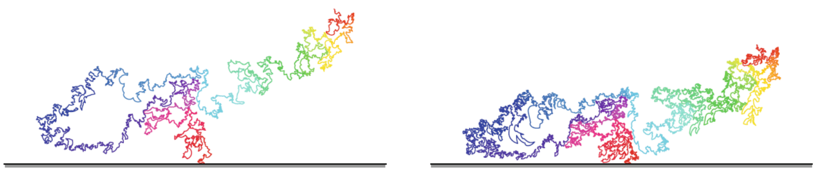
$$\dim_{\mathcal{H}}(\text{SLE}_\kappa) = \min(2, 1 + \kappa/8).$$



(a) SLE($\frac{1}{2}$) (b) SLE(1)



(c) SLE(2) (d) SLE(4)



(a) SLE(6) (b) SLE(8)

(c) SLE(16) (d) SLE(32)

Schramm-Loewner evolution for different values of κ

The curves SLE_κ satisfy two fundamental properties called *conformal invariance* and *domain markov*. These properties have been used to show that SLE_κ arises as scaling limits of a variety of two-dimensional lattice models in statistical mechanics:

① SLE_6 is the scaling limit of critical percolation on the triangular lattice, a theorem of Stas Smirnov that earned him the Fields medal in 2010.

② SLE_3 is the scaling limit of interfaces for the Ising model.

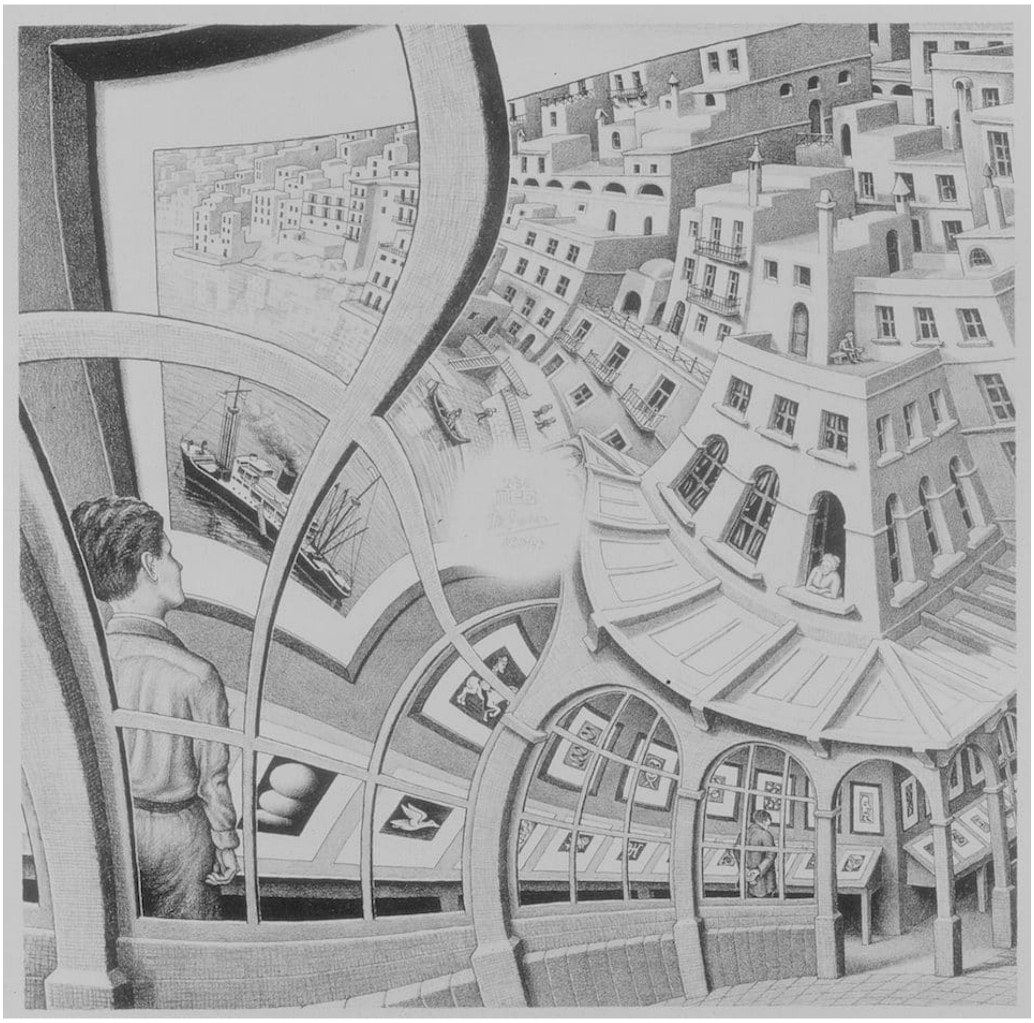
③ $\text{SLE}_{8/3}$ is the outer boundary of two-dimensional Brownian motion. This was proved by Lawler, Schramm and Werner in the early 2000s. In particular the outer boundary of two-dimensional Brownian motion has Hausdorff dimension $4/3$. Werner received the Fields medal in 2006 partly for this work.

④ SLE_2 is the scaling limit of loop-erased random walk.

Nowadays Schramm–Loewner evolution remains a very exciting subject at the intersection of complex analysis and probability.

A.12 Complex analysis and art

Conformal maps and their angle-preserving property were used by the Dutch artist M.C. Escher to create amazing art and used by others to better understand, and even improve on, Escher's work.



Prententententoonstelling, M.C. Escher, 1956

Prententententoonstelling completed, De smit–Lenstra, 2004