MATHEMATICS DEPARTMENT ASSESSMENT EXAM

1. Part II: Basic Proofs and Examples.

Problem 1. The relation $\sim$ on $\mathbb{R}^2$ defined by $(x_1, y_1) \sim (x_2, y_2)$ if $x_1^2 - y_1^2 = x_2^2 - y_2^2$ is an equivalence relation (You may prove this).
  (1) Find the equivalence class of $(1, 0)$.
  (2) Describe the collection of all equivalence classes. It should be clear from your description that the equivalence classes form a partition $\mathbb{R}^2$.

Problem 2. A relation $\sim$ is defined on the set of positive integers $\mathbb{N}$ by $n \sim m$ if either n divides m or m divides n. Determine, with proof, whether $\sim$ is or is not:
  (1) reflexive
  (2) symmetric
  (3) anti-symmetric
  (4) transitive.

Problem 3. Prove by induction that for all $n \geq 2$:
$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{1}{2} \cdot \frac{n+1}{n} \, .$$

Problem 4. Let A be an $m \times n$ matrix. Prove that the null space $\left\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = 0\right\}$ of A is a subspace of $\mathbb{R}^n$.

Problem 5. Show that a bounded monotonic sequence of real numbers converges.

Problem 6. Let E be a non–empty and bounded subset of $\mathbb{R}$, and let $x_0 = \sup E$. Show that $x_0 \in E$ or that x_0 is an accumulation point of E.

Problem 7. Let $f : D \to \mathbb{R}$ be continuous with D compact. Prove that $f(D)$ is compact.

Problem 8. Give an example, with justification, of a bounded real–valued function on $[0, 1]$ that is not Riemann integrable.

Problem 9. Let A, B, and C be sets. Define the symmetric difference of A and B, written as $A + B$, by $A + B = (A \cup B) \setminus (A \cap B)$. Prove or disprove $(A + B) \cap C = (A \cap C) + (B \cap C)$.

Problem 10. Let $f : X \to Y$ be a function. Prove or disprove the equivalence of the following two statements.

(1) f is injective.

(2) If Z is any set and $h_1 : Z \to X$ and $h_2 : Z \to X$ are any two functions, then $f \circ h_1 = f \circ h_2$ implies $h_1 = h_2$.