

MATHEMATICS DEPARTMENT ASSESSMENT EXAM

One may expect students to spend one hour answering the questions for any of the 400 level classes and to address at least the questions of two 400 level classes. Students are most certainly invited to spend more time on this exam, and they may answer questions from courses that they have not taken.

1. PART III PROBLEMS FROM SENIOR COURSES

3.1. Obligatory.

Problem 1. List all senior ($4xx$ -level) classes which you have taken.

3.2. Math 402–403.

Problem 2. Find the solution $u(x, t)$ of the following heat problem for a rod with insulated ends:

$$(D.E.) \quad u_t = 2u_{xx}$$

$$(B.C.) \quad u_x(0, t) = 0$$

$$u_x(3, t) = 0$$

$$(I.C.) \quad u(x, 0) = \sin^2(\pi x/6).$$

Problem 3. Give the formula for the solution $u(x, t)$ of the wave equation $u_{tt} = a^2 u_{xx}$ for the infinite string $-\infty < x < \infty$, which meets the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. Verify that the formula yields a C^2 solution of $u_{tt} = a^2 u_{xx}$ provided that $f(x)$ is C^2 and g is C^1 .

Problem 4. What is a spherical harmonic $f(\varphi, \theta)$? How can one obtain a spherical harmonic $f(\varphi, \theta)$ from a given homogeneous harmonic polynomial $p(x, y, z)$. How is the eigenvalue λ of $f(\varphi, \theta)$ (where $\Delta_s f + \lambda f = 0$) related to the degree n of $p(x, y, z)$?

3.3. Math 407.

Problem 5. Consider the simple initial value problem

$$y' = -100y \quad y(0) = 1$$

which has solution $y(t) = e^{-100t}$. Use induction to show that Euler's method with step size h gives a solution $y_j = (1 - h100)^j$.

Problem 6. Give a formula for a one step IVP solver with local truncation error that is $\mathcal{O}(h^3)$. Justify your answer.

Problem 7. Consider the system of equations

$$\begin{cases} u^2 + v^2 = 1 \\ (u - 1)^2 + v^2 = 1 \end{cases}.$$

Apply one step of Newton's method with starting point $(1, 1)$. Show all your work.

Problem 8. Let $\epsilon > 0$. For the matrix

$$A = \begin{bmatrix} 1/\epsilon & 1 \\ 1 & 1 \end{bmatrix}$$

Find the $PA = LU$ (i.e., decomposition with partial pivoting).

- (1) What are P, L, U ?
- (2) Use your answer to solve $Ax = \begin{bmatrix} 1/\epsilon \\ 2 \end{bmatrix}$ using back-substitution, showing each intermediate step. In particular, solve $Lc = Pb$ and $Ux = c$.

Problem 9. Estimate the error e_{i+1} in terms of the previous error e_i as Newton's method converges to the root $r = -1$ of the function $f(x) = x^5 - 2x^4 + 2x^2 - x$? Is the convergence linear or quadratic?

3.4. Math 411.

Problem 10. The linear operator T defined by

$$T(p(x)) = (x+1) \frac{d}{dx} p(x)$$

acts on the vector space $\mathcal{P}_3(\mathbb{R})$ of polynomials with real coefficients of degree at most 3. List all eigenvectors.

Problem 11. Let A be a complex 5×5 matrix with characteristic polynomial $f(x) = (x-2)^3(x+7)^2$ and minimal polynomial $p(x) = (x-2)^2(x+7)$.

- (1) What is the Jordan canonical form of A ?
- (2) What is the trace and determinant of A ?

Problem 12. Find an orthogonal matrix U such that $U^T A U$ is diagonal, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Problem 13. Let V be a vector space and W its subspace. Show that the dimension of W is smaller than or equal to the dimension of V . Discuss the finite and infinite dimensional cases.

3.5. Math 412–413.

Problem 14. Consider the ring $R = \mathbb{Z}_{12}$.

- (1) List all zero divisors in R
- (2) List all units in R
- (3) How many solutions does the equation $x^2 = 1$ have in R ?
- (4) Show that $\mathbb{Z}_{12} \simeq \mathbb{Z}_3 \times \mathbb{Z}_4$.

Problem 15. Let R be a commutative ring and $a \in R$. Define

$$\text{ann}(a) = \{x \in R \mid xa = 0\}.$$

Prove that $\text{ann}(a)$ is an ideal in R .

Problem 16. For each positive integer k , let $k\mathbb{Z}$ be the subring of \mathbb{Z} consisting of all integer multiples of k . Prove that if $m \neq n$, then the rings $m\mathbb{Z}$ and $n\mathbb{Z}$ are not isomorphic.

Problem 17. Prove that

$$f : \mathbb{Z}_m \rightarrow \mathbb{Z}_n, \quad f([a]_m) = [a]_n$$

is a well-defined function if and only if $n \mid m$. In the case where f is well-defined, prove that f is a ring homomorphism.

Problem 18. Prove that $\mathbb{Z}_5[x]/(x^2 + x + 1)$ is a field of order 25.

Problem 19. Suppose that G is a group in which every nonidentity element has order 2. Prove G is abelian.

Problem 20. For each pair of real numbers a, b with $a \neq 0$, consider the function

$$T_{a,b} : \mathbb{R} \rightarrow \mathbb{R}, \quad T_{a,b}(x) = ax + b.$$

Prove that the set $G = \{T_{a,b} \mid a, b \in \mathbb{R}, a \neq 0\}$ is a group under function composition.

Problem 21. Find a permutation in the symmetric group S_5 with order 6. Prove that there are no permutations in S_5 with order 7.

Problem 22. Suppose K is a subfield of $\mathbb{Q}(\sqrt[5]{6})$. Prove that either $K = \mathbb{Q}$ or $K = \mathbb{Q}(\sqrt[5]{6})$.

3.6. Math 420.

Problem 23. Let ϕ denote Euler's ϕ -function. Find all integers n such that $\phi(n) = 6$.

Problem 24. Let p be a prime. Prove that $2(p-3)! + 1 \equiv 0 \pmod{p}$.

Problem 25. Let p be a prime, and let g be a primitive root modulo p . Find the residue of $g^{\frac{p-1}{2}}$ modulo p .

Problem 26. Consider the sequence of integers a_n for $n \geq 0$ defined recursively as follows:

$$a_0 = a_1 = a_2 = 0; \quad a_3 = 1.$$

For $n \geq 4$,

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}.$$

Find a closed formula for a_n . (Note that $x = 1$ is a root of the polynomial $6x^3 - 11x^2 + 6x - 1$.)

Problem 27. Let $x > 1$ be a positive real number, and $x = [a_0; a_1, a_2, \dots]$ be its continued fraction representation. Find the continued fraction representation for the real number $1/x$.

Problem 28. A positive integer n leaves a remainder of 3 when divided by 4. Show that n cannot be written as the sum of the squares of two integers.

Problem 29. Determine the *two smallest* positive solutions of the system of congruence relations

$$x \equiv 2 \pmod{81}$$

$$x \equiv 3 \pmod{71}.$$

3.7. Math 421.

Problem 30. Let \mathbb{R}^ω consist of all functions $\mathbb{Z}_+ \rightarrow \mathbb{R}$ or, equivalently, all infinite sequences $x = (x_1, x_2, x_3, \dots)$ of real numbers. Topologize \mathbb{R}^ω by taking as a basis open sets of the form $U_1 \times U_2 \times U_3 \times \dots$, where each U_i is open in \mathbb{R} . This defines the “box” topology on \mathbb{R}^ω .

- (A) Determine whether the map $\mathbb{R} \rightarrow \mathbb{R}^\omega$ given by $t \mapsto (t, \frac{1}{2}t, \frac{1}{3}t, \dots)$ is continuous at 0.
- (B) Show \mathbb{R}^ω is not connected. [*Hint:* The set of bounded sequences is closed and open.]

Problem 31. Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X .

Problem 32. Let $p : X \rightarrow Y$ be a closed continuous surjective map. Show, if X is normal, then so is Y .

Problem 33. Consider the labelling scheme

$$abchdec^{-1}fgh^{-1}adbg^{-1}ef^{-1}$$

for the edges of a polygon. A letter stands for an edge that is directed counter clockwise, its inverse stands for an edge that is directed clockwise. We construct a surface by identifying directed edges with the same name.

What is the Euler characteristic of the surface? Is the surface orientable? Why? Describe it topologically by naming and sketching it, if possible.

3.8. Math 431.

Problem 34. Consider the following conditions for the real valued function f defined on $[a, b]$:

- (1) f is continuous at each x in $[a, b]$
- (2) f is differentiable at each x in $[a, b]$
- (3) $\int_a^b f(x) dx$ exists.

Do the following:

- (1) There are six statements of the type ‘ X implies Y ’ where X and Y are each A , B , or C . (Nine if you allow things like ‘ A implies A .’) Which of the six are true and which are false?
- (2) Prove one of the true ones.
- (3) Give a counter example for one of the false ones.

Problem 35. Let K be a non-empty compact subset of \mathbb{R}^n , and let $C(K)$ denote the space of continuous functions $f : K \rightarrow \mathbb{R}$ equipped with the supremum norm

$$\|f\| := \sup_{x \in K} |f(x)|.$$

Say that a function $f : K \rightarrow \mathbb{R}$ is 1-*Lipschitz* if

$$|f(x) - f(y)| \leq d(x, y)$$

for all $x, y \in K$ (here d is the usual Euclidean distance). Show that the collection L of all 1-Lipschitz functions is closed in $C(K)$.

Problem 36. Give an example of a metric space X and a closed and bounded subset of X that is not compact. Justify your answer.

Problem 37. Give an example of a sequence of real-valued continuous functions (f_n) with domain $[0, 1]$ such that $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in [0, 1]$, but such that $\int_0^1 f_n(x) dx = 1$ for all n .

3.9. Math 432.

Problem 38. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^1 function such that $f(0) = 0$, and let (x_n) be a sequence in \mathbb{R}^2 such that $x_n \neq 0$ for all n and $x_n \rightarrow 0$ as $n \rightarrow \infty$. Assume that the derivative $f'(0)$ is invertible. Show that $f(x_n) \neq 0$ for all but finitely many n .

Problem 39. Let $GL(n, \mathbb{R})$ denote the collection of invertible $n \times n$ matrices with real number entries, considered as a subset of the finite dimensional real vector space

$$\mathbb{R}^{n \times n} = \{(a_{ij})_{i,j=1}^n \mid a_{ij} \in \mathbb{R} \text{ for all } i, j \in \{1, \dots, n\}\}$$

of $n \times n$ matrices with real number entries equipped with the norm

$$\|(a_{ij})\| = \sqrt{\sum_{i,j=1}^n a_{ij}^2}.$$

- (1) Show that $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n \times n}$. *Hint: use that polynomials are continuous functions on $\mathbb{R}^{n \times n}$ for the above norm.*
- (2) Show that if $i : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ is the map defined by $i(A) = A^{-1}$, then its derivative Di is given by the formula

$$(Di)(A) : B \mapsto -A^{-1}BA^{-1}.$$

3.10. Math 442.

Problem 40. Consider a particle of mass m that is constrained to move on a right circular cylinder of radius A .

- (1) Find a set of generalized coordinates for the configuration space and describe how the coordinates are used to represent the position of the particle in \mathbb{R}^3 .
- (2) Find an expression for the kinetic energy of the particle in terms of the generalized coordinates from part 1.

Problem 41. Let $f : \mathbb{R}^3 \rightarrow [0, \infty)$ be an integrable function with compact support. View f as defining the mass density of a rigid body B , which is the support of f , as follows: for any compact set $K \subset \mathbb{R}^3$, the mass of $K \cap B$ is

$$\mu(K) = \iiint_K f(x, y, z) \, dx dy dz.$$

Let $\mathbf{r} = \langle x, y, z \rangle$ denote the position vector in \mathbb{R}^3 . Define the *inertia tensor* $\mathbb{I} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$\mathbb{I}(\mathbf{u}) = \iiint_B \mathbf{r} \times (\mathbf{u} \times \mathbf{r}) f(x, y, z) \, dx dy dz.$$

Prove that \mathbb{I} has the following properties:

- (1) \mathbb{I} is a linear transformation.
- (2) For all $\mathbf{u} \in \mathbb{R}^3$, $\mathbf{u} \cdot \mathbb{I}(\mathbf{u}) \geq 0$.

- (3) For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $\mathbf{u} \cdot \mathbb{I}(\mathbf{v}) = \mathbf{v} \cdot \mathbb{I}(\mathbf{u})$.

3.11. Math 443.

Problem 42. (1) Consider the parametrized curve

$$\gamma(t) = \left\langle \frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{1}{\sqrt{2}}t \right\rangle, \quad -1 < t < 1.$$

- Show that γ is a *unit speed* curve.
 - Find (simplified) expressions for the vector fields along γ that constitute the *Frenet frame*. That is, find the unit tangent $T(t)$, the unit normal $N(t)$, and the binormal $B(t)$.
 - Find (simplified) expressions for the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the curve γ .
 - Consolidate the previous results into the *Frenet equations* for the unit speed curve γ .
- (2) Consider the parametrized surface

$$\mathbf{x}(u, v) = \left\langle u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2 \right\rangle.$$

- Find (simplified) expressions for the partial derivatives \mathbf{x}_u and \mathbf{x}_v .
- Find a simplified expression for the *Gauss map*, $G(\mathbf{x}(u, v))$.
- Find the 2×2 matrix that represents the *first fundamental form* in the ordered basis $\{\mathbf{x}_u, \mathbf{x}_v\}$.
- Find the 2×2 matrix that represents the *second fundamental form* in the ordered basis $\{\mathbf{x}_u, \mathbf{x}_v\}$.
- Find (simplified) expressions for the mean curvature $H(\mathbf{x}(u, v))$ and for the Gaussian curvature $K(\mathbf{x}(u, v))$.

3.12. Math 444.

Problem 43. Show that $f(z) = 1/\bar{z}$ is not analytic in any subset of the plane.

Problem 44. On what domain does the Taylor series of $f(z) = 1/z$ in powers of $(z - 2)$ converge to f ?

Problem 45. Find $\int_C f(z) dz$ if C is the circle of radius 1 centered at 2. Give a reason for your answer.

Problem 46. Find $\int_C \frac{e^z}{z-2} dz$ if C is the circle of radius 1 centered at 2.

3.13. Math 454.

Problem 47. The following statements are equivalent over ZF:

- (A) Every set has a choice function (the axiom of choice).
- (B) Every nonempty partial order in which chains have upper bounds has a maximal element (Zorn's lemma).
- (C) Every set can be well-ordered.
- (D) The Cartesian product of nonempty sets is nonempty.

Prove two of the implications among them.

Problem 48. Prove that for every ordinal α , $\alpha \leq \aleph_\alpha$.

Problem 49. Prove that every ordinal α can be written uniquely as $\beta + n$, where β is a limit ordinal and n is a natural number.

3.14. Math 455.

Problem 50. Show that a theory has a computably enumerable set of axioms if and only if it has a computable set of axioms.

Problem 51. Let $\mathfrak{A} = (\mathbb{R}, \leq)$ and let $T = \text{Th}(\mathfrak{A})$.

- (1) What is the smallest size that a model of T can have? Explain.
- (2) What subsets of \mathbb{R} are definable in this structure? Justify your answer.

Problem 52. Show that if a theory has arbitrarily large finite models, then it has an infinite model.

3.15. Math 471–472.

Problem 53. The density function of a random variable X is given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ x + 1 & \text{if } -1 < x < 0 \\ \frac{3}{16}x^2 & \text{if } 0 \leq x < 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$

- (a) Find the distribution function of X .
- (b) Find the density function for the random variable $Y = |X|$.

Problem 54. Suppose X is a normally distributed random variable with mean 0 and standard deviation σ .

- (1) Write down the probability density function of X .

- (2) Write down the likelihood function $L(\sigma)$ for the sample
- $$(X_1, X_2, X_3) = (x_1, x_2, x_3) = (0, 1, -1).$$
- (3) Find the maximum likelihood estimate of the standard deviation σ for the sample (2).

3.16. Math 475.

Problem 55. Let S be an n -element set and let

$$R = \{(A, B) : A \subseteq B \subseteq S\}$$

(that is R is the order relation \subseteq on the powerset of S). Find $|R|$.

Problem 56. Let G be the complete graph on 6 vertices. (This means that every pair of vertices has an edge connecting them.) Suppose each edge of G is colored either red or blue. Prove that there are 3 vertices such that the 3 edges connecting them all have the same color.