FIBONACCI NUMBERS: AN APPLICATION OF LINEAR ALGEBRA

1. Powers of a matrix

We begin with a proposition which illustrates the usefulness of the diagonalization. Recall that a square matrix A is dioganalizable if there is a non-singular matrix S of the same size such that the matrix $S^{-1}AS$ is diagonal. That means all entries of $S^{-1}AS$ except possibly diagonal entries are zeros. The numbers which show up on the diagonal of $S^{-1}AS$ are the *eigenvalues* of A. For a diagonal matrix, it is very easy to calculate its powers.

Proposition 1. *Let A a diaganalizable matrix of size $m \times m$, and assume that*

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & \dots & \dots & \\ & \dots & \dots & \\ & \dots & \dots & \\ 0 & 0 & \dots & \lambda_m \end{pmatrix}$$

for a non-singular matrix S.

Then for an integer $n \geq 0$

$$A^n = S \begin{pmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & \lambda_2^n & \dots & 0 \\ & \dots & \dots & \\ & \dots & \dots & \\ & \dots & \dots & \\ 0 & 0 & \dots & \lambda_m^n \end{pmatrix} S^{-1}.$$

Proof. Denote the diagonal matrix by

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & \dots & \dots & \\ & \dots & \dots & \\ & \dots & \dots & \\ 0 & 0 & \dots & \lambda_m \end{pmatrix},$$

and observe that, since it is very easy to multiply diagonal matrices, we have

$$\Lambda^n = \begin{pmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & \lambda_2^n & \dots & 0 \\ & \dots & \dots & \\ & \dots & \dots & \\ & \dots & \dots & \\ 0 & 0 & \dots & \lambda_m^n \end{pmatrix} \tag{1}$$

At the same time since

$$\Lambda = S^{-1}AS,$$

we find that

$$\Lambda^n = (S^{-1}AS)^n = S^{-1}AS\ S{-}1AS\ \dots S^{-1}AS = S^{-1}A^nS$$

because of the obvious cancellations. We thus conclude that

$$A^n = S\Lambda^n S^{-1},$$

and the required identity follows from (1). □

2. Fibonacci numbers and Kepler's observation

The sequence of Fibonacci numbers F_n

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \dots$$

is defined recursively as follows. One begins with

$$F_0 = 0,\ \text{and}\ F_1 = 1.$$

After that every number is defined to be sum of its two predecessors:

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

The sequence of Fibonacci numbers attract certain interest for various reasons (see, for instance, `http://en.wikipedia.org/wiki/Fibonacci_number`).

In particular, Johannes Kepler (1571 1630) , one of the greatest astronomers in the history, observed that the ratio of consecutive Fibonacci numbers converges to the *golden ratio*.

Theorem (Kepler).

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

In this note, we make use of linear algebra in order to find an explicit formula for Fibonacci numbers, and derive Kepler's observation from this formula. More specifically, we will prove the following statement.

Proposition 2. *For* $n \geq 1$

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

We now show how to derive Kepler's observation from Proposition 2.

Proof of Kepler's observation. We simply calculate the limit as follows:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{F_{n+1}}{F_n} & \\
&= \lim_{n\to\infty} \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} \frac{2^n\sqrt{5}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n} \\
&= \frac{1}{2} \lim_{n\to\infty} \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n} \\
&= \frac{1}{2} \lim_{n\to\infty} \frac{(1+\sqrt{5}) - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n (1-\sqrt{5})}{1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n} = \frac{1+\sqrt{5}}{2}
\end{aligned}
$$

since

$$\lim_{n\to\infty} \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n = 0$$

as soon as

$$\left|\frac{1-\sqrt{5}}{1+\sqrt{5}}\right| < 1.$$

$\square$

The rest of the note is devoted to the proof of Proposition 2 with the help of Linear Algebra, and Proposition 1 in particular.

3. Linear Algebra interpretation of Fibonacci numbers

Let L be the linear operator on $\mathbb{R}^2$ represented by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the standard basis of $\mathbb{R}^2$. For any vector $\mathbf{v} = (x, y)^T$, we have that

$$L(\mathbf{v}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x \end{pmatrix}.$$

In particular, for the vector $\mathbf{u_k}$ whose coordinates are two consecutive Fibonacci numbers $(F_k, F_{k-1})^T$, we have that

$$L(\mathbf{u_k}) = A \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_k \\ F_{k-1} \end{pmatrix} = \begin{pmatrix} F_k + F_{k-1} \\ F_k \end{pmatrix} = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = \mathbf{u_{k+1}}.$$

Thus we can produce a vector whose coordinates are two consecutive Fibonacci numbers by applying L many times to the vector $\mathbf{u_1}$ with coordinates $(F_1, F_0)^T = (1, 0)$:

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

Equation 2 is nothing but a reformulation of the definition of Fibonacci numbers. This equation, however, allows us to find an explicit formula for Fibonacci numbers as soon as we know how to calculate the powers A^n of the matrix A with the help of the diagonalization.

4. Diagonalization of the matrix A and proof of Proposition 2

We begin with finding the eigenvalues of A as the roots of its characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{pmatrix} = \lambda^2 - \lambda - 1.$$

We make use of the quadratic formula to find the roots as

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}, \tag{3}$$

and we conclude, since the two eigenvalues are real and distinct, that the matrix A dioganalizable. In order to dioganalize it, we need to find a basis which consists of eigenvectors of the linear operator L.

Let us find the eigenvectors from the equations

$$L(\mathbf{v_1}) = \lambda_1 \mathbf{v_1}, \quad \text{and} \quad L(\mathbf{v_2}) = \lambda_2 \mathbf{v_2}$$

or, in coordinates with respect to the standard basis of $\mathbb{R}^2$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

We solve these equations and find eigenvectors:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

The transition matrices between the standard basis and the basis of eigenvectors is thus

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

and

$$S^{-1} = \begin{pmatrix} \frac{1}{\lambda_1 - \lambda_2} & \frac{\lambda_2}{\lambda_2 - \lambda_1} \\ \frac{1}{\lambda_2 - \lambda_1} & \frac{\lambda_1}{\lambda_1 - \lambda_2} \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

We can now check that, as expected,

$$S^{-1} A S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{4}$$

We are now in a position to prove Proposition 2 with the help of the dioganalization (4).

Proof of Proposition 2. Proposition 1 now implies that

$$A^n = S \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} S^{-1},$$

and we combine this with equation (2) to obtain that

$$\begin{aligned} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} &= A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = S \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} S^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}. \end{aligned}$$

Equating the entries of the vectors in the last formula we obtain in view of (3) that

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

as claimed in Proposition 2. □

Remark 1. Using the explicit formula from Proposition 2 one may address some other questions about Fibonacci numbers.

Remark 2. It was Linear Algebra, specifically the diagonalization procedure, which allowed us to obtain the explicit formula in Proposition 2. This is not the only way to prove the formula.

Remark 3. The sequence of Fibonacci is a very simple example of a sequence given by a *recursive relation*. One may apply similar methods in order to investigate other sequences given by recurrence relations.