Counting the relations compatible with an algebra

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Four finiteness conditions

Proof-by-picture: An easy example

Proof-by-picture: Two general conditions for ‘infiniteness’

A family of ‘finite’ examples

Compatible relations

A compatible relation on a finite algebra $\mathbf{A}$ is a non-empty subuniverse of $\mathbf{A}^n$, for some $n \in \mathbb{N}$.

There are several natural finiteness conditions on $\mathbf{A}$ that are based on 'how many' compatible relations it has.

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- The finite relatedness of **A** only depends on Var(**A**).[3]

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- (2) $\Rightarrow$ (1): Few subpowers implies finitely related.[6]

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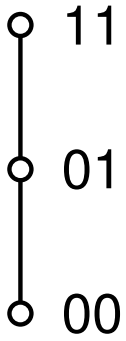
Condition (4): Finitely many relations

There is a finite set of compatible relations on **A** such that every other compatible relation on **A** is interdefinable with one of these relations, via conjunctions of atomic formulæ.

An example

Two compatible relations on the 2-element bounded lattice $\mathbf{L} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$:

$\leqslant \subseteq L^2$



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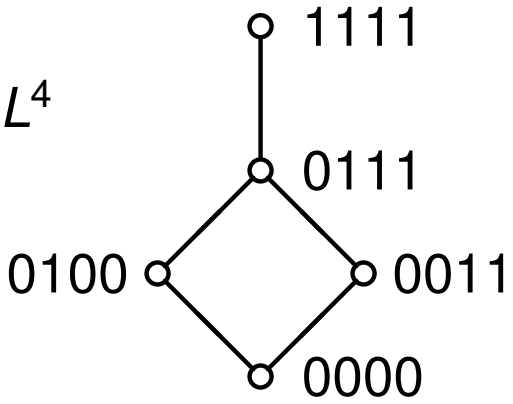

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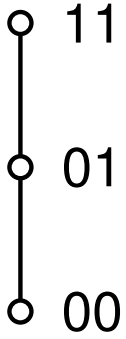

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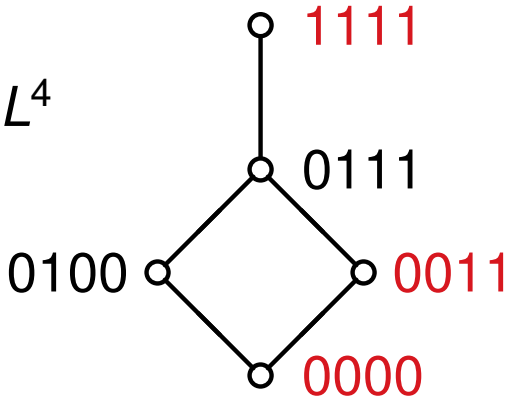

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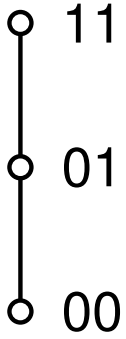

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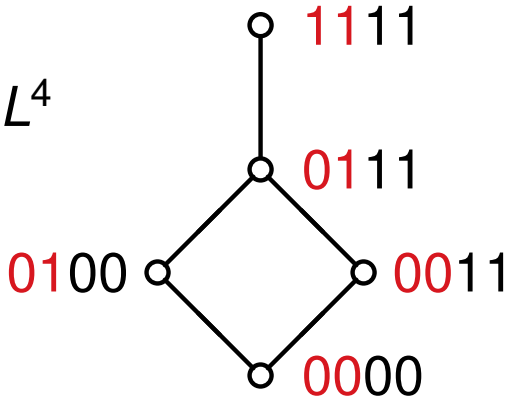

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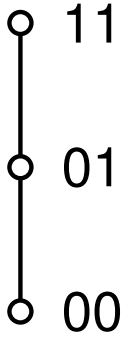

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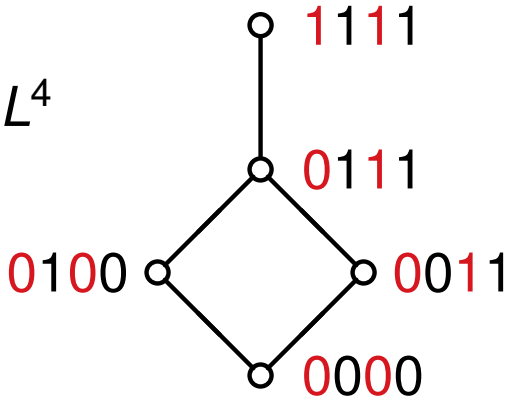

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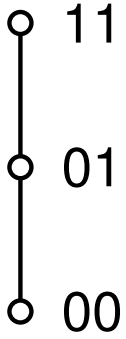

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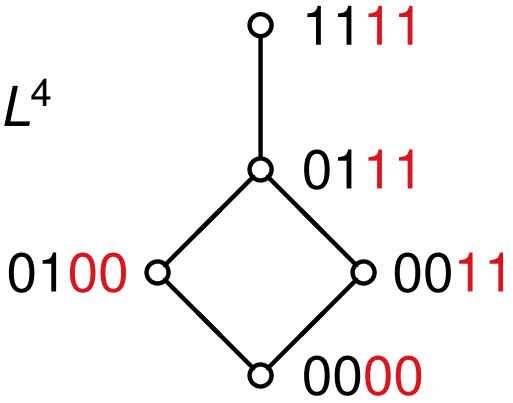

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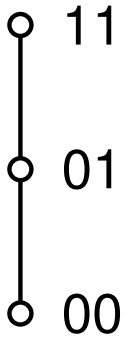

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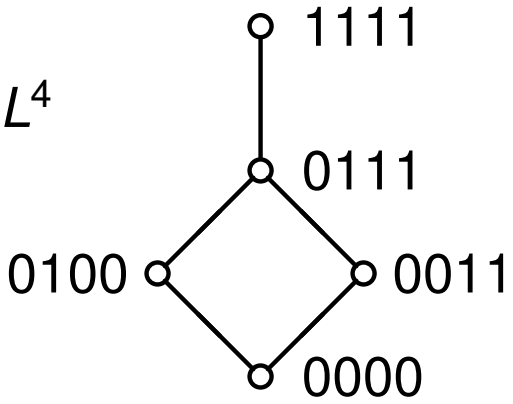

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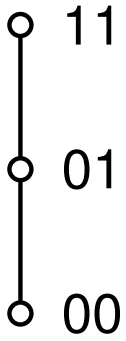

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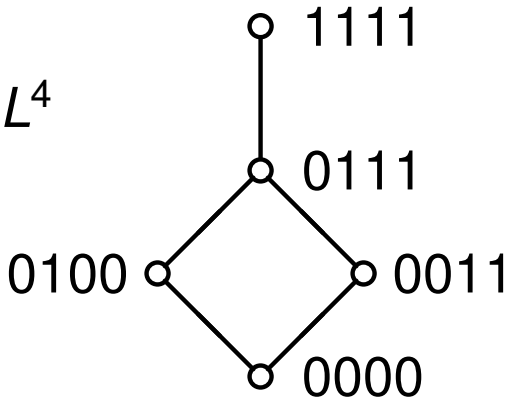

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Every compatible relation on $\mathbf{L}$ is equivalent to either $\leqslant$ or Δ_L.

Basic definitions

Two compatible relations on **A** are equivalent if each is conjunct-definable from the other.

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Question

Which finite algebras admit only finitely many relations?

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Two compatible relations: Δ_L, $\leqslant$.

2-element lattice $\mathbf{2} = \langle \{0,1\}; \vee, \wedge \rangle$

Eight compatible relations: Δ_2, $\leqslant$, $\{0\}$, $\{1\}$, $\{(0,1)\}$, $\leqslant \times \{0\}$, $\leqslant \times \{1\}$, $\leqslant \times \{(0,1)\}$.

More examples

Finitely many relations:

- 3-element p-algebra $\mathbf{S} = \langle \{0, d, 1\}; \vee, \wedge, ^*, 0, 1 \rangle$,
- 4-element Boolean algebra with constants,
- ring with unity $\mathbf{Z}_{pq}$, for distinct primes p, q,
- every quasi-primal algebra (with R. Willard),
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(If $\mathbf{A}$ admits infinitely many relations and $\mathbf{A} \in \mathrm{HS}(\mathbf{B})$, then $\mathbf{B}$ admits infinitely many relations.)

Proof-by-picture: An easy example

Consider the 2-element algebra $\mathbf{M} = \langle \{0, 1\}; m \rangle$, where $m \colon \{0, 1\}^3 \to \{0, 1\}$ is the majority operation.

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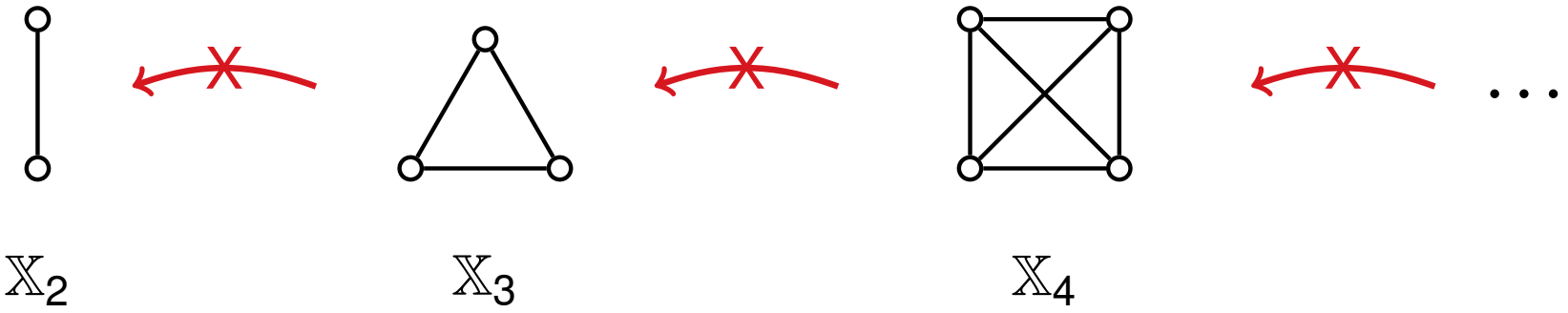
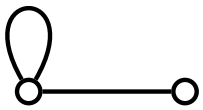
This picture proves that $\mathbf{M}$ admits infinitely many relations:



$\mathbb{M}$ 

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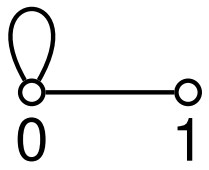

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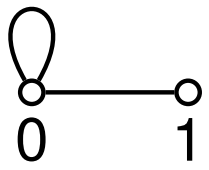

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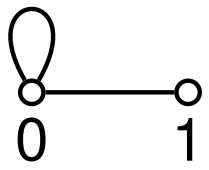

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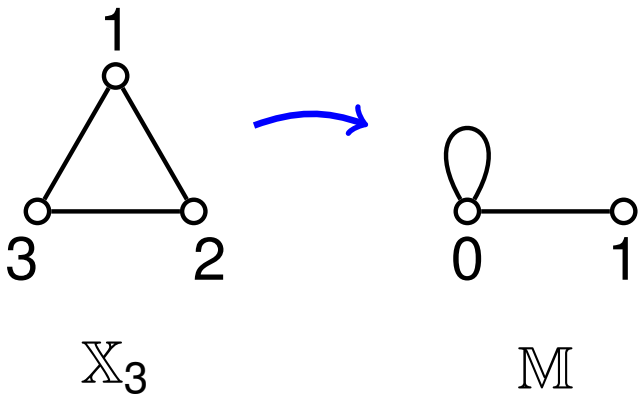

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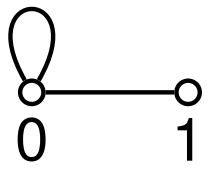

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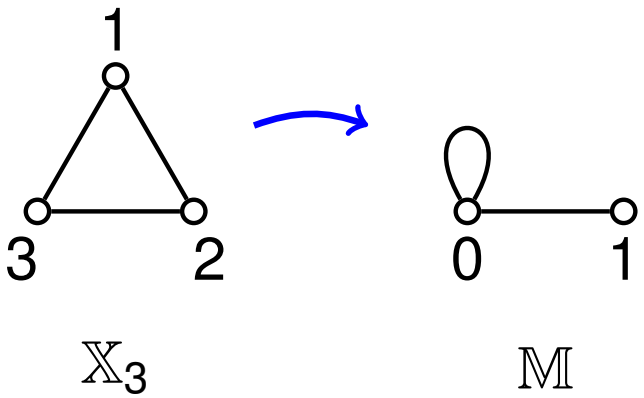

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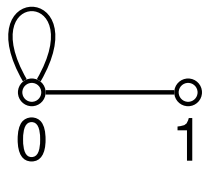

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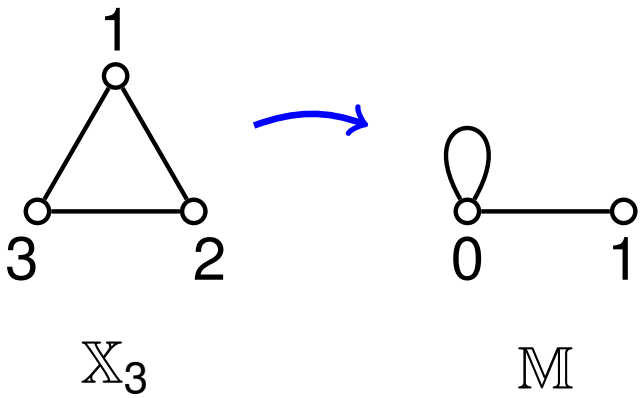

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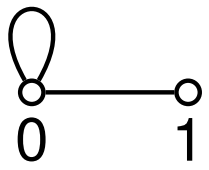

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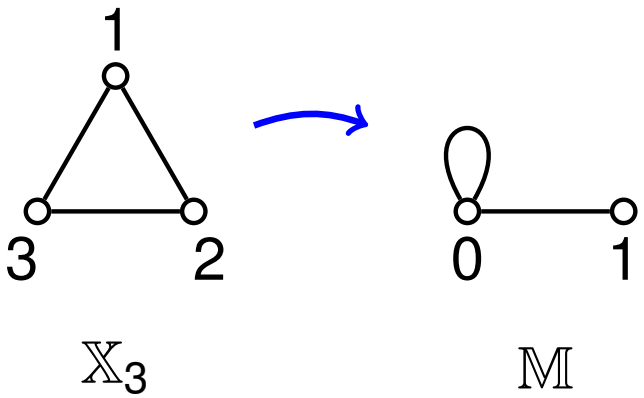

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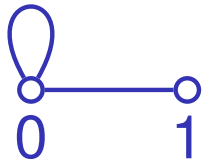
$\mathbb{M}$ 

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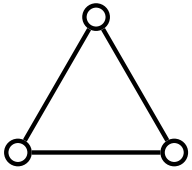

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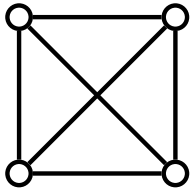

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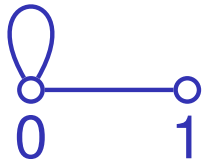
$\mathbb{M}$ 

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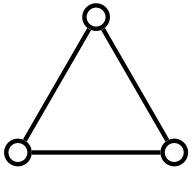

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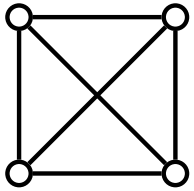

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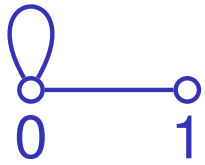
$\mathbb{M}$ 

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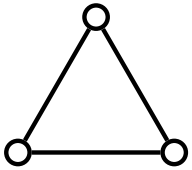

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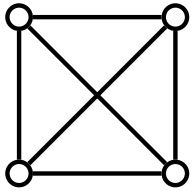

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- We want to show that the relations $r_2, r_3, r_4, \dots$ are pairwise non-equivalent.
- We will just check that r_3 is not ca-definable from r_4.
- It suffices to find a map $\varphi\colon Z \to \{0,1\}$, for some $Z \subseteq \{0,1\}^n$, such that φ preserves r_4 but not r_3.

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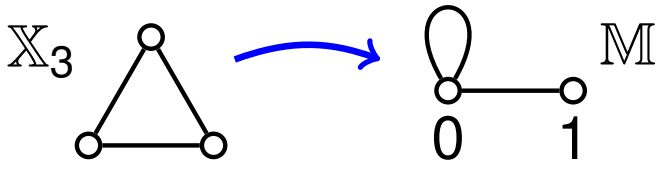

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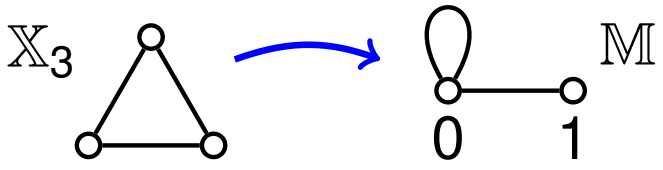

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- In fact, we have $\mathbb{X}_3 \in \mathrm{ISP}(\mathbb{M})$. Therefore $\mathbb{X}_3 \cong \mathbb{Z} \leqslant \mathbb{M}^4$.
- The map $\varphi\colon X_3 \to \{0,1\}$ does not preserve r_3, because it is not a graph homomorphism, i.e., $\varphi \notin \hom(\mathbb{X}_3, \mathbb{M})$.

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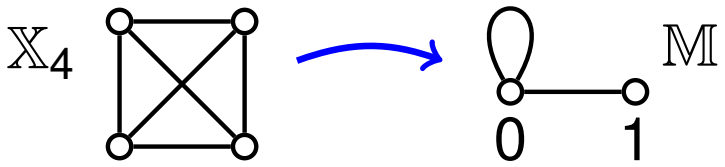

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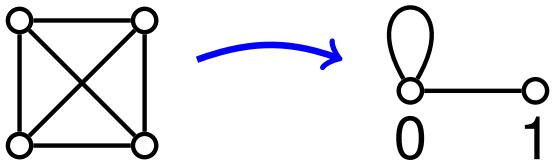

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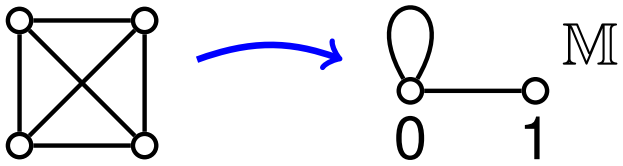
$\mathbb{X}_4$ 

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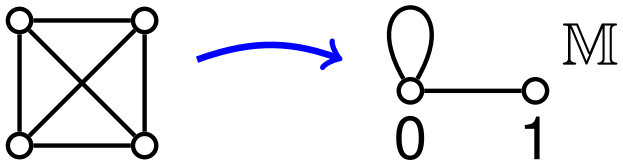

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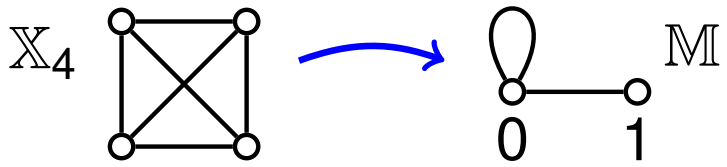

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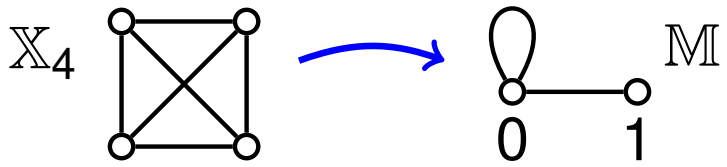

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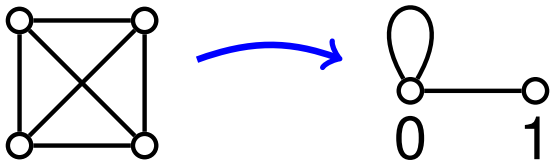

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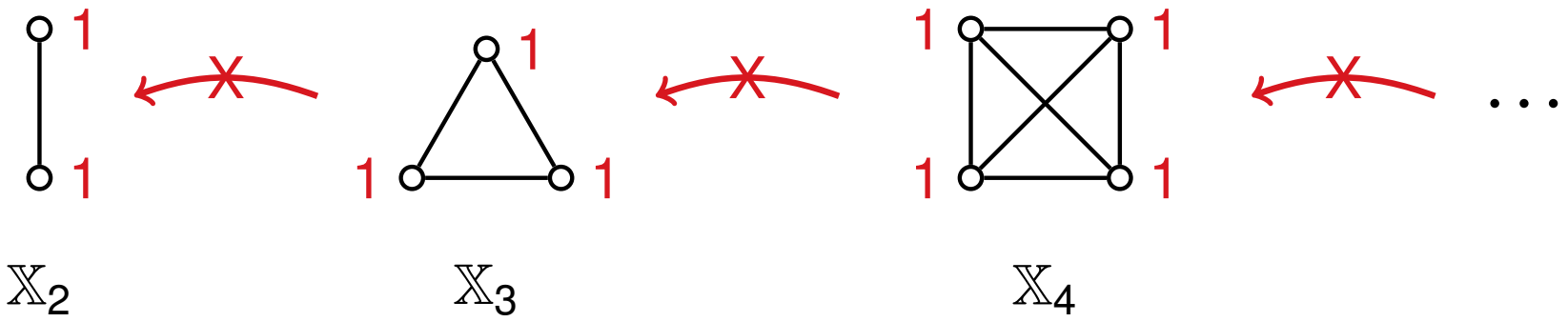
- But $\mathbb{X}_3 \leqslant \mathbb{M}^4$, so we really want: If $\omega\colon \mathbb{X}_4 \to \mathbb{X}_3$, then $\varphi \circ \omega\colon \mathbb{X}_4 \to \mathbb{M}$ is a graph homomorphism. True vacuously.

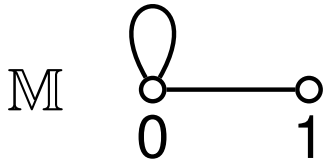
Proof-by-picture: Recap

Consider the 2-element algebra $\mathbf{M} = \langle \{0,1\}; m \rangle$, where $m\colon \{0,1\}^3 \to \{0,1\}$ is the majority operation.

This picture below proves that $\mathbf{M}$ admits infinitely many relations.

The red labels on $\mathbb{X}_n$ yield a map $\varphi_n\colon X_n \to \{0,1\}$ that is not a graph homomorphism.





Proof-by-picture: General approach

Lemma

To show that **A** *admits infinitely many relations,*
it suffices to find

- *a structure* $\mathbb{A} = \langle A; R \rangle$ *that is compatible with* **A**, *and*
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Proof-by-picture: General approach

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such that, either for all $k < \ell$ *or for all* $k > \ell$, *the following condition holds:*

- *for every morphism* $\omega \colon \mathbb{X}_\ell \to \mathbb{X}_k$, *the map* $\varphi_k \circ \omega \colon X_\ell \to A$ *is a morphism from* $\mathbb{X}_\ell$ *to* $\mathbb{A}$

  (i.e., the relation r_k is not ca-definable from r_ℓ).

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A bad relation

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- For non-permuting congruences: $r = \theta_1 \cdot \theta_2$.
- For $\mathbf{A} = \mathbf{B}^2$: $r = \{\, ((x_1, x_2), (y_1, y_2)) \in A^2 \mid x_2 = y_1 \,\}$.

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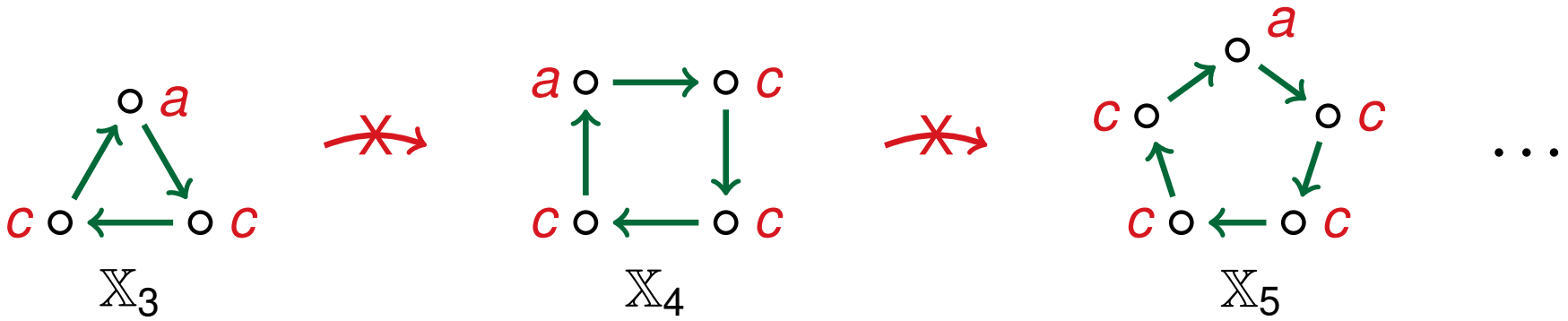

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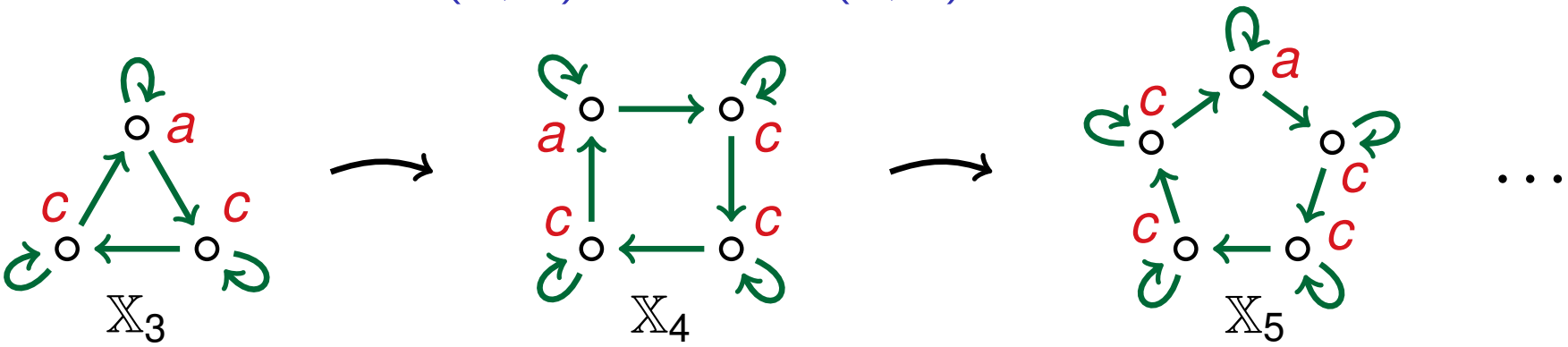


A general condition for 'infiniteness'

Applications

Corollary

If one of the following holds, then $\mathbf{A}$ *admits infinitely many relations:*

1. $\mathbf{A}$ *has a pair of non-permuting congruences;*
2. $\mathbf{A}$ *is isomorphic to* $\mathbf{B}^2$*, with* $\mathbf{B}$ *non-trivial;*
3. $\mathbf{A}$ *is a subalgebra of* $\mathbf{B}^2$ *such that* $\{(0,0),(0,1),(1,0)\} \subseteq A$*, for some* $0 \neq 1$ *in* $\mathbf{B}$*.*

This result covers:

- Boolean algebras of size $\geqslant 4$,
- lattices of size $\geqslant 3$,
- p-algebras of size $\geqslant 4$,
- non-chain Heyting algebras, ...

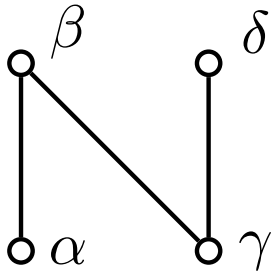
Another general condition for ‘infiniteness’

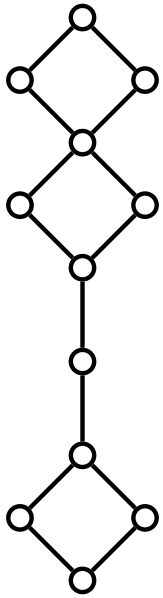
Theorem and example

Theorem

If $\mathbf{A}$ *admits only finitely many relations, then* $\mathrm{Con}(\mathbf{A})$ *is an* $\mathcal{N}$*-free distributive lattice.*

$\mathcal{N}$





An $\mathcal{N}$-free distributive lattice

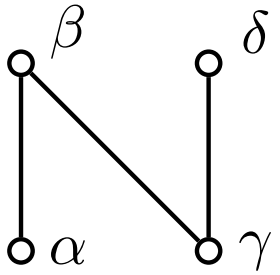
Another general condition for ‘infiniteness’

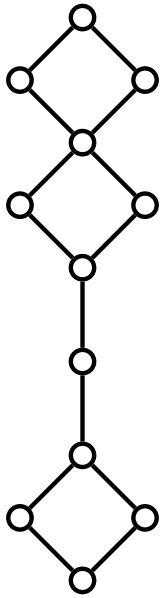
Theorem and example

Theorem

If $\mathbf{A}$ *admits only finitely many relations, then* $\mathrm{Con}(\mathbf{A})$ *is an* $\mathcal{N}$*-free distributive lattice.*

$\mathcal{N}$





An $\mathcal{N}$-free distributive lattice

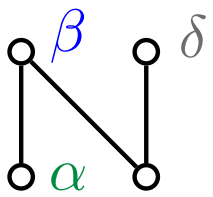
Example

Assume $\mathbf{M} = \mathbf{A} \times \mathbf{B} \times \mathbf{C}$, for non-trivial finite algebras $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Then $\mathbf{M}$ admits infinitely many relations.

Another general condition for ‘infiniteness’

Proof-by-picture

Proof

Assume Con(**A**) contains 

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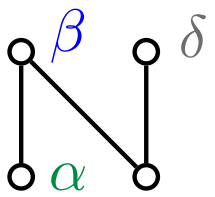
Another general condition for ‘infiniteness’

Proof-by-picture

Proof

Assume Con($\mathbf{A}$) contains 

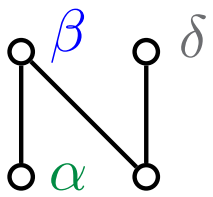
.

Take $\mathbb{A} = \langle A; \alpha, \beta, \delta \rangle$ and choose $(a, b) \in \delta \setminus \beta$.

Another general condition for ‘infiniteness’

Proof-by-picture

Proof

Assume Con(**A**) contains 

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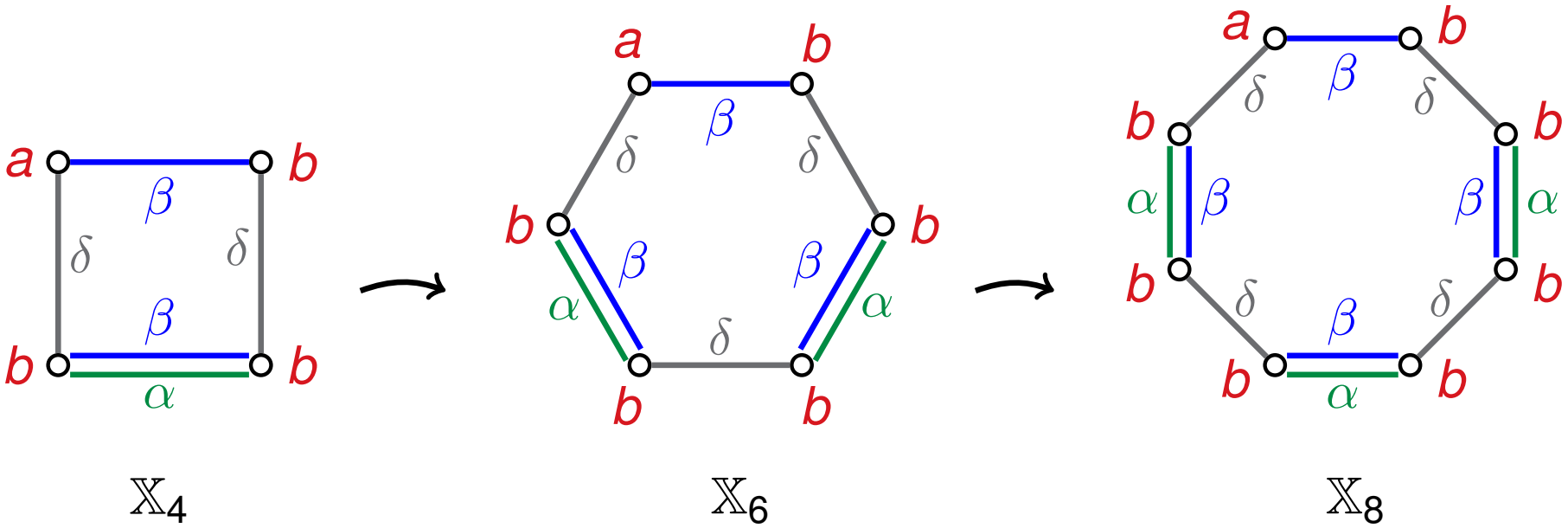

$\mathbb{X}_4$ $\mathbb{X}_6$ $\mathbb{X}_8$

Another general condition for ‘infiniteness’

Proof-by-picture

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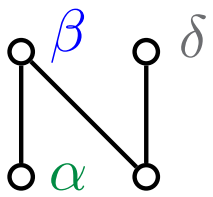
Assume Con(**A**) contains 

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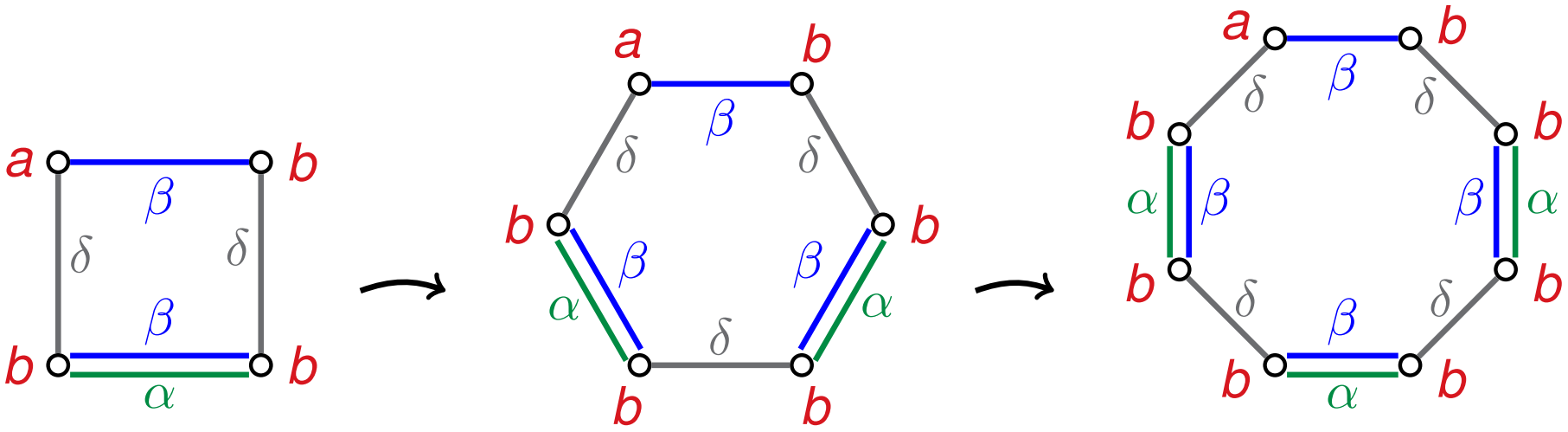

$\mathbb{X}_4$ $\mathbb{X}_6$ $\mathbb{X}_8$

So **A** admits infinitely many relations.

Finding examples of ‘finiteness’

A definition

A finite algebra $\mathbf{A}$ is strictly affine complete if:

for all $n \in \mathbb{N}$ and all $X \subseteq A^n$, every function $f\colon X \to A$ that preserves each $\theta \in \mathrm{Con}(\mathbf{A})$ extends to a polynomial of $\mathbf{A}$.

Finding examples of ‘finiteness’

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Example

Every finite algebra that generates an arithmetical variety is strictly affine complete (Pixley).

Finding examples of ‘finiteness’

A theorem

Theorem

Let $\mathbf{A}$ *be a finite algebra such that*

1. $\mathbf{A}$ *is strictly affine complete,*
2. *each* $a \in A$ *is the value of a constant term function of* $\mathbf{A}$*, and*
3. $\mathrm{Con}(\mathbf{A})$ *is an* $\mathcal{N}$*-free lattice.*

Then $\mathbf{A}$ *admits only finitely many relations.*

Finding examples of ‘finiteness’

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Proof

We use the duality given by the alter ego $\mathbb{A} := \langle A; \mathrm{Con}(\mathbf{A}), \mathcal{T} \rangle$ to represent all the compatible relations on $\mathbf{A}$.

Finding examples of ‘finiteness’

Application of the theorem

Example

Let $\mathbf{L}$ be a finite $\mathcal{N}$-free distributive lattice.

There exists a finite algebra $\mathbf{A}$ with $\mathrm{Con}(\mathbf{A}) \cong \mathbf{L}$ such that $\mathbf{A}$ admits only finitely many relations.

Finding examples of ‘finiteness’

Application of the theorem

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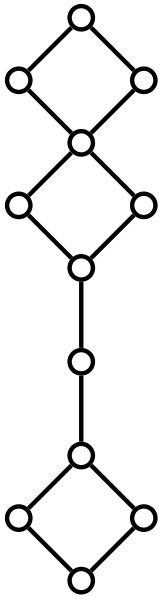

L

Finding examples of ‘finiteness’

Application of the theorem

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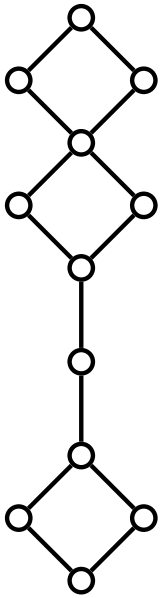
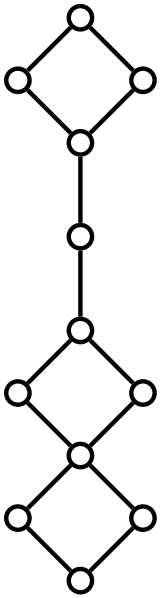


L



$\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, a_1, a_2, \ldots, a_n \rangle$

Finding examples of ‘finiteness’

Another application of the theorem

Example

Assume that $\mathbf{M} = \mathbf{A} \times \mathbf{B}$ is the independent product of two primal algebras. Then $\mathbf{M}$ admits only finitely many relations.

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For example, the following algebras admit only finitely many relations:

- the four-element Boolean algebra enriched with all elements as constants;
- the ring with unity $\mathbb{Z}_{pq}$, for all distinct primes p and q.

Finding examples of ‘finiteness’

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Mahalo!