Math 100 – worksheet 9 –

Loan Payments and Mortgages

1. We consider some variations related to exercise 26 (p.251) in the book.

   For a loan of \$24,000 at a fixed APR of 8% for 15 years, find the amount of each monthly payment which goes toward principal and interest for the first two months.

   The exercise was solved in class; to warm up, we recall steps here.

   - We make use of loan payment formula

     $$PMT = \frac{P \times \left(\frac{APR}{n}\right)}{\left[1 - \left(1 + \frac{APR}{n}\right)^{(-nY)}\right]}$$

     with with $P = 24000$, $APR = 0.08$, $n = 12$, and $Y = 15$ to calculate monthly payments and find

     $$PMT \approx \$229.36$$

     After that we begin a month-by-month calculation.

   - End of first month:

     $$\text{The interest:} \quad 24000 \times \frac{0.08}{12} = \$160$$

     $$\text{Payment:} \quad \$229.36 = \$160 + \$69.36$$

     $$\text{New principal:} \quad \$24.000 - \$69.36 = \$23,930.64$$

   - End of second month:

     $$\text{The interest:} \quad 23930.64 \times \frac{0.08}{12} = \$159.54$$

     $$\text{Payment:} \quad \$229.36 = \$159.54 + \$69.82$$

     $$\text{New principal:} \quad \$23,930.64 - \$69.82 = \$23,860.82$$

   - Even with these two first months we can see that although the payment is constant, the break-off between the parts of the payment which go towards the principal and towards the interest are changing. That raises two questions which we will address.

2. Calculate the total amount paid over the loan period, and the amount paid towards interest.

3. It becomes clear even with the first two months calculations above (see also the Figure 4.8 on p. 247) that the portion of payment which goes towards the principal is growing with time.

   In other words, at the beginning of loan term a smaller fraction of our payments is counted towards the principal.

   That does not actually matter for those who simply pay certain amount monthly till the end (that is what primarily installment loans are designed for). However, there are situations when that matters.

   Assume that 5 years after the start of loan term we won a lottery and want to pay the loan off with a lump sum. How much do we actually owe at this moment?

   - Explain why the following calculation does not work.

     Over a period of 5 years, we have paid

     $$229.36 \times 12 \times 5 = \$13,761.6$$

     while the total was $\$24,000$. We thus owe $24,000 - 13761.6 = \$10,238.4$.

   - Explain why the following calculation does not work either.

     While we have already paid $\$13,761.6$, and the total amount we were supposed to pay over a period of 15 years is $\$41,284.8$, we owe after 5 years

     $$41,284.8 - 13,761.6 = \$27,523.2$$

   - Your last explanation suggests to think about our payments as a saving plan (with the same APR of 8%). This point of view is actually adopted in the derivation of the loan payment formula (p.240). Specifically, the sum we paid after 5 years should be counted not just a bare amount of $\$13,761.6$: we made monthly payments, but also interest rate must be taken into the account. That is exactly what saving plan formula does.

     Calculate this sum using saving plan formula

     $$A = PMT \times \frac{\left[\left(1+\frac{APR}{n}\right)^{(nY)} - 1\right]}{\left(\frac{APR}{n}\right)}$$

- At this point, a bank may argue that it planned to receive the *future value* of \$41,284.8 in any event towards this loan, and you thus owe

$$41,284.8 - 16,852.65 = \$24,432.15$$

  Is that, in your opinion, a valid argument?

- Yet another way to approach this question is to continue the line-by-line calculations we started with. That is quite boring, and in order to accelerate the process one may create a computer script which does that.

  As mathematicians, we better use yet another formula for this case. Such formula is not available in the book. However, no book contains all possible formulas, and the formulas available are not given for granted but *derived* (cf. mathematical insights on pp.218,240). Here it is. Let

$$R = \text{remaining debt after } y \text{ years}$$

  then

$$R = P \times \left(\left(1 + \frac{APR}{n}\right)^{(ny)} - \left(1 + \frac{APR}{n}\right)^{(nY)} \times \frac{\left(1 + \frac{APR}{n}\right)^{(ny)} - 1}{\left(1 + \frac{APR}{n}\right)^{(nY)} - 1} \right)$$

  Calculate remaining debt after 5 years using this formula.

- The knowledge of the remaining debt allows us to calculate the amounts which go towards the principal and interest out of the first payment after 5 years (that is the 61st payment). Do that and compare to the splitting for the first two payments.