Separating Clones Near the Top of the Clone Lattice

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Separation Theorem for a Clone $\mathcal{Q}$

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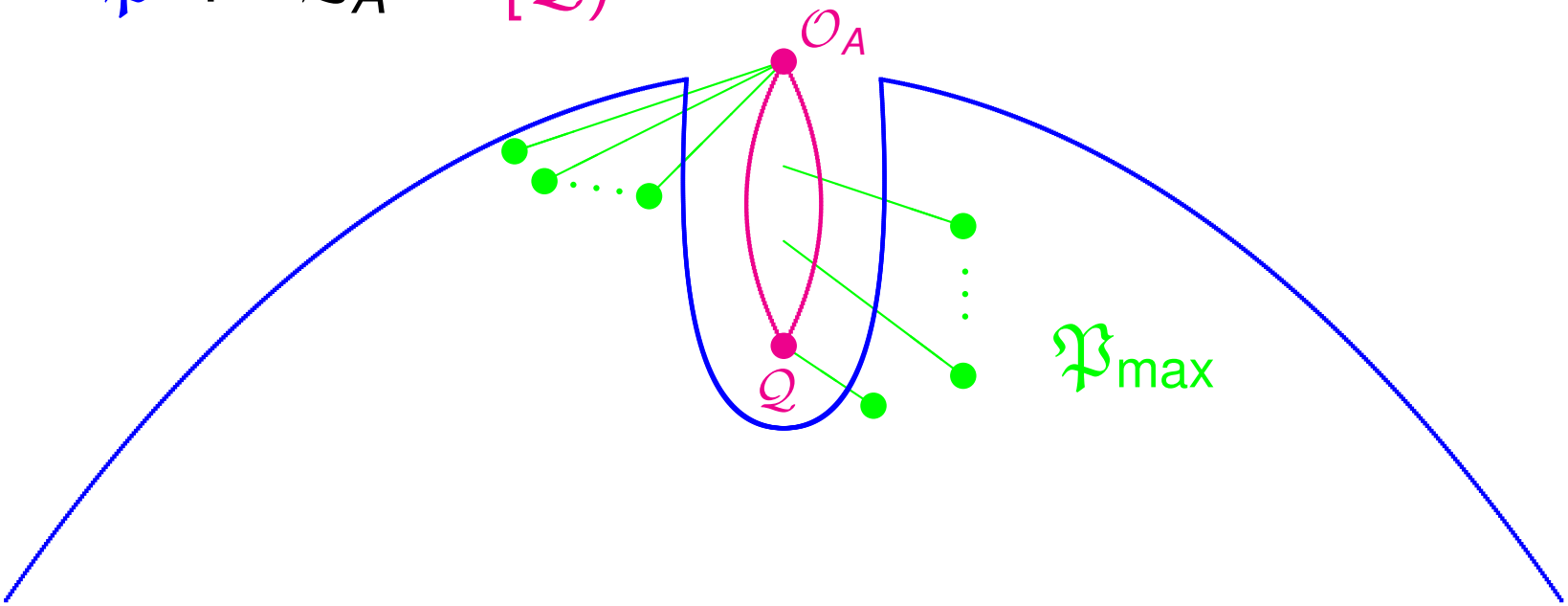

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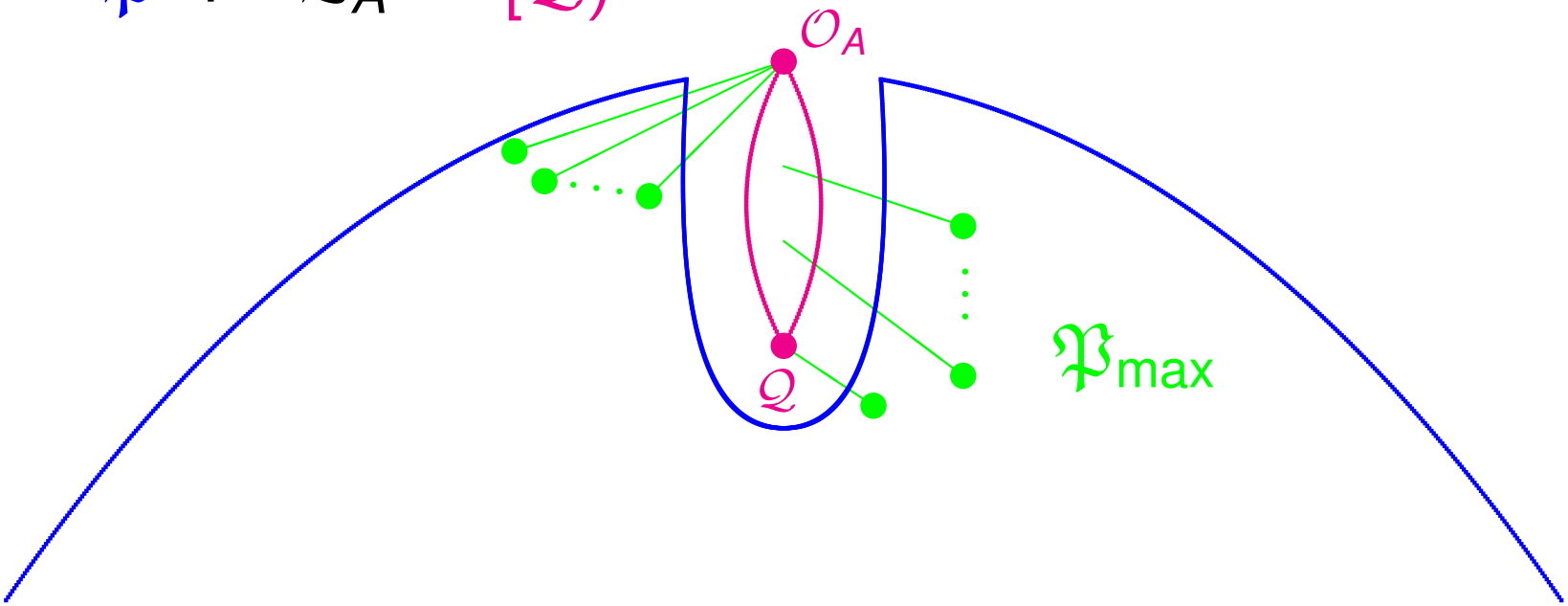

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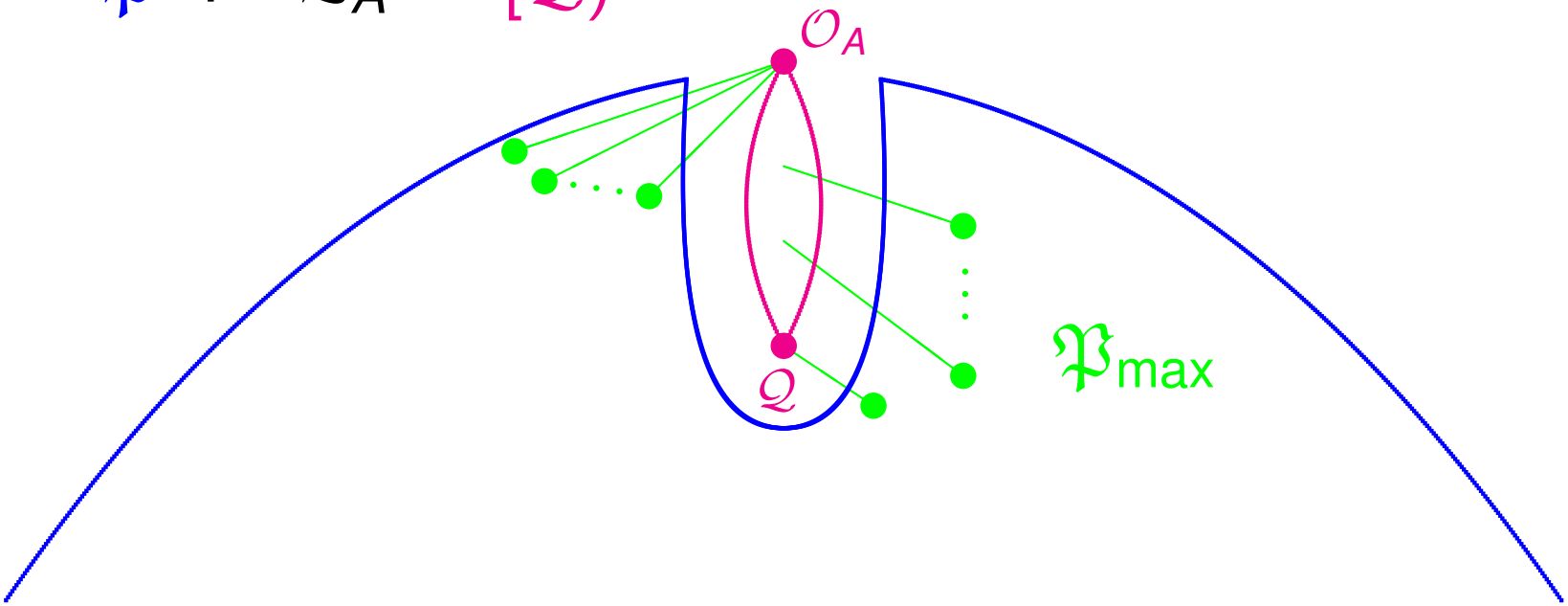

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$[\mathcal{Q})$ finite $\Longrightarrow$ • $\mathfrak{P}_{\max}$ is finite

Example 1: The Maximal Subclones of $\mathcal{O}_A$

$\mathcal{Q} = \mathcal{O}_A$

Rosenberg's Thm. The maximal clones on A are of the clones $\{\rho\}^{\perp} = \{f \in \mathcal{O}_A : f \text{ preserves } \rho\}$ for one of six types of rels ρ:

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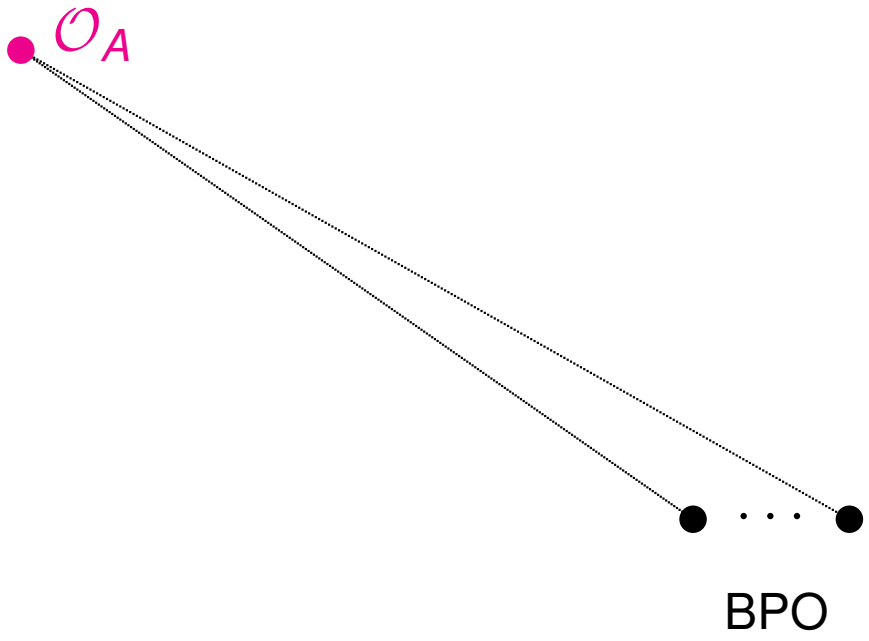

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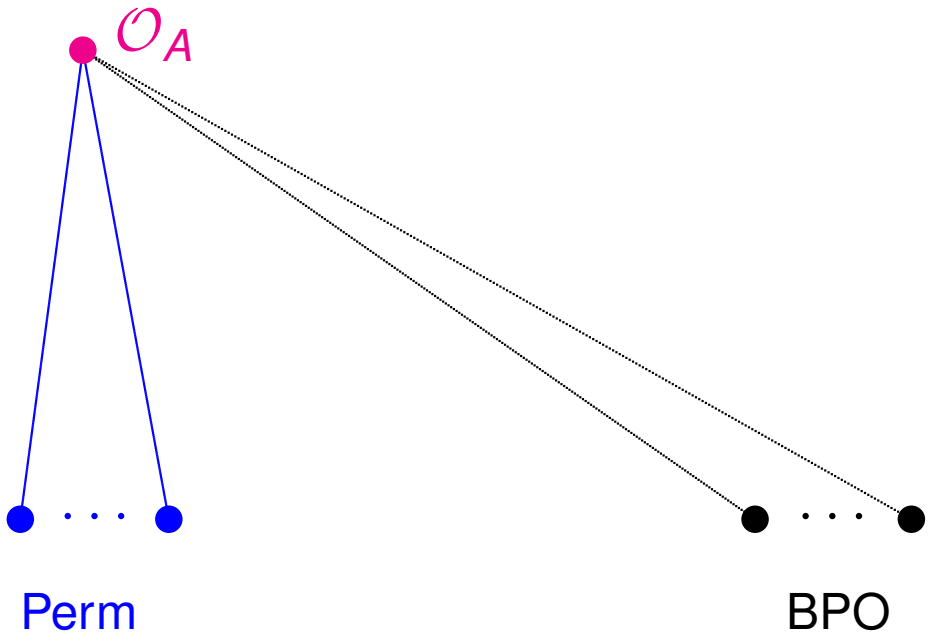

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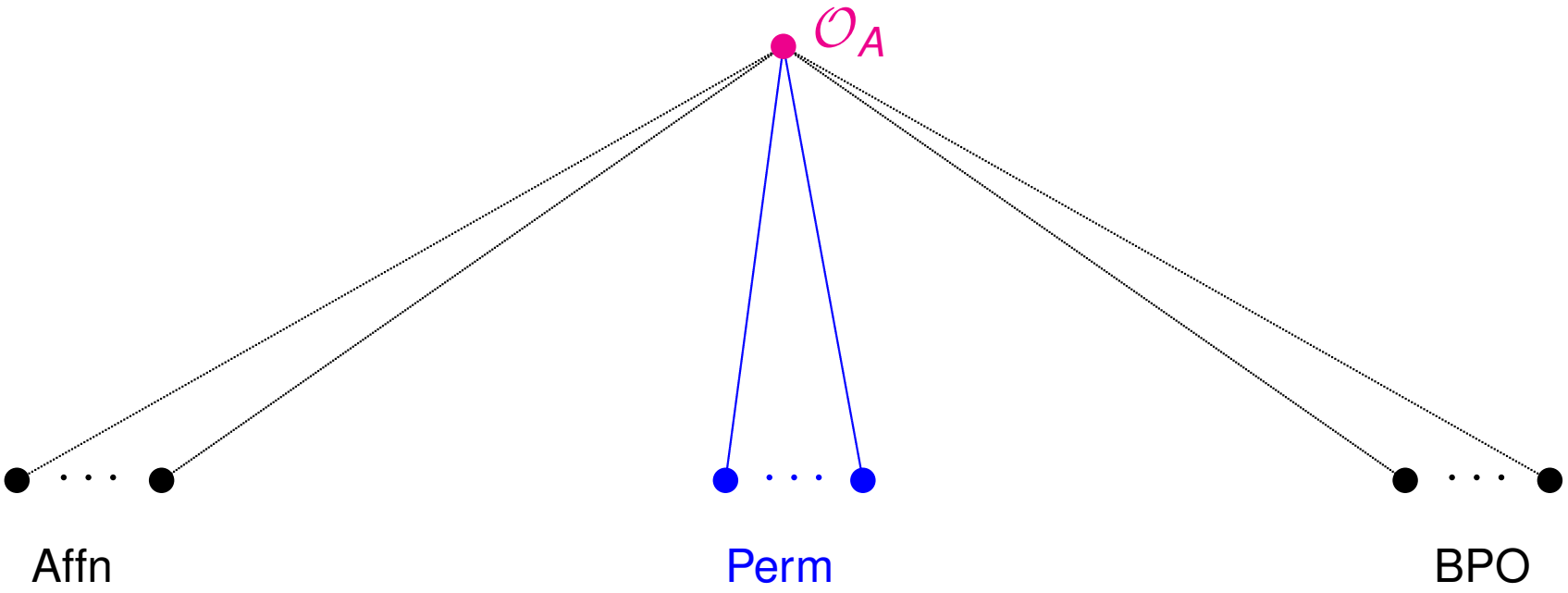

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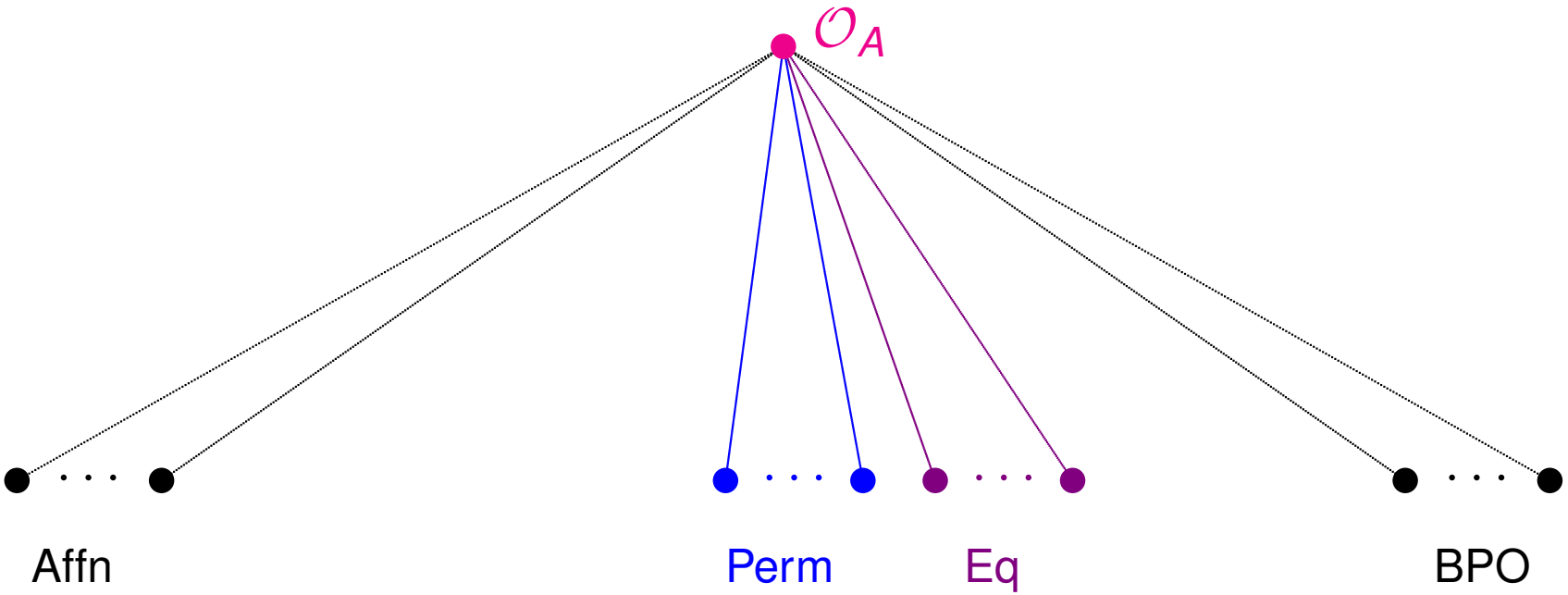

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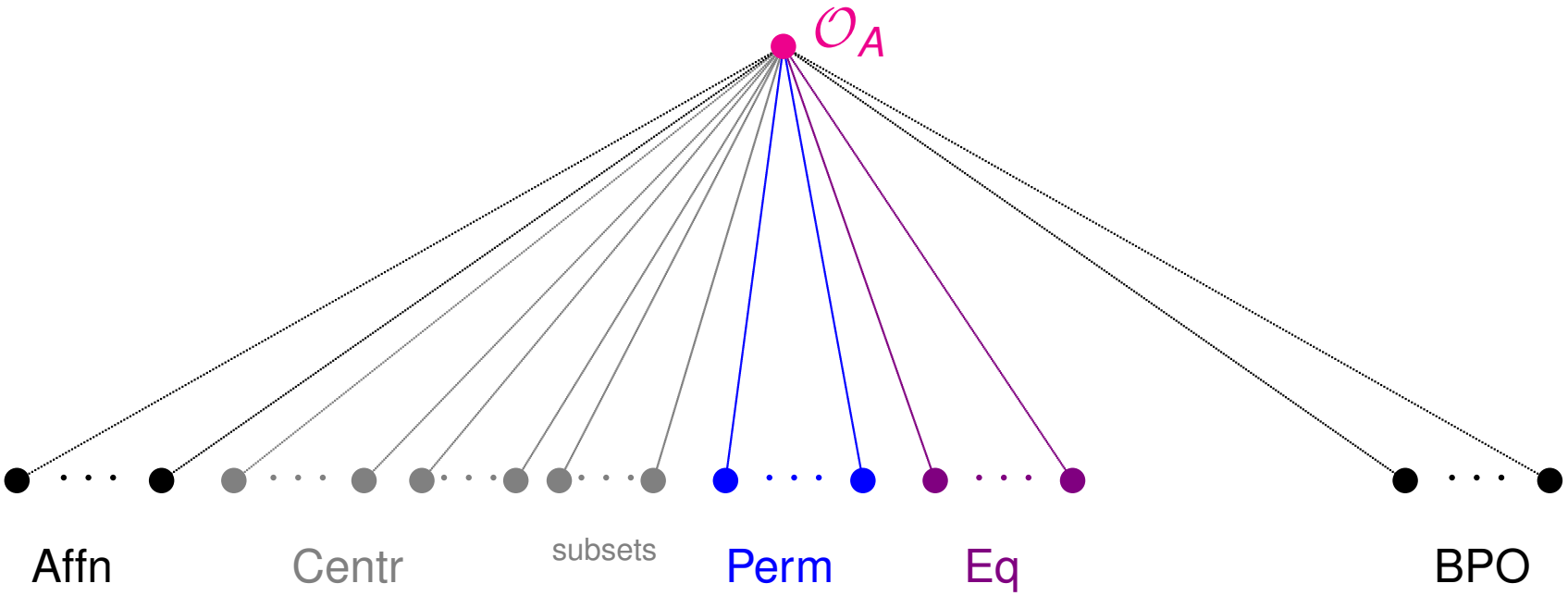

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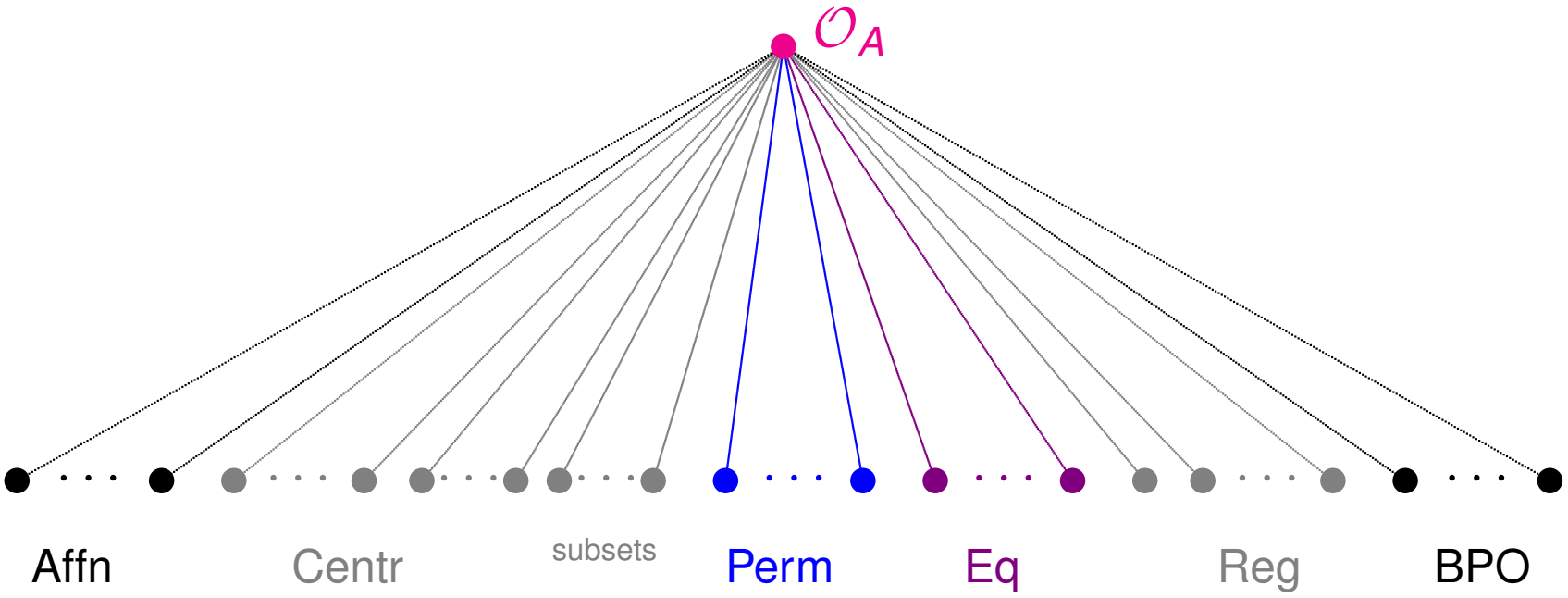

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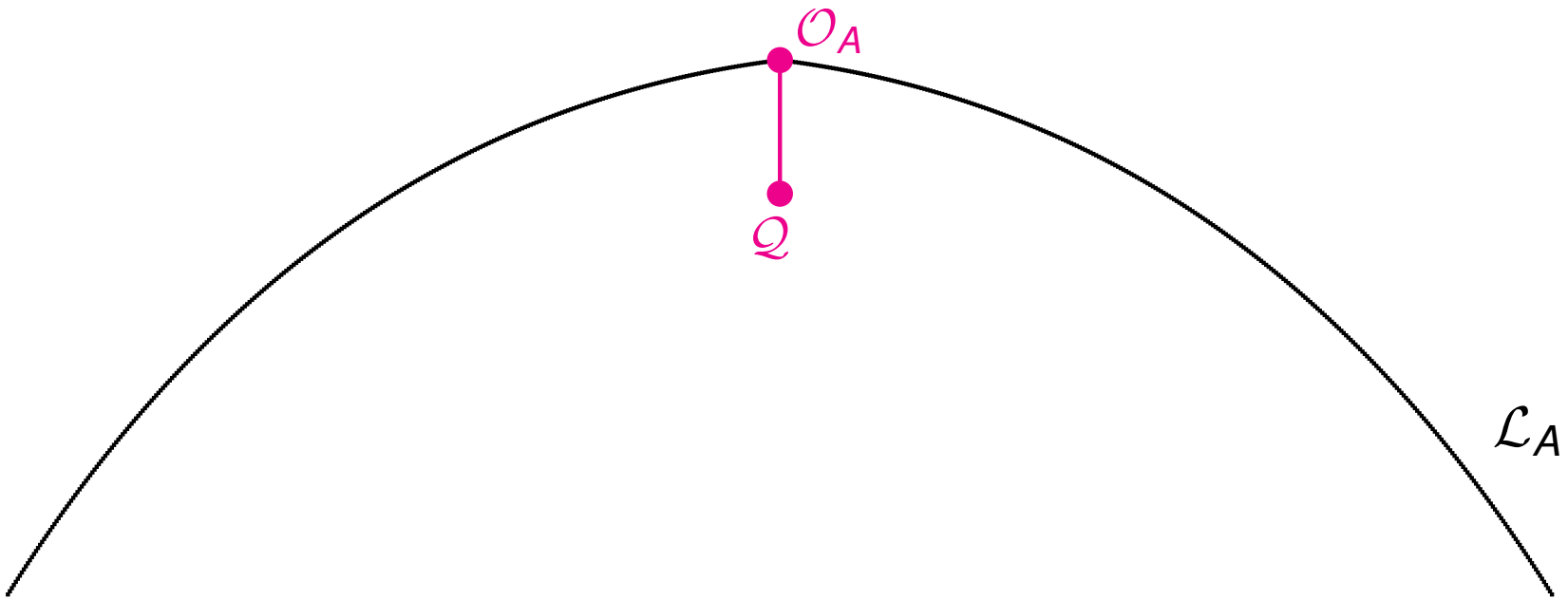

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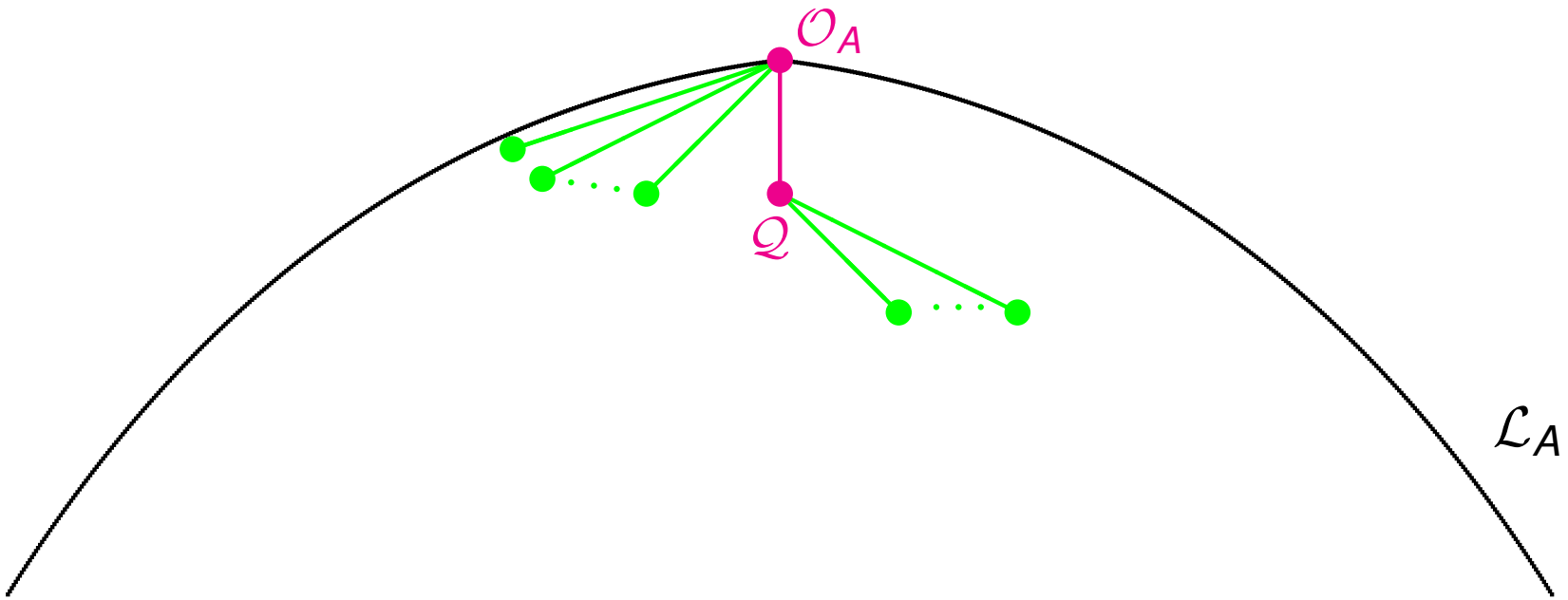

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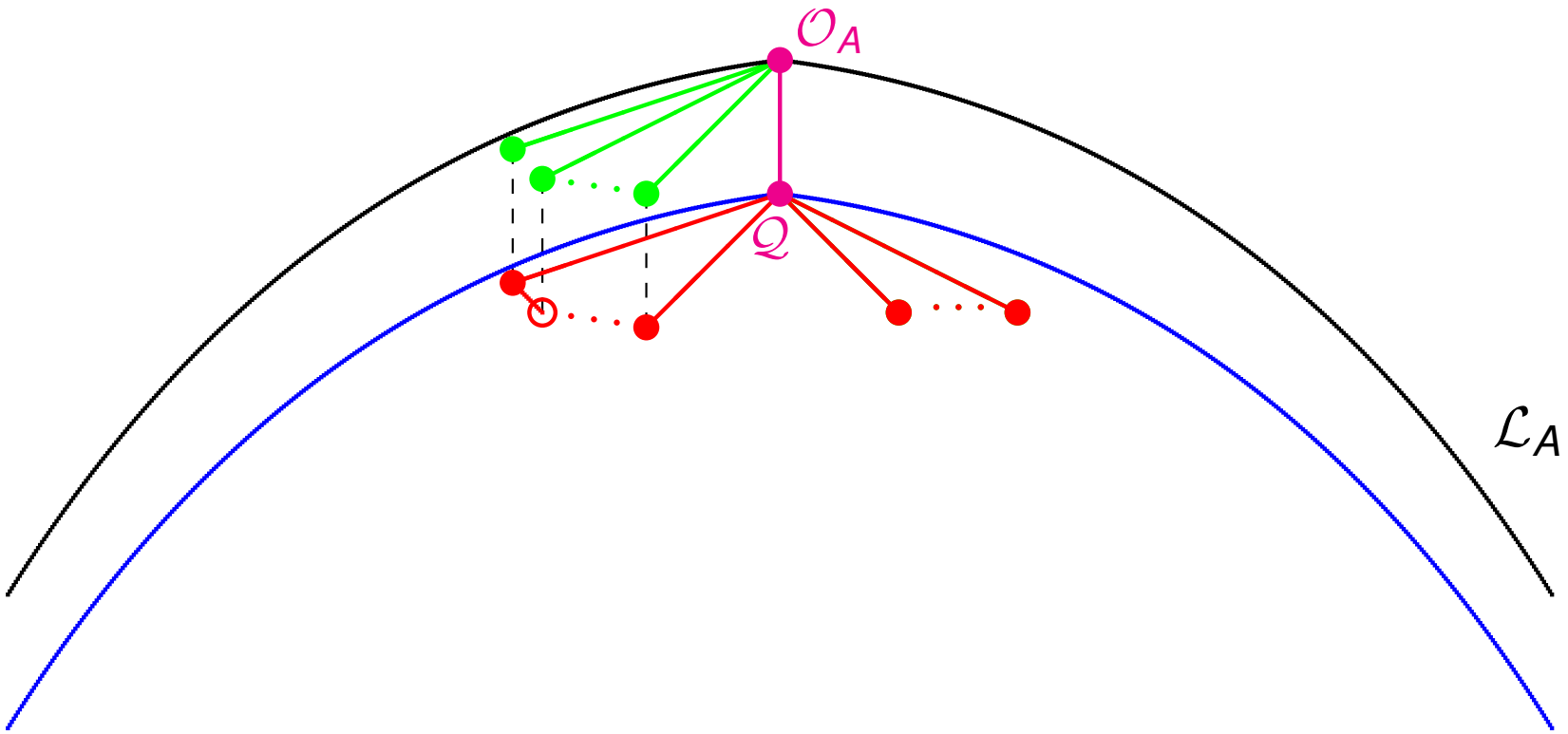

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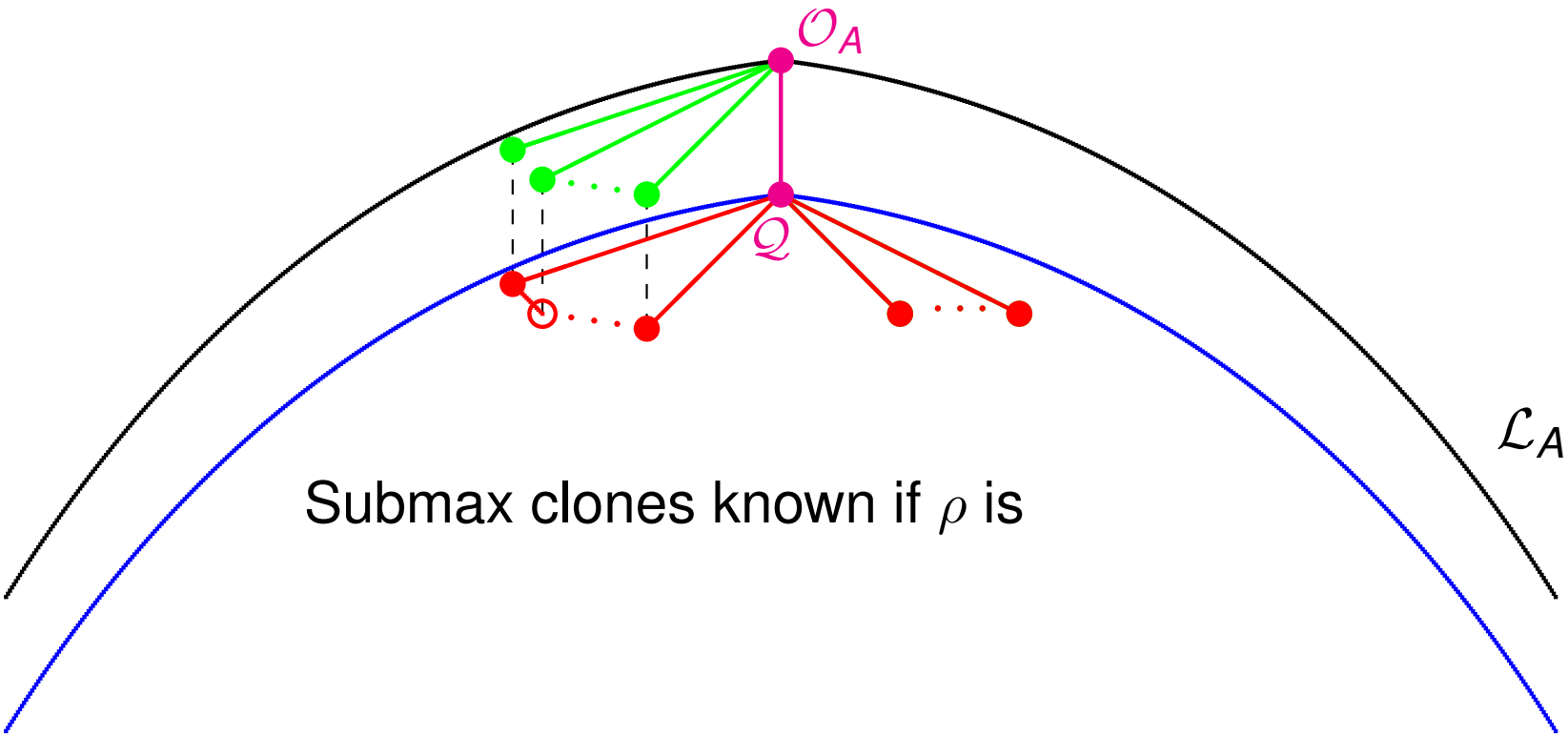

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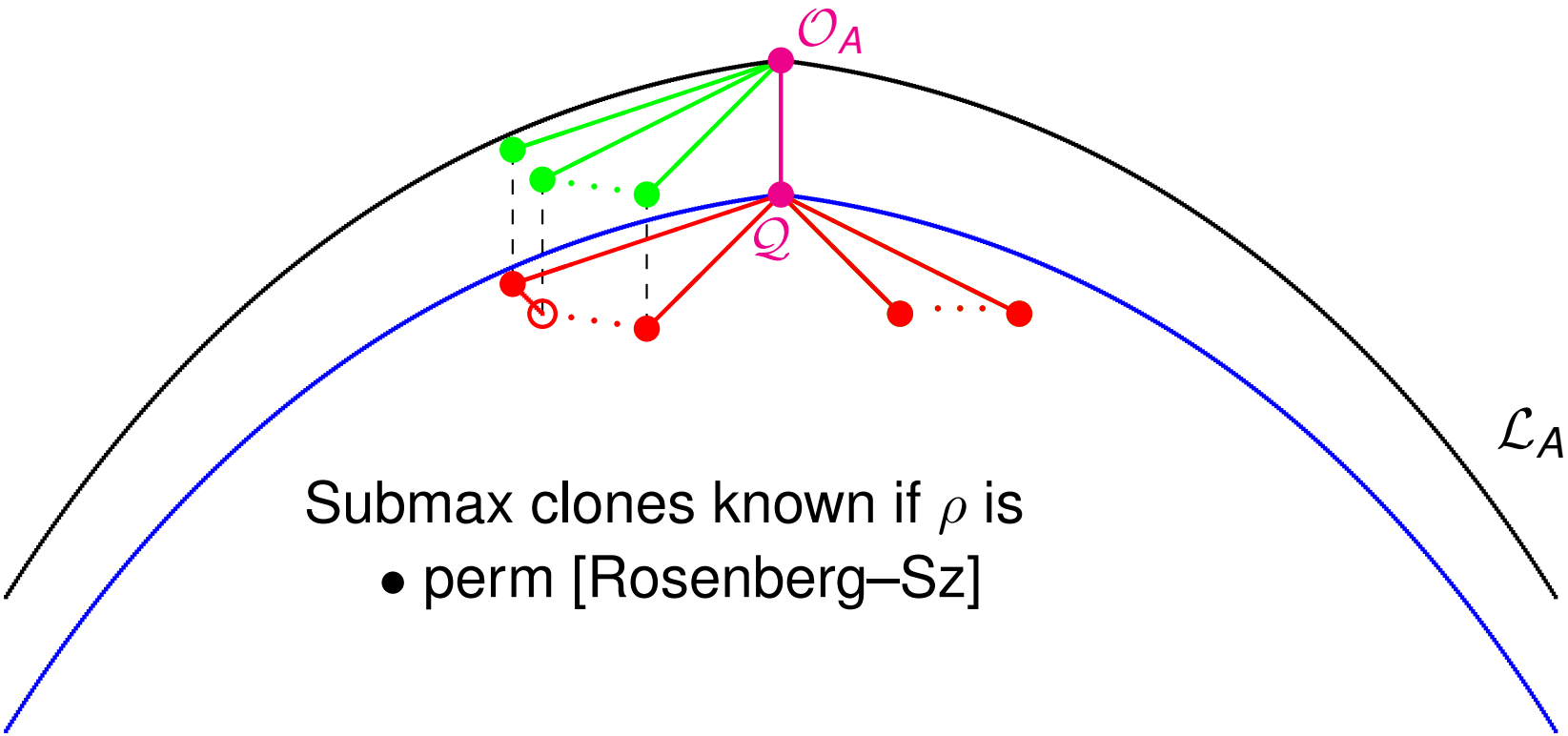

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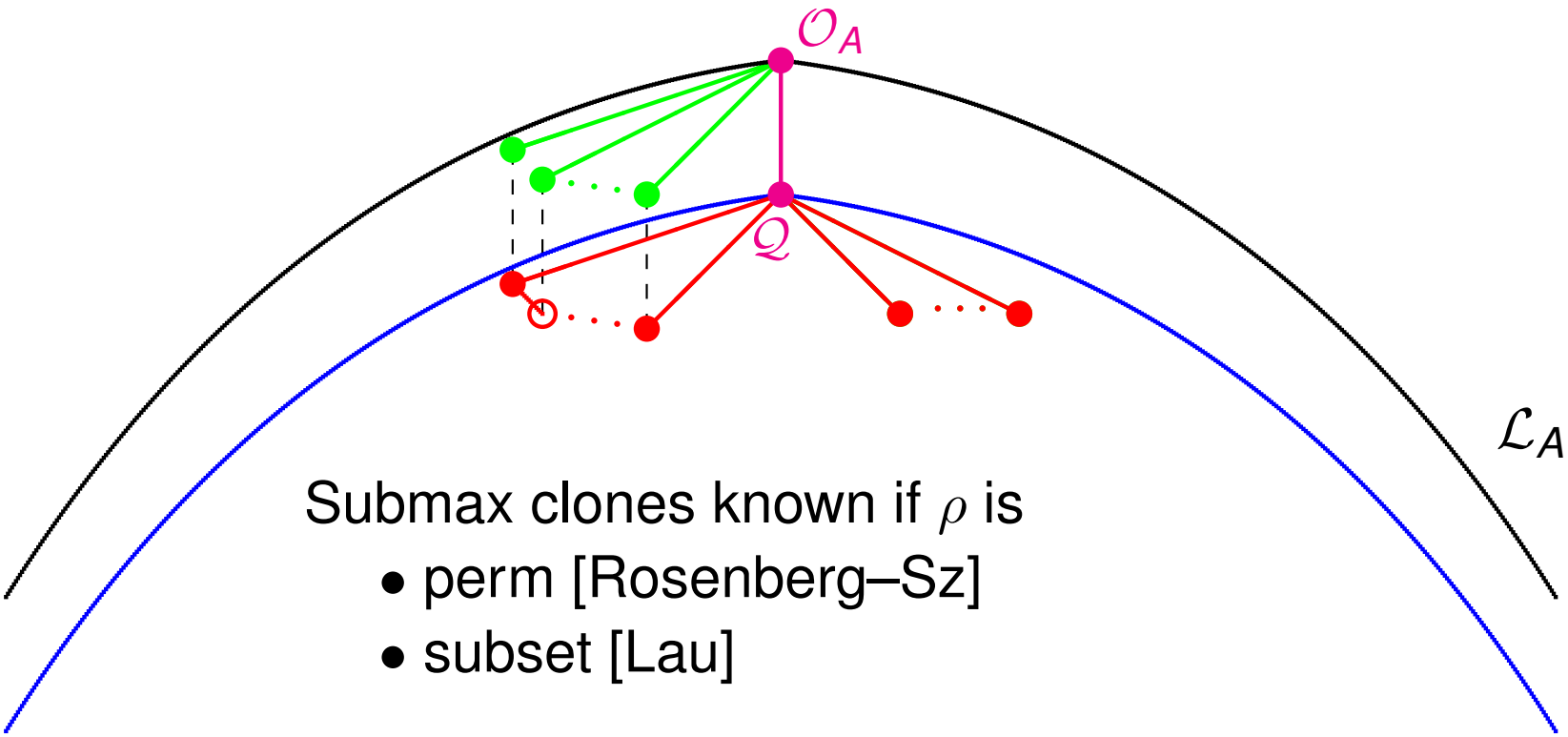

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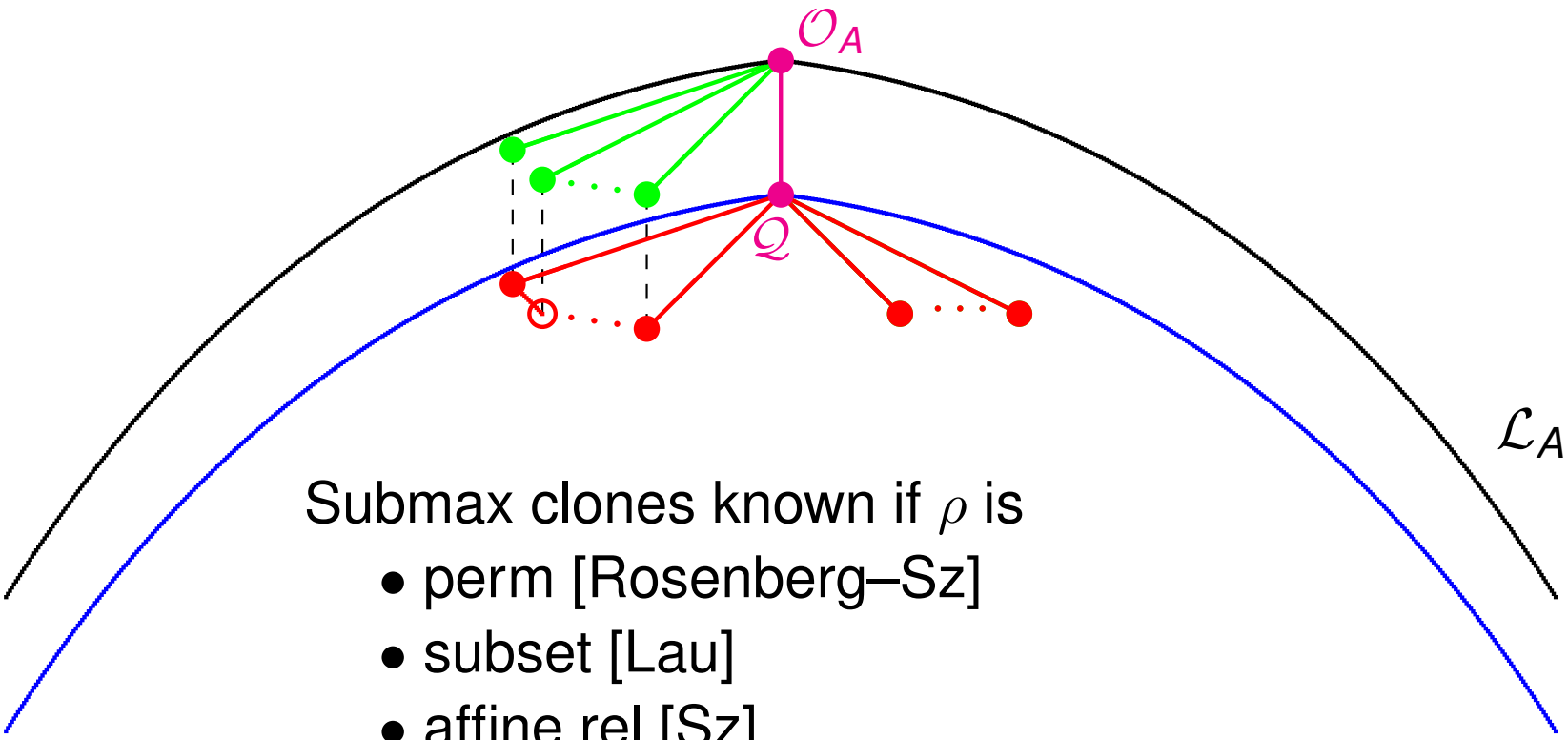

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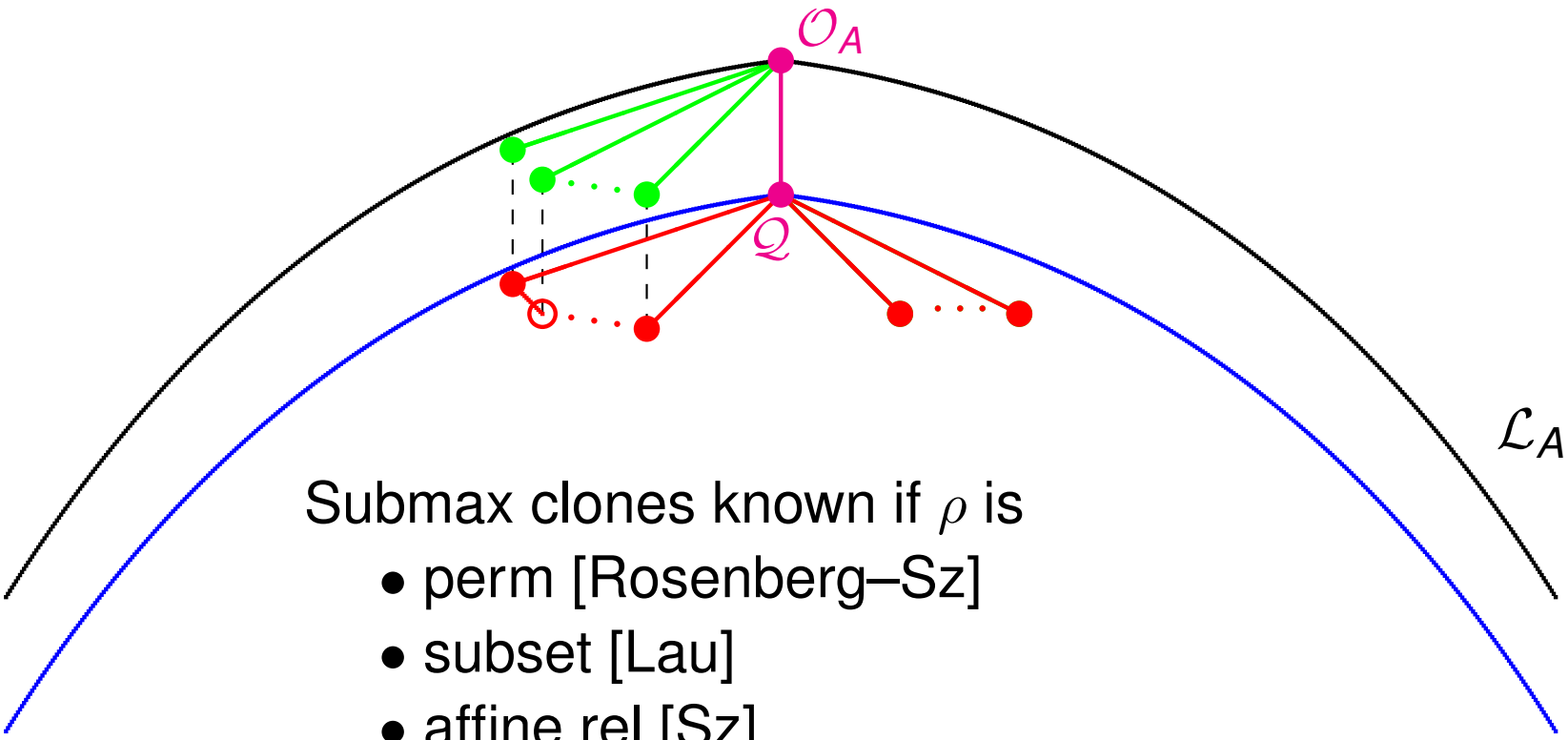

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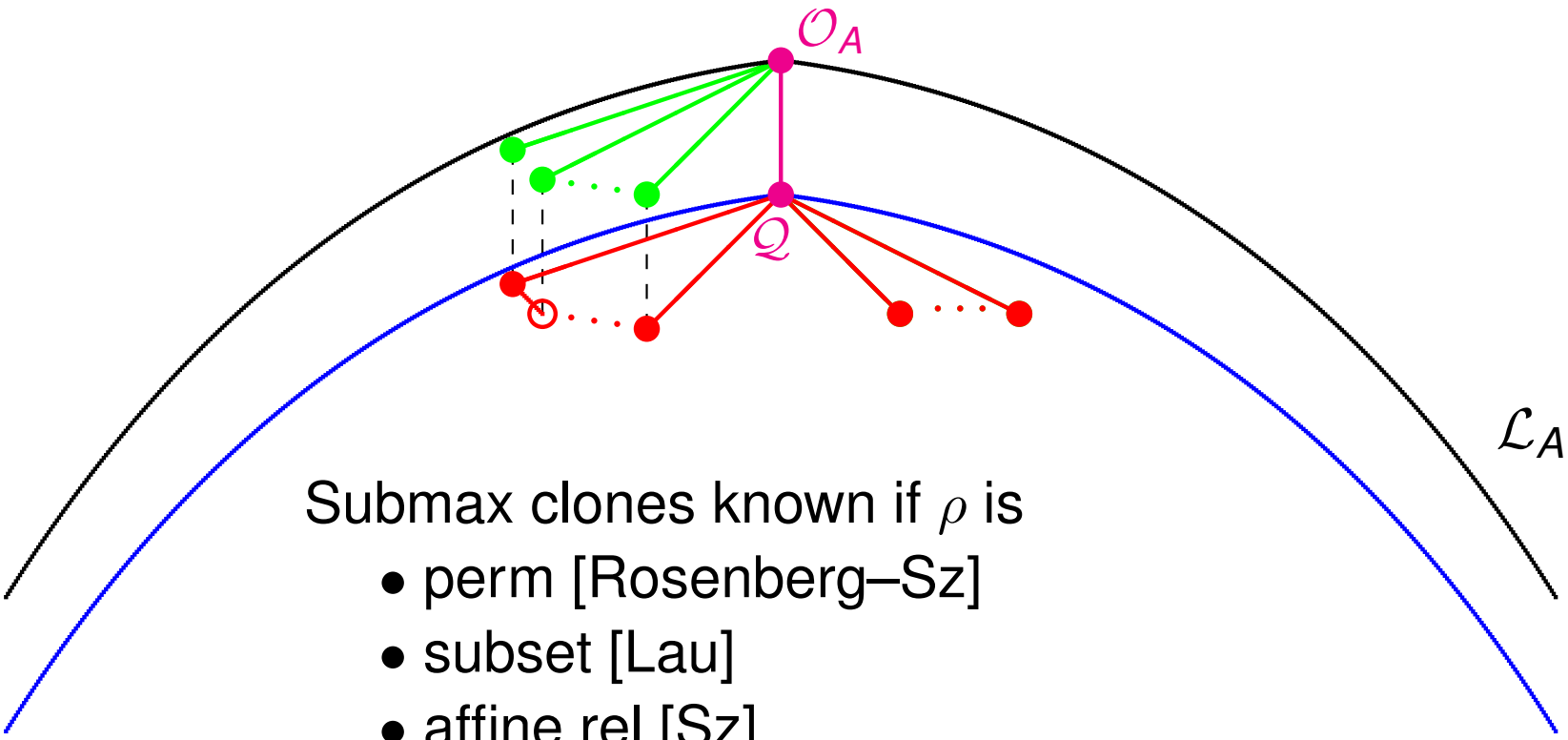

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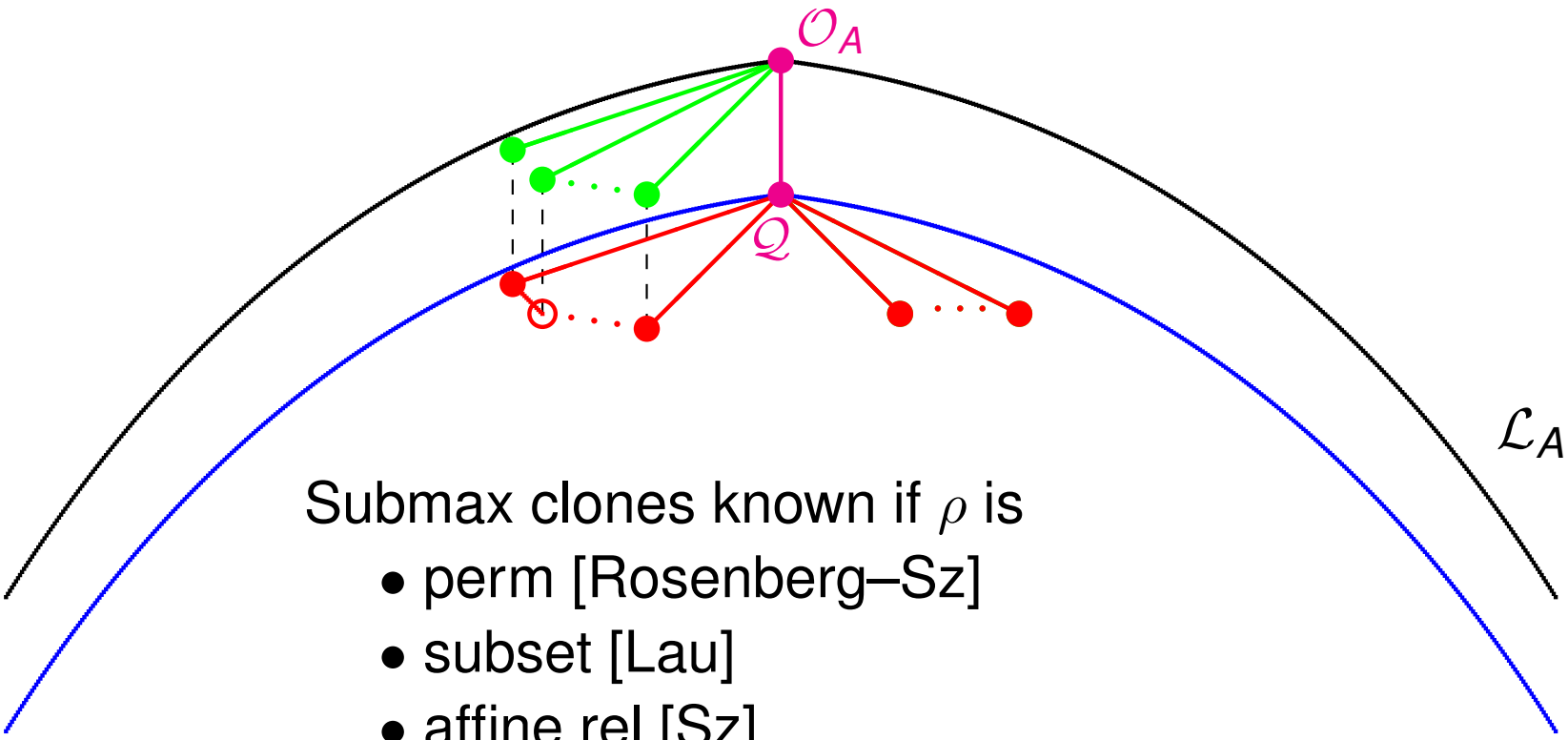

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or if $|A| = k = 3$ [Lau].

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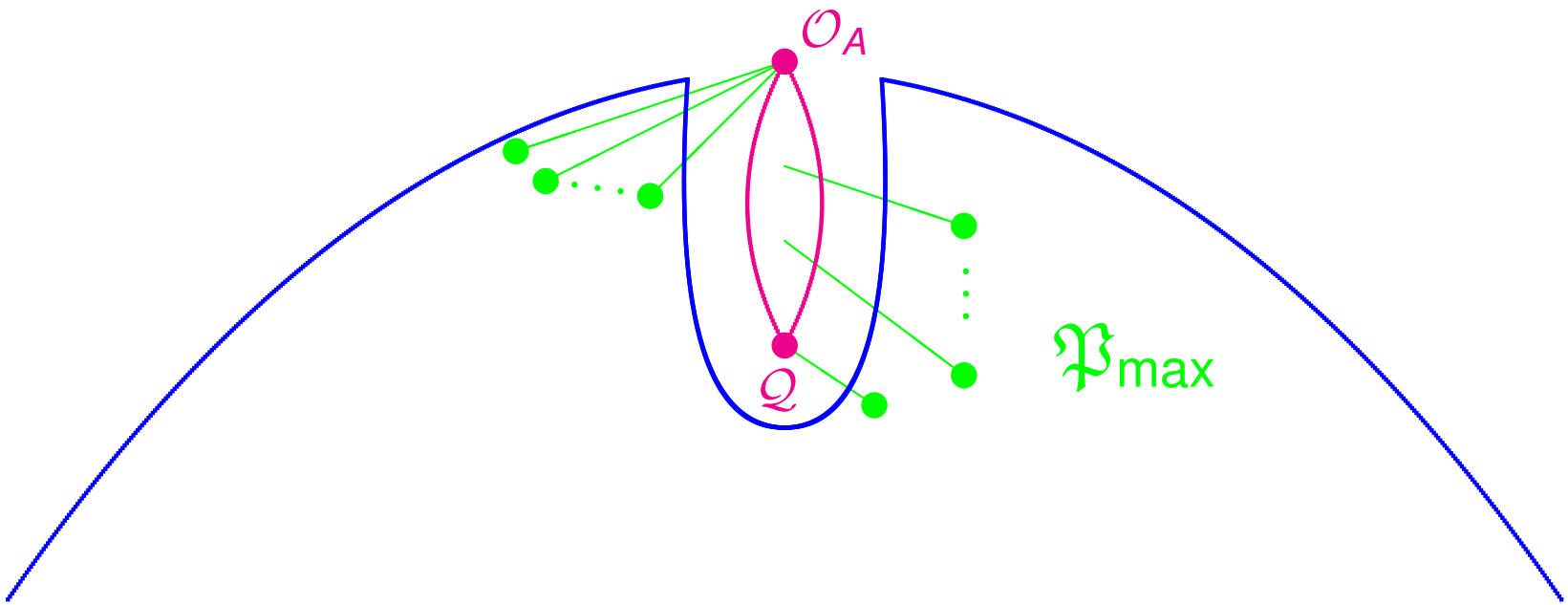

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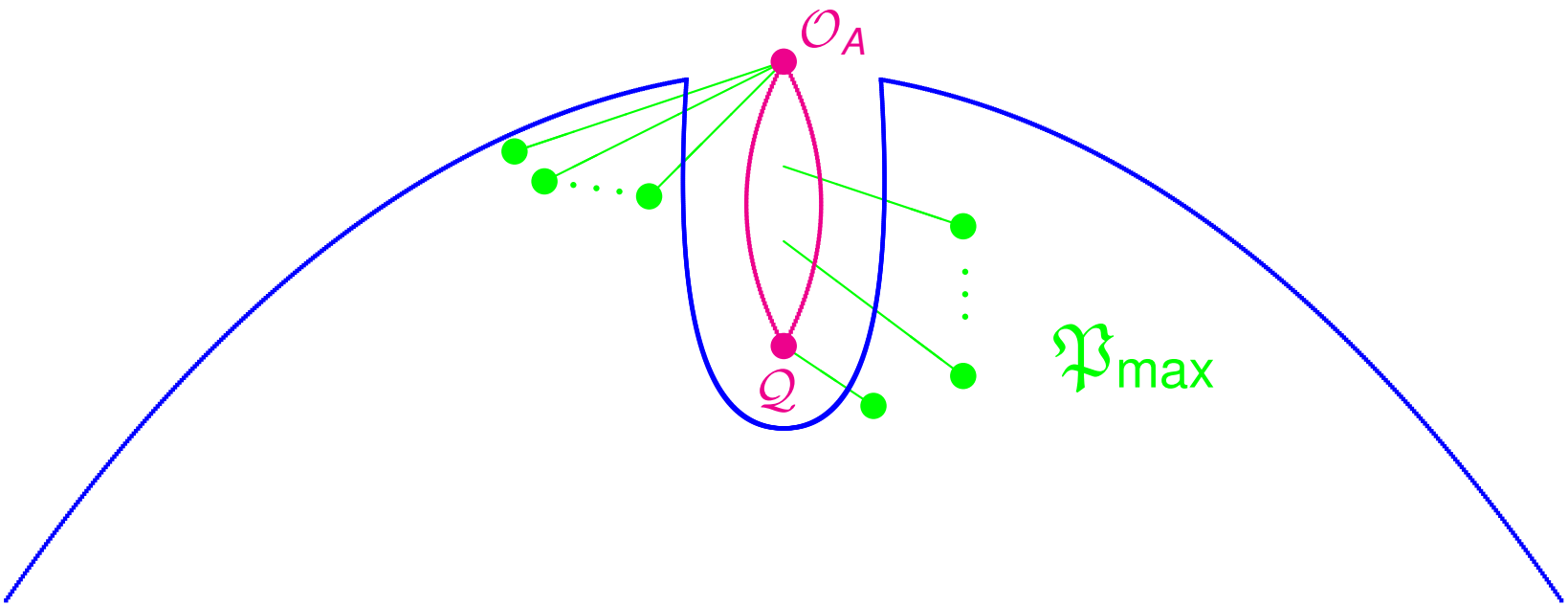

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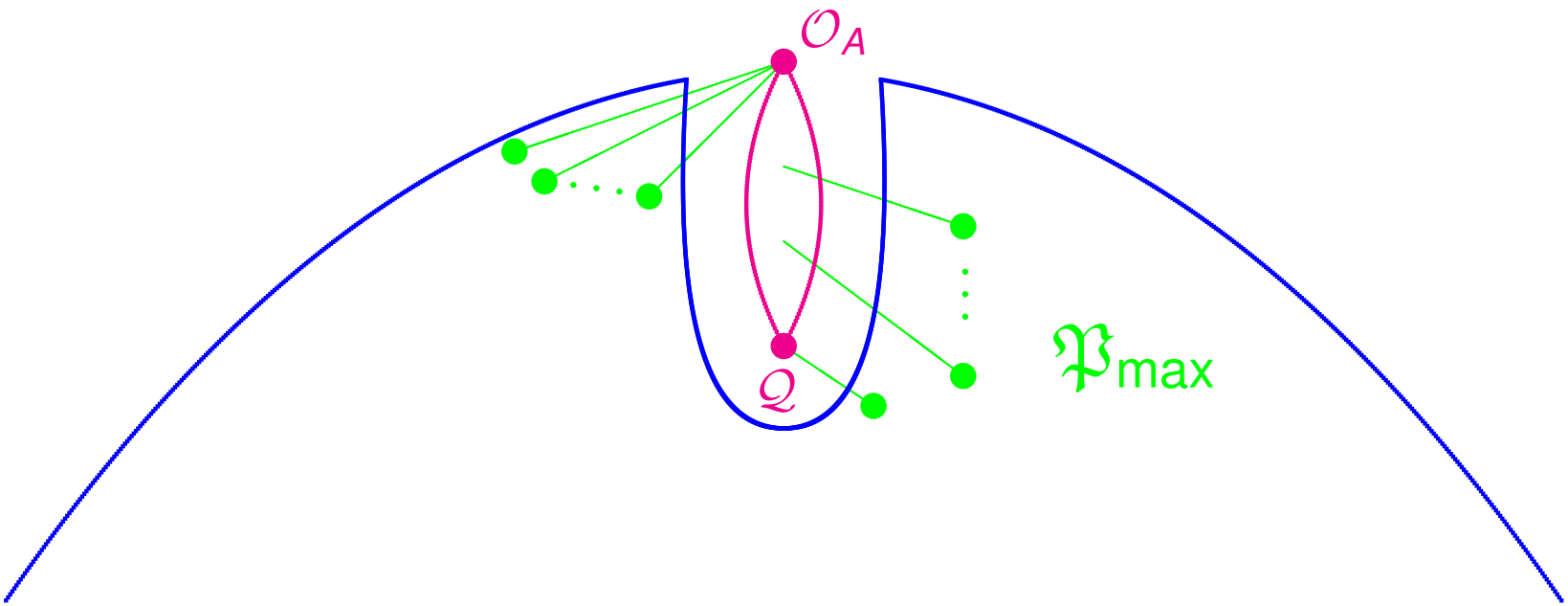

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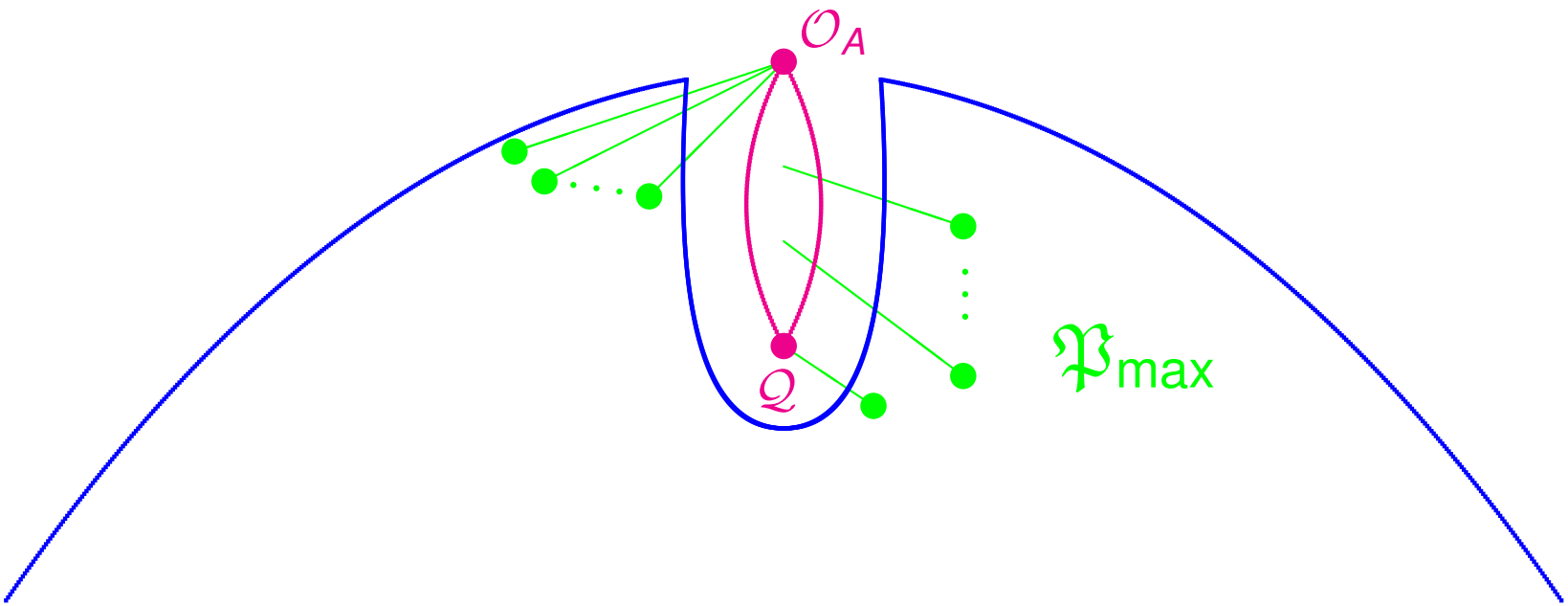

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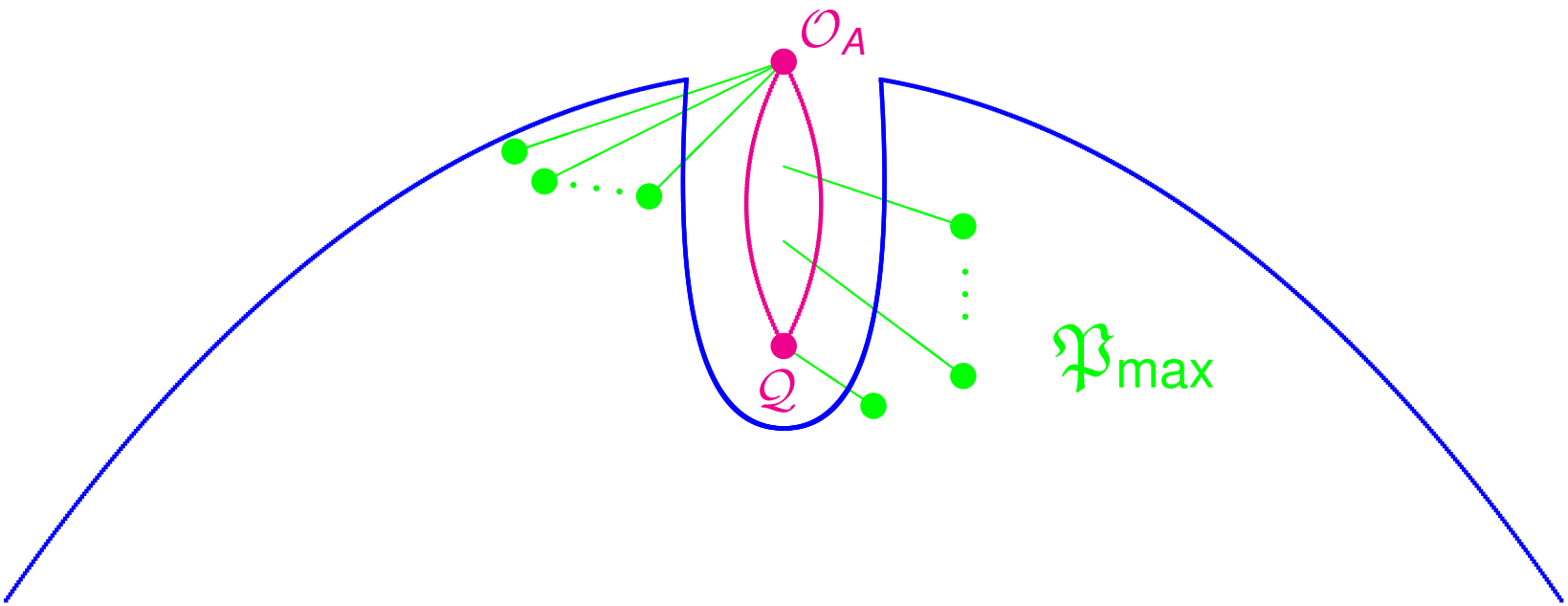

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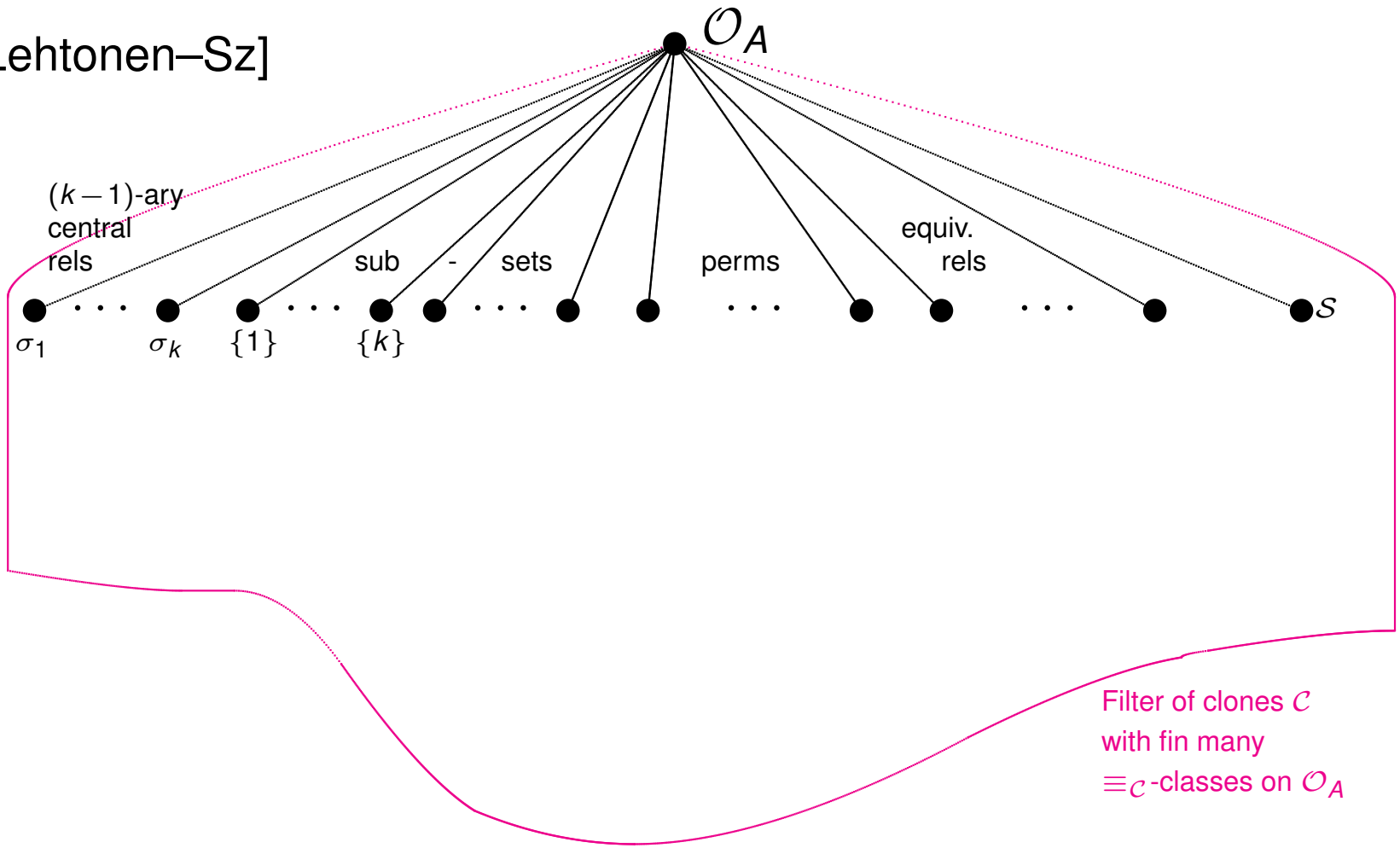

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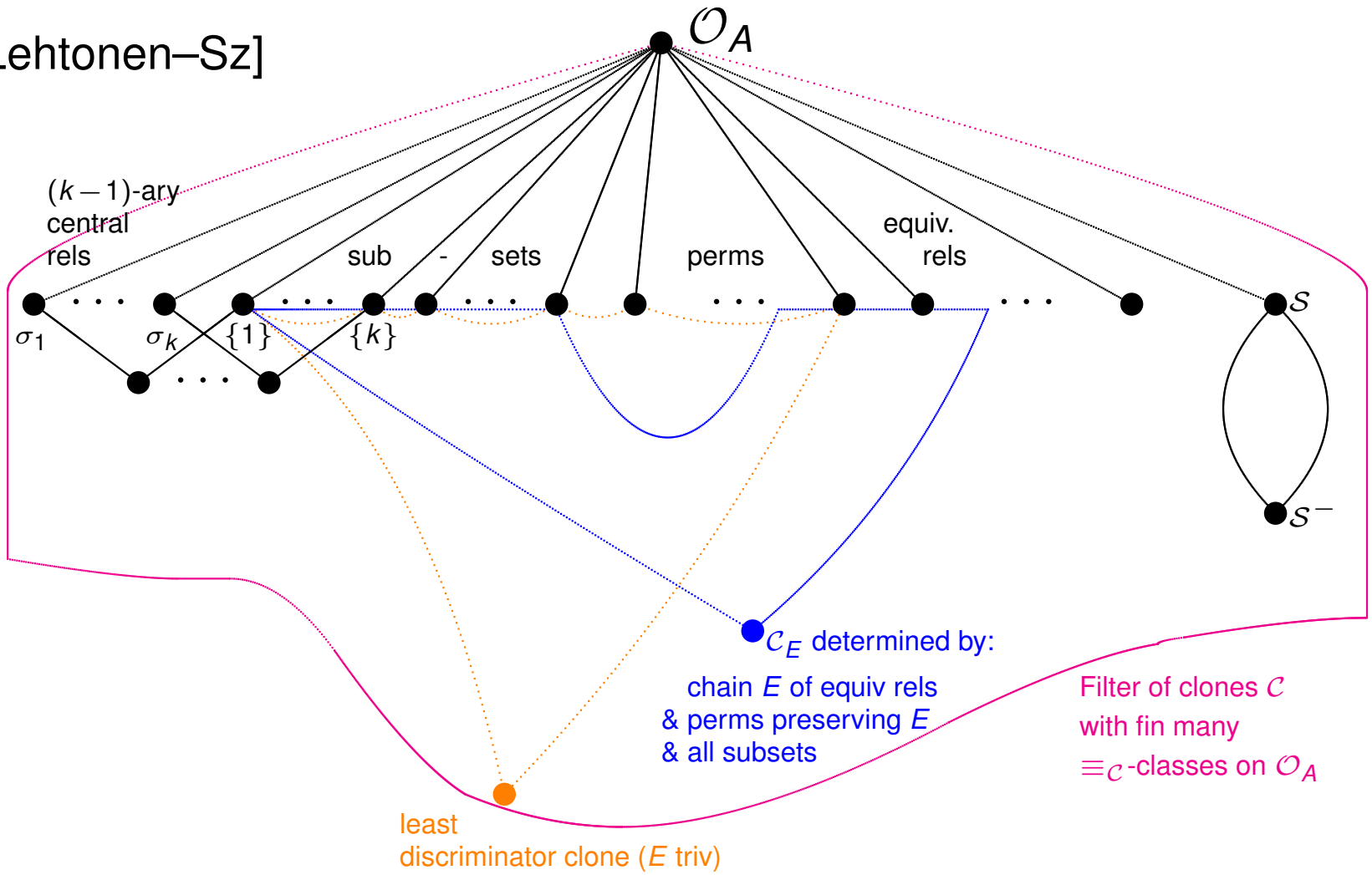

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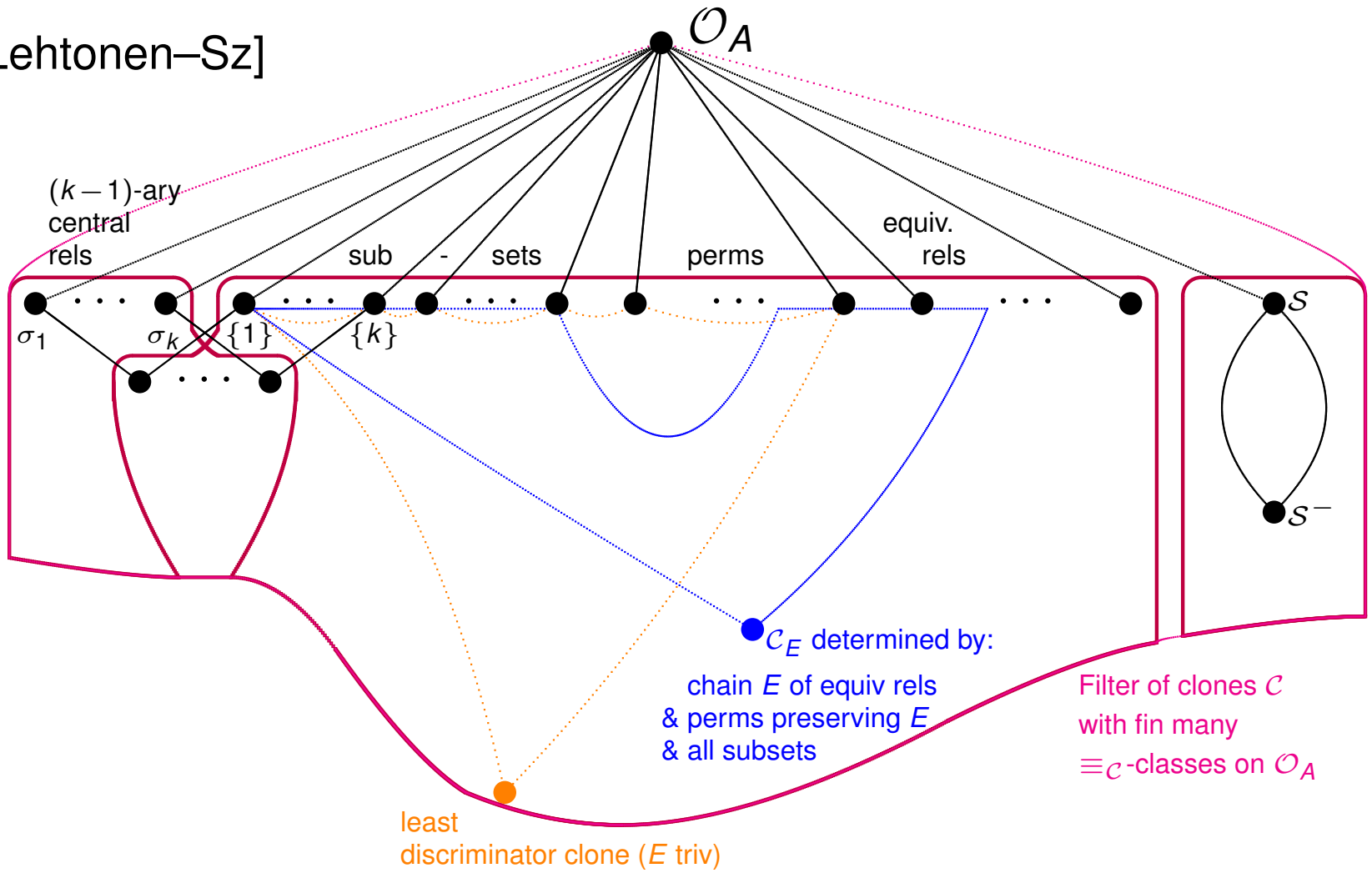

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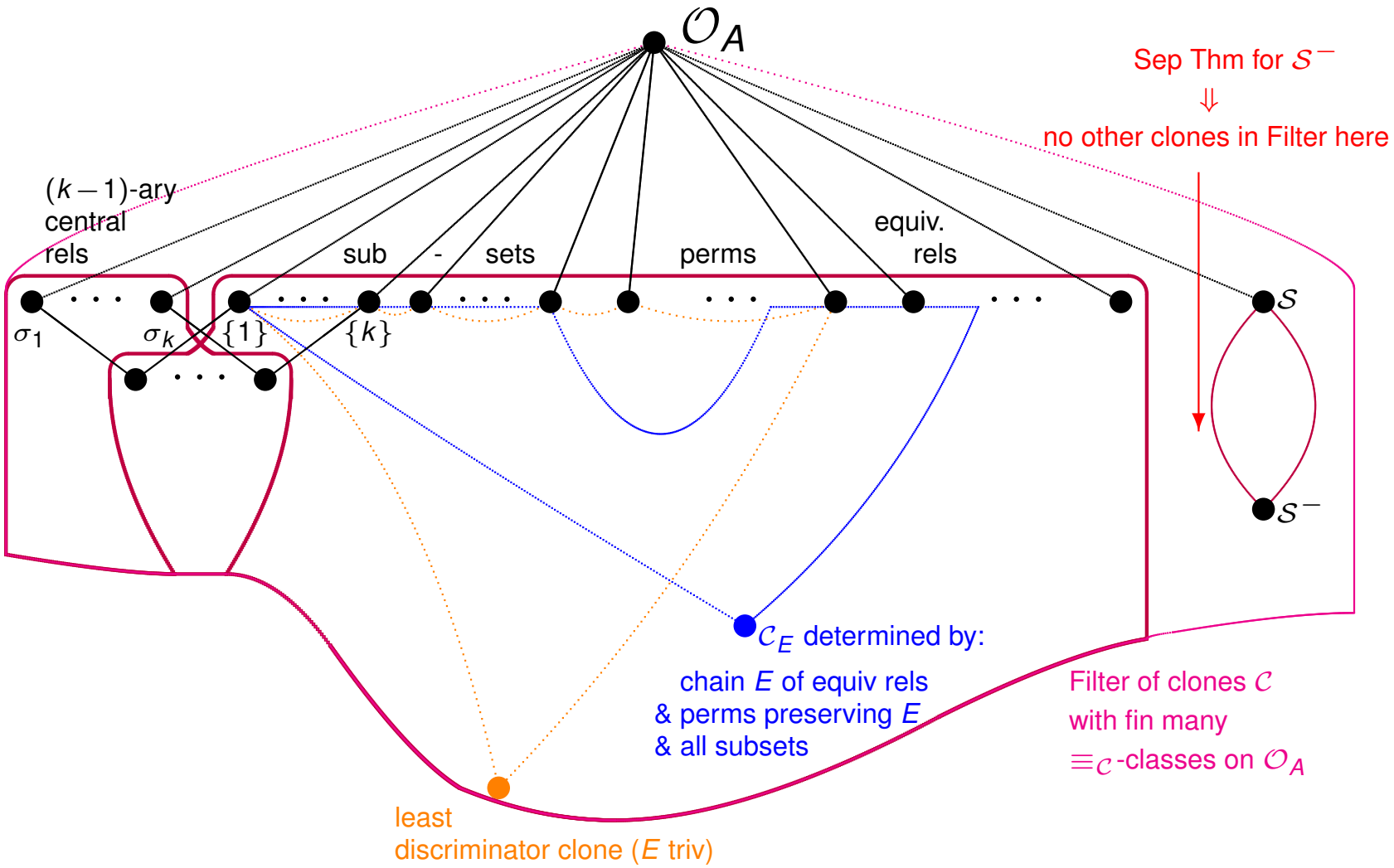

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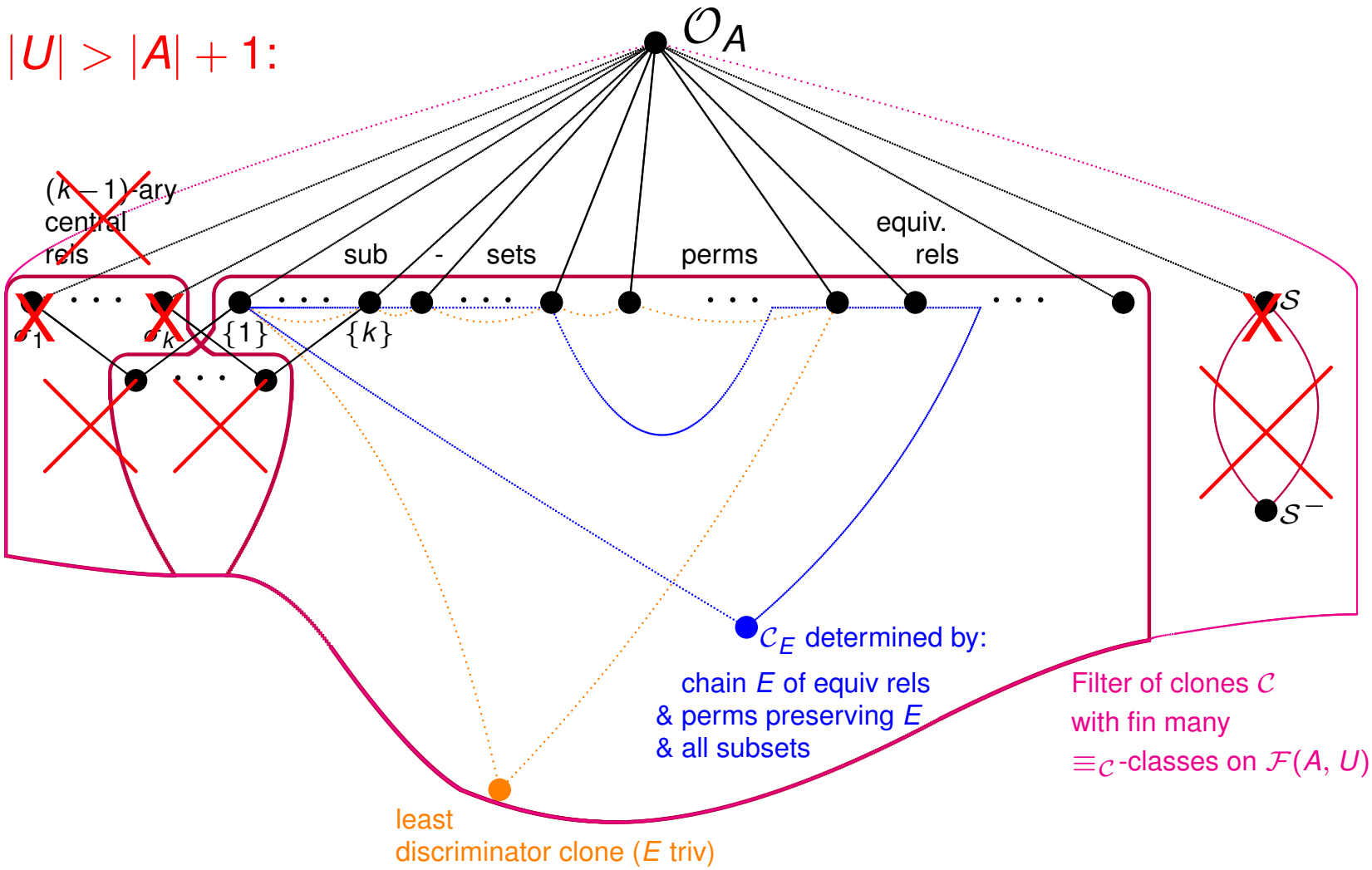

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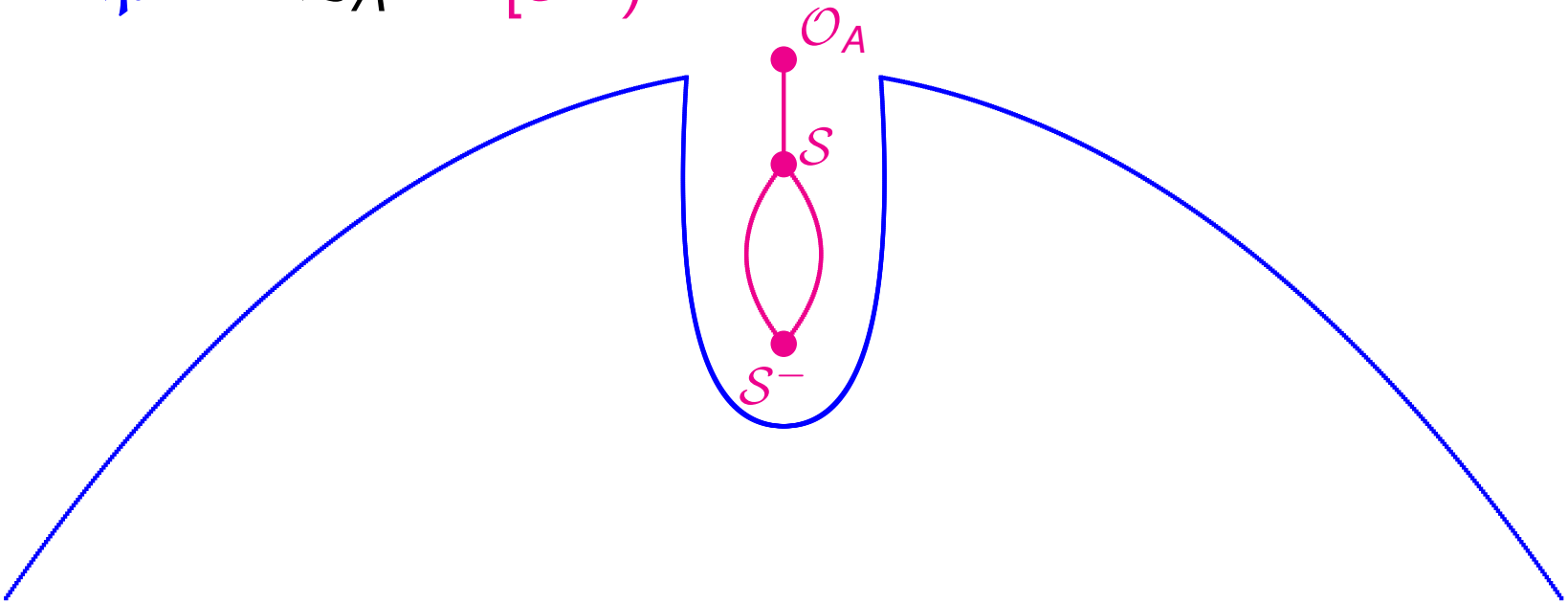

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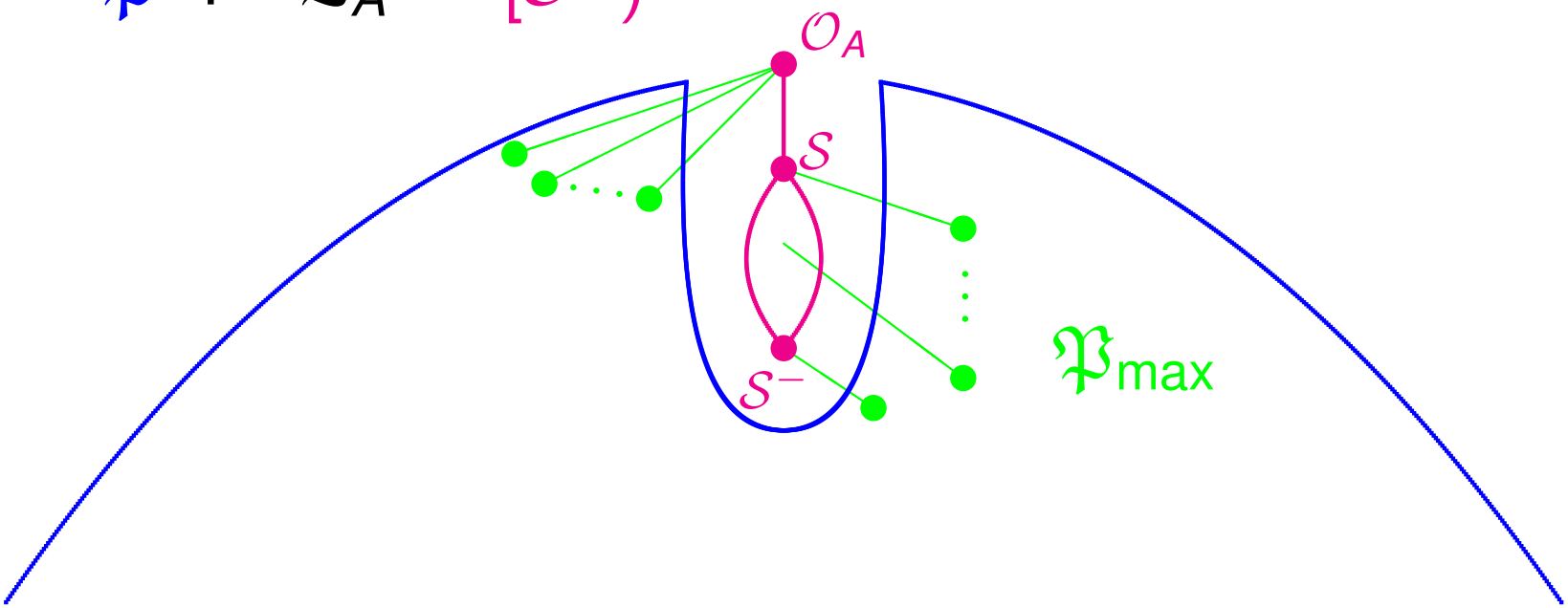

Subclones of $\mathcal{S}$: Some Facts

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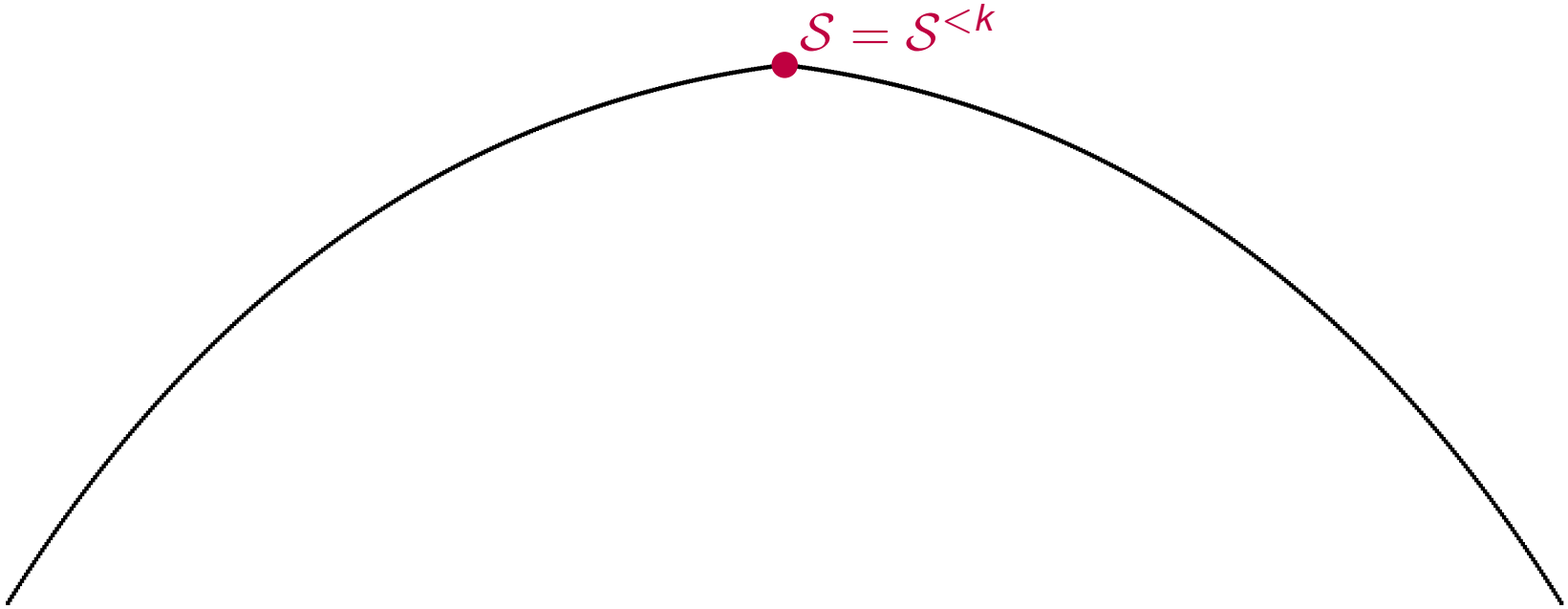

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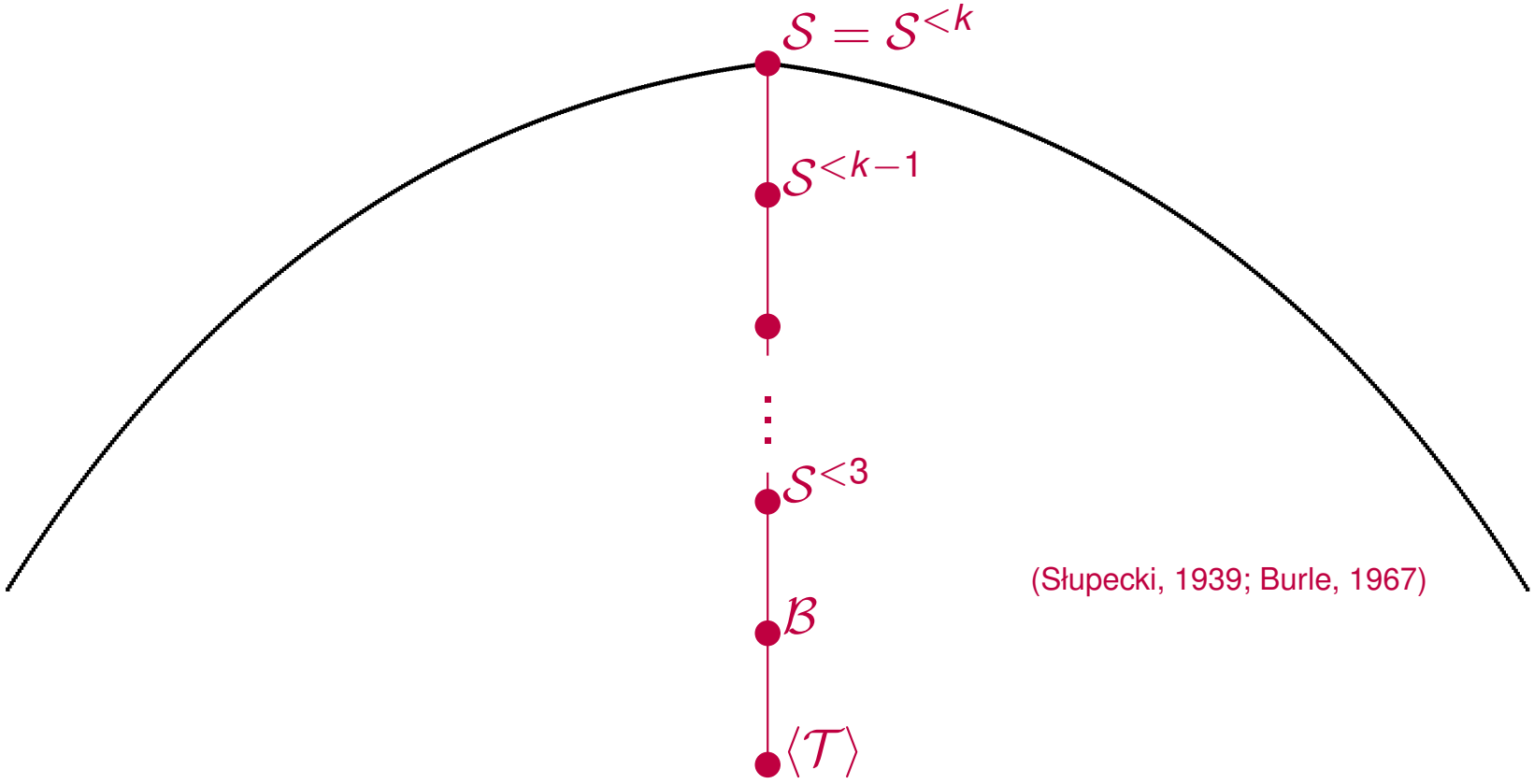

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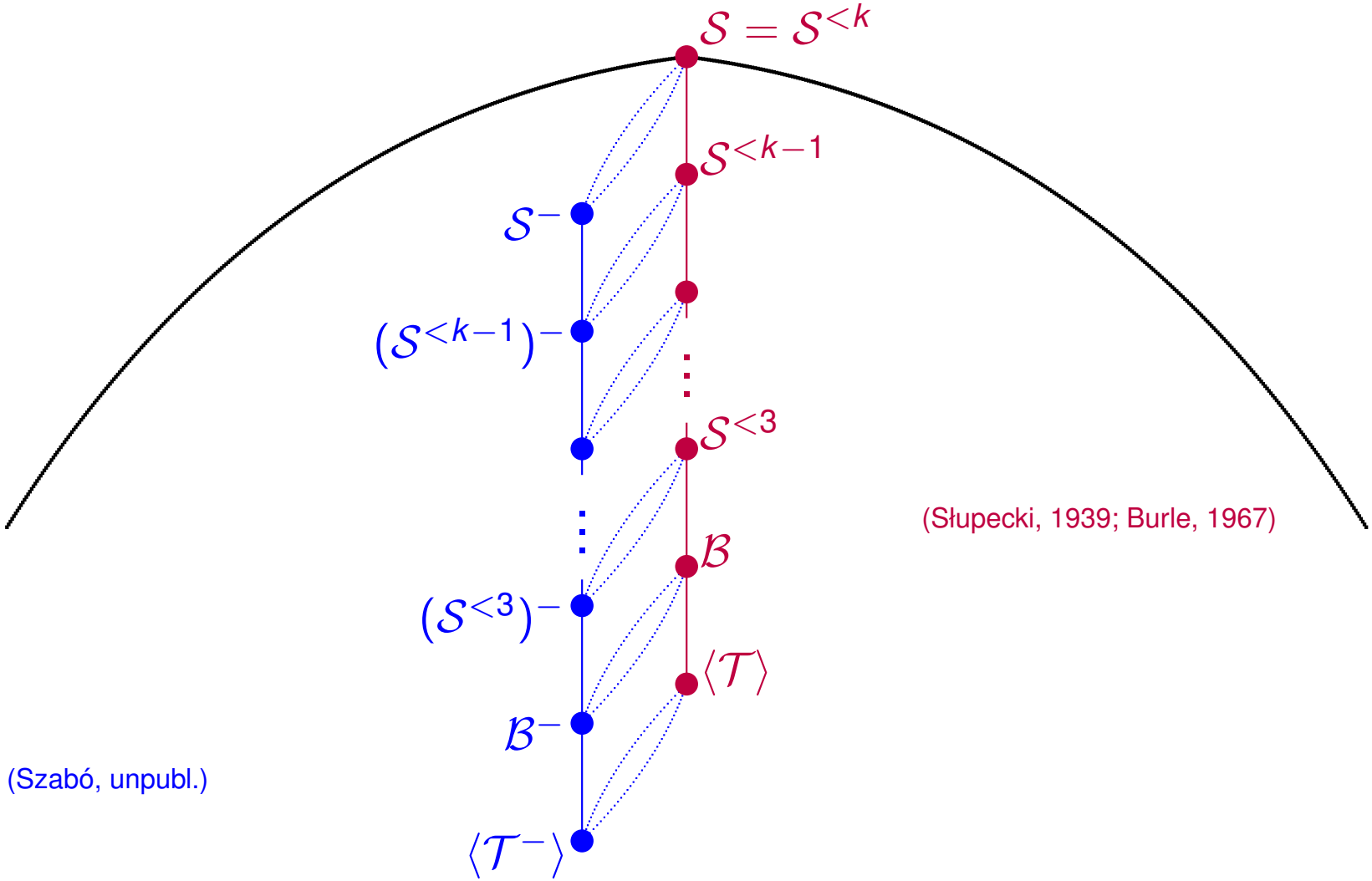

(Słupecki, 1939; Burle, 1967)

(Szabó, unpubl.)

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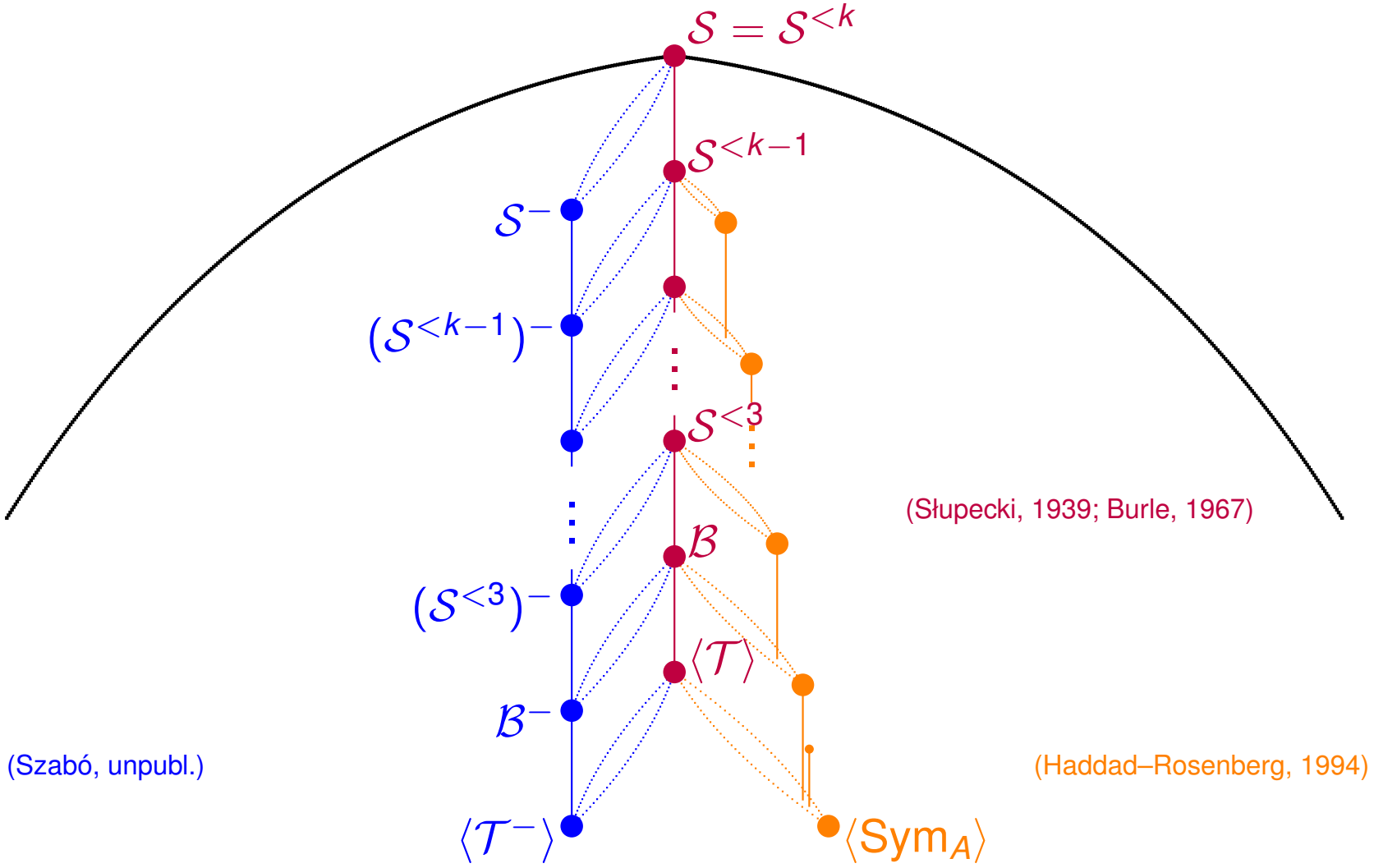

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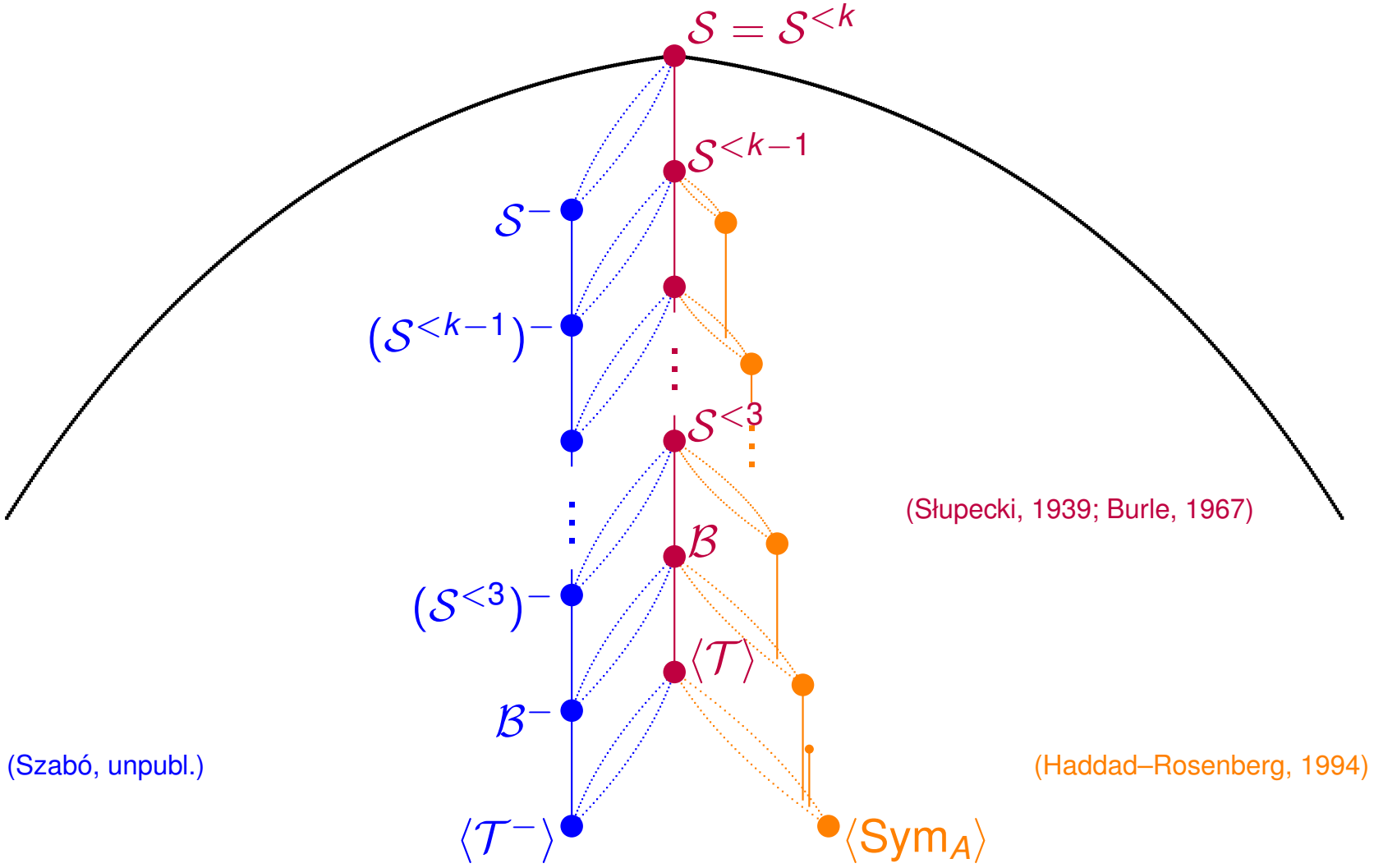

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- $\mathcal{S}^{<k-1}$ *if $k \geq 4$, and $\mathcal{B}$ if $k = 3$.*

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For a set E of equiv rels on A, with exactly m blocks each, let

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The Maximal Subclones of $\mathcal{U}$, $\mathcal{S}^- \subseteq \mathcal{U} \subseteq \mathcal{S}$

$\mathcal{S}^- \subseteq \mathcal{U} \subseteq \mathcal{S} \quad \Rightarrow \quad \mathcal{U} = \mathcal{S}[G]$ for some subgroup $G \subseteq \mathrm{Sym}_A$

Corollary

Every maximal subclone of $\mathcal{S}[G]$ is of the form

(1) *$\mathcal{S}[H]$ for a maximal subgroup H of G, or*

(2) *$\mathcal{S} \cap \{\rho\}^\perp$ for some*

$$\rho \in \mathrm{BPO} \cup \mathrm{Eq} \cup \mathrm{Centr} \cup \mathrm{Reg}^* \cup \begin{cases} \mathrm{aCentr} \cup \mathrm{aReg} & \text{if } k \geq 4, \\ \{\beta\} & \text{if } k = 3 \end{cases}$$

such that $G \subseteq \{\rho\}^\perp$.

For some $\rho \in \mathrm{BPO}, \mathrm{Centr}, \mathrm{Reg}^*$ in (2), $\mathcal{S} \cap \{\rho\}^\perp$ is not max in $\mathcal{S}[G]$.