The model companion of the class of pseudo-complemented semilattices is finitely axiomatizable

(joint work with Regula Rupp and Jürg Schmid)

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Contents

Contents

Contents

Contents

Contents

Contents

Contents

Basic notions

Algebraic and existential closedness

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Definition: (i) An algebra $A \in K$ is called *algebraically closed* in K - for short: ac - iff for every finite set $\phi_1, \ldots, \phi_n$ of positive $\exists_1$-sentences from $\mathcal{L}_{\mathsf{K},A}$ the following holds: Whenever $\bigwedge_1^n \phi_i$ is satisfied in some extension $A \subseteq A' \in \mathsf{K}$, then it is satisfied already in A.

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(ii) $A \in K$ is called *existentially closed* - for short: ec - iff (the same as sub (i), with "positive" removed).

Basic notions

Algebraic and existential closedness for the working mathematician

A is ac in K iff every finite set of equations with parameters from A which is solvable in some extension $A \subseteq A' \in$ K already has a solution in A, and it is ec in K iff the same holds for every such finite set of equations *and* negated equations.

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Explicitly, we are dealing with finite sets of formulae of type $s(x_1, \ldots, x_m, a_1, \ldots, a_n) = t(x_1, \ldots, x_m, a_1, \ldots, a_n)$ resp. $s(x_1, \ldots, x_m, a_1, \ldots, a_n) \neq t(x_1, \ldots, x_m, a_1, \ldots, a_n)$, where s and t are $\mathcal{L}$-terms and $a_1, \ldots, a_n \in A$.

Basic notions

Algebraic versus existential closedness

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The two notions can coincide as in the class of fields:

- If K is a field and $p(\overrightarrow{x})$ and $q(\overrightarrow{x})$ are polynomials over K, then the satisifiability of the inequality $p(\overrightarrow{x}) \neq q(\overrightarrow{x})$ is equivalent to the satisfiability of the equation

$$x \cdot (p(\overrightarrow{x}) - q(\overrightarrow{x})) = 1,$$

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This is not the general situation:

- In the class of boolean algebras every boolean algebra is algebraically closed whereas boolean algebra is existentially closed if and only if it is atomfree.
- An abelian group is algebraically closed if and only if it is divisible, whereas it is existentially closed if and only if it is divisible and contains an infinite direct sum of copies of $\mathbb{Q}/\mathbb{Z}$ (as a module).

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A semantic characterization of finite axiomatizability of a class of $\mathcal{L}$-structures:

Theoreom: An finitely axiomatizable class of $\mathcal{L}$-structures is *finitely axiomatizable* iff both the class itself as well as its complementary class are closed under elementary equivalence and ultraproducts.

Basic notions

The classes K^{ac} and K^{ec}

- If BA is the class of boolean algebras, then adding the sentence $(\forall x)(\exists y)(0 < x \to 0 < y < x)$ to the axioms of boolean algebras yields a finite axiomatization of BA^{ec}.

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- If GA is the class of abelian groups, then adding the infinite set of sentence $\{(\forall x)(\exists y)(x = ny) \mid n \in \mathbb{N}\}$ to the axioms of abelian groups yields an infinite axiomatization of GA^{ac}. There is no finite axiomatization of GA^{ac}

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- Let $\mathcal{L}$ a be the first-order language without relation, function and constant symbols and K be the class of all $\mathcal{L}$-models (of the empty theory). Then K is simply the class of sets. We obtain $K^{ac} = K$ and $K^{ec} = \{x \in K \mid x \text{ is infinite}\}$. Thus K^{ac} is finitely axiomatizable and K^{ec} is not.

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The class PCS of pseudo-complemented semilattices satisfies these assumptions.

Basic notions

Pseudo-complemented semilattices: definition

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Definition: A pseudo-complemented semilattice (for short: p-semilattice) $(P;\ \wedge,^*, 0, 1)$ is a meet-semilattice $(P;\ \wedge)$ with least element 0 and top element 1, equipped with an unary operation $a \mapsto a^*$ such that for all $x \in P$, $x \wedge a = 0$ iff $x \leq a^*$. With PCS we denote the class of all p-semilattices.

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PCS is a variety.

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Theorem (Frink 1962)
PCS is a variety.

Definition: A distributive pseudo-complemented lattice (for short: p-algebra) $(L;\ \vee,\wedge,^*,0,1)$ is a distributive lattice $(L;\ \vee,\wedge)$ with least element 0 and top element 1, equipped with an unary operation $a \mapsto a^*$ such that for all $x \in L$, $x \wedge a = 0$ iff $x \leq a^*$. Let PALG be the class of all p-algebras.

Basic notions

Pseudo-complemented semilattices: examples

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- Every boolean algebra $\mathbf{B}$ considered as a $(B; \wedge, ^*, 0, 1)$-reduct is a pcs. $\mathbf{2}$, $\mathbf{F}_n$ and $\mathbf{A}$ denote the two-element boolean algebra, the n-atom boolean algebra and the countable atomless boolean algebra, respectively.

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- For an arbitrary boolean algebra $\mathbf{B}$ the ordered structure $\widehat{\mathbf{B}}$ obtained from $\mathbf{B}$ by adding a new top element is a pcs. $\mathbf{3}$ denotes three element chain which is obtained from $\mathbf{2}$ by adding a new top element.

Basic notions

Pseudo-complemented semilattices: special elements

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Let $x \in A \in PCS$.

- x is *dense* iff $x^* = 0$. $\mathsf{D}(A) = \{x \in A : \ x \text{ dense}\}$ $=$ the *dense filter* of A.

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- x is *dense* iff $x^* = 0$. $\mathsf{D}(A) = \{x \in A : \ x \text{ dense}\}$ $=$ the *dense filter* of A.
- x is *skeletal* iff $x^{**} = x$. $\mathsf{Sk}(A) = \{x \in A : \ x \text{ skeletal}\} =$ the *skeleton* of A.

Basic notions

Pseudo-complemented semilattices: some facts

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- If $\mathbf{P} = \langle P; \wedge, ^*, 0, 1\rangle$ is a pcs, then $\langle \mathsf{Sk}(P); \wedge, \underline{\vee}, ^*, 0, 1\rangle$ putting $a \underline{\vee} b := (a^* \wedge b^*)^*$ is a boolean algebra.

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- $\widehat{\mathbf{B}}$, where $\mathbf{B}$ is any boolean algebra, are exactly the subdirectly irreducible members of PCS.

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- If $\mathbf{P} = \langle P; \wedge, ^*, 0, 1\rangle$ is a pcs, then $\langle \mathsf{Sk}(P); \wedge, \underline{\vee}, ^*, 0, 1\rangle$ putting $a \underline{\vee} b := (a^* \wedge b^*)^*$ is a boolean algebra.
- $\widehat{\mathbf{B}}$, where $\mathbf{B}$ is any boolean algebra, are exactly the subdirectly irreducible members of PCS.
- The three-element chain $\mathbf{3}$ generates the variety PCS, that is $PCS = \mathbb{HSP}(\mathbf{3})$. The subclass $BA = \mathbb{HSP}(\mathbf{2})$ of boolean algebras is the only non-trivial subvariety.

Point of departure

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which is the same as asking

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In the above paper the authors answered the same question for the class of *p*osets and the class of *s*emilattices in the affirmative.

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For the following classes K of algebras with a pseudo-complement the above question was answered in the affirmative:

- K the class of pseudo-complemented distributive lattices (J. Schmid, 1982).
  The same author showed that this also holds for all Lee classes B_n which are subvarieties of K.

- K the class of the so-called *Stone semilattices* alias the class $\mathbb{ISP}(\mathbf{3}) \subseteq PCS$, which is a quasi-subvariety of PCS (Gerber,Schmid,1991).

A finite axiomatization of PCS^{ac}

A semantic characterization of PCS^{ac}

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The following theorem is crucial for finding a finite axiomatization of PCS^{ac} as well as PCS^{ec}.

(J. Schmid, 1985) A p-semilattice $\mathbf{P}$ is algebraically closed iff for any finite subalgebra $\mathbf{F} \leq \mathbf{P}$ there exists $r, s \in \mathbb{N}$ and a p-semilattice $\mathbf{F}'$ isomorphic to $\mathbf{2}^r \times \widehat{\mathbf{A}}^s$ such that $\mathbf{F} \leq \mathbf{F}' \leq \mathbf{P}$.

A finite axiomatization of PCS^{ac}

The list of axioms

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- (AC1) $(\forall a, b, c \in P)(\exists x, y \in P)(c \geq a \wedge b \longrightarrow x \geq a,\ y \geq b\ \&\ x \wedge y = c)$

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- (AC1) $(\forall a, b, c \in P)(\exists x, y \in P)(c \geq a \wedge b \longrightarrow x \geq a,\ y \geq b\ \&\ x \wedge y = c)$
- (AC2) $(\forall a, b, c, t \in P)(\exists x \in P)(a^* = b^* = c^* = 0,\ c < b < a,\ t \wedge c < t \wedge b < t \wedge a \longrightarrow c < x < a,\ x \wedge b = c,\ t \wedge c < t \wedge x < t \wedge a)$

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- (AC3) $(\forall d, d_m \in \mathsf{D}(P), f, f_m, x \in P, k \in \mathsf{Sk}(P))(\exists z_x \in \mathsf{Sk}(P))(d \| d_m,\ f \leq d_m,\ f_m \leq d,\ f_m \not\leq d_m,\ k \leq d,\ k^* \wedge f \not\leq d,\ x^* \leq d_m \longrightarrow k \leq z_x \leq d,\ z_x^* \wedge f \not\leq d,\ z_x \wedge f_m \not\leq d_m\ \&\ (z_x \wedge x)^* \leq d_m)$.

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- (AC4) $(\forall d \in \mathsf{D}(P), b_1 \in \mathsf{Sk}(P)(\exists b_2 \in \mathsf{Sk}(P))(b_1 < d < 1 \longrightarrow b_1 < b_2 < d\ \&\ b_1 \underline{\vee} b_2^* < d)$

A finite axiomatization of PCS^{ac}

The theorem

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The theorem

Finally, a finite axiomatization of PCS^{ac}:

(–,R. Rupp,J. Schmid,2012) Let $\mathcal{A}$ be a finite list of first-order sentences axiomatizing PCS. Then the finite set $\mathcal{A} \cup \{$(AC1),(AC2),(AC3),(AC4)$\}$ axiomatizes PCS^{ac}.

A finite axiomatization of PCS^{ac}

Proof sketch

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Let P be a p-semilattice satisfying (AC1), (AC2), (AC3) and (AC4) and $F_0 \leq P$, $|F_0| < \omega$. By repeatedly applying axiom (ACi), $i = 1, \ldots, 4$, we construct a chain $F_0 \leq F_1 \leq F_2 \leq F_3 \leq F_4 \leq P$ s.t.

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- there is $r \in \mathbb{N}$ s.t. $F_3 \cong \mathbf{2}^r \times \prod_{i=1}^{s} \widehat{\mathbf{F}_{f(i)}}$, $f(i) \in \mathbb{N}, i = 1, \ldots, s$,

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- $F_4 \cong \mathbf{2}^r \times \left(\widehat{\mathbf{A}}\right)^s$.

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Existential closedness revisited

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- P is ec iff for any extension Q of P and any quantifier-free formula φ from $\mathcal{L}(P)$ the following holds: If $Q \models (\exists x_1, \ldots, x_n)\varphi(x_1, \ldots, x_n, a_1, \ldots, a_m)$ then there are $u_1, \ldots, u_n \in P$ such that $P \models \varphi(\overrightarrow{u}, \overrightarrow{a})$

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- P is ec iff for any extension Q of P, arbitrary $S := \{a_1, \ldots, a_m\} \subset P$ and $V := \{v_1, \ldots, v_n\} \subset Q$ there is $U := \{u_1, \ldots, u_n\} \subset P$ and an isomorphism $f : S \cup U \to S \cup V$ with $f_{|S} = \mathrm{id}_S$.

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$$T \cong \underline{2}^p \times \widehat{F}_u^q \ \ (p, q, u \in \mathbb{N})$$

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- (EC1) $(\forall b_1, b_2)(\exists b_3)((\mathsf{Sk}(b_1) \;\&\; \mathsf{Sk}(b_2) \;\&\; b_1 < b_2) \rightarrow (\mathsf{Sk}(b_3) \;\&\; b_1 < b_3 < b_2))$

A finite axiomatization of PCS^{ec}

The list of axioms

- (EC1) $(\forall b_1, b_2)(\exists b_3)((\mathsf{Sk}(b_1) \;\&\; \mathsf{Sk}(b_2) \;\&\; b_1 < b_2) \rightarrow (\mathsf{Sk}(b_3) \;\&\; b_1 < b_3 < b_2))$
- (EC2) $(\forall b_1, d)(\exists b_2)((\mathsf{Sk}(b_1) \;\&\; \mathsf{D}(d) \;\&\; b_1 < d \;\&\; b_1^* \parallel d) \rightarrow (\mathsf{Sk}(b_2) \;\&\; b_1 < b_2 \parallel d \;\&\; b_2 < 1 \;\&\; b_1 \dot{\vee} b_2^* < d \;\&\; b_1^* \wedge b_2 \parallel d))$

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- (EC2) $(\forall b_1, d)(\exists b_2)((\mathsf{Sk}(b_1) \mathbin{\&} \mathsf{D}(d) \mathbin{\&} b_1 < d \mathbin{\&} b_1^* \parallel d) \rightarrow (\mathsf{Sk}(b_2) \mathbin{\&} b_1 < b_2 \parallel d \mathbin{\&} b_2 < 1 \mathbin{\&} b_1 \mathbin{\dot{\vee}} b_2^* < d \mathbin{\&} b_1^* \wedge b_2 \parallel d))$
- (EC3) $(\exists d)(\mathsf{D}(d) \mathbin{\&} d < 1)$

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- (EC2) $(\forall b_1, d)(\exists b_2)((\mathsf{Sk}(b_1) \;\&\; \mathsf{D}(d) \;\&\; b_1 < d \;\&\; b_1^* \parallel d) \rightarrow (\mathsf{Sk}(b_2) \;\&\; b_1 < b_2 \parallel d \;\&\; b_2 < 1 \;\&\; b_1 \dot{\vee} b_2^* < d \;\&\; b_1^* \wedge b_2 \parallel d))$
- (EC3) $(\exists d)(\mathsf{D}(d) \;\&\; d < 1)$
- (EC4) $(\forall d_1, d_2)(\exists d_3)((\mathsf{D}(d_1) \;\&\; d_1 < d_2) \rightarrow (d_1 < d_3 < d_2))$

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The list of axioms

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- (EC4) $(\forall d_1, d_2)(\exists d_3)((\mathsf{D}(d_1) \;\&\; d_1 < d_2) \rightarrow (d_1 < d_3 < d_2))$
- (EC5) $(\forall b, d_1)(\exists d_2)((\mathsf{D}(d_1) \;\&\; \mathsf{Sk}(b) \;\&\; 0 < b < d_1) \rightarrow (\mathsf{D}(d_2) \;\&\; d_2 < d_1 \;\&\; b \parallel d_2 \;\&\; d_1 \wedge b^* = d_2 \wedge b^*))$

A finite axiomatization of PCS^{ec}

The theorem

A finite axiomatization of PCS^{ec}

The theorem

At last, a finite axiomatization of PCS^{ec}:

(–,2012) Let $\mathcal{A}$ be a finite list of first-order sentences axiomatizing PCS. Then the finite set $\mathcal{A} \cup \{$(AC1),. . . ,(AC4),(EC1),. . . ,(EC5)$\}$ axiomatizes PCS^{ec}.