Name: ________________________________

Question 1

Let $K \supset F$ be a field extension. Let $u \in K$, and let $p \in F[x]$ be the minimal polynomial of u. Let $f \in F[x]$ be such that $f(u) = 0$.

Circle True if the statement must be true, and False if the statement may be false.

TRUE FALSE If K is finite-dimensional over F then K is a simple extension of F.

TRUE FALSE If $K = F(u)$ is a simple extension of F, then p splits completely in K.

TRUE FALSE If K is finitely-generated over F then K is an algebraic extension of F.

TRUE FALSE If K is transcendental over F, then K is infinite dimensional over F.

TRUE FALSE If K is algebraic and separable over F, then K is normal over F.

TRUE FALSE If K is normal over F, then f splits completely in $K[x]$.

TRUE FALSE If K is normal over F, then p splits completely in $K[x]$.

TRUE FALSE If K is separable over F, then f splits completely in $K[x]$.

TRUE FALSE If F has characteristic zero, then f is separable.

TRUE FALSE If K has characteristic zero, then F is of characteristic zero.

TRUE FALSE $F(u) \subseteq F(u^2)$.

TRUE FALSE $F(u) \supseteq F(u^2)$.

TRUE FALSE If $v, w \in K$ are algebraic over F and $w \neq 0$, then vw^{-1} is algebraic over F.

Question 2

Let K be an extension of a field F, and let E be an intermediate field so that $F \subseteq E \subseteq K$. Assume that both field extensions $F \subseteq E$ and $E \subseteq K$ are finite dimensional.

Circle True if the statement must be true, and False if the statement may be false.

TRUE FALSE The field K is algebraic over F.

TRUE FALSE If F is of characteristic zero then there exists $p \in F[x]$ such that $K = F[x]/(p)$.

TRUE FALSE If K is normal over F, then K is normal over E.

TRUE FALSE If K is separable over F, then K is separable over E.

TRUE FALSE If K is simple over F, then K is simple over E.

TRUE FALSE If K is normal over F, then E is normal over F.

Question 3

Does there exist an irreducible polynomial $p \in \mathbb{Q}[x]$ with multiple roots? Verify your answer.

Question 4

Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{-2})$.

a) Extend the set $\{\sqrt{2}, \sqrt{-2}\}$ to a basis of K as a vector field over $\mathbb{Q}$.

b) Find $\mathrm{G}al_{\mathbb{Q}}K$.

Question 5

Let p be a prime, and let $n \geq 1$ be an integer. Let K be a field of order p^n, and let $\mathbb{Z}_p$ be its prime subfield. This question is about the field extension $K \supseteq \mathbb{Z}_p$ and its Galois group.

a) What is the characteristic of the field K?

b) What is the dimension $[K : \mathbb{Z}_p]$?

c) Is the field extension $K \supseteq \mathbb{Z}_p$ separable? Explain your answer.

d) Is the field extension $K \supseteq \mathbb{Z}_p$ normal? Explain your answer.

e) What can we conclude about the order $|Gal_{\mathbb{Z}_p} K|$ out of the above **a**–**d**? Explain why we need all these statements for your conclusion.

f) Prove that the group $Gal_{\mathbb{Z}_p} K$ is cyclic.

Consider the map $f : K \to K$ defined by $f(t) = t^p$ for $t \in K$. State without proof the properties of the map f which allow you to conclude that $f \in Gal_{\mathbb{Z}_p} K$. Then prove that f generates the group $Gal_{\mathbb{Z}_p} K$.

g) Let $n = 12$. List all integers m such that K has a subfield E of order p^m. Make use of Galois correspondence to justify your answer.