PARTIALLY ORDERED SETS AND LATTICES

Partially Ordered Sets

A *parially ordered set*, or *poset*, is a pair $\mathbf{P} = (X, \leq)$, where X is a nonempty set and $\leq$ is a partial order on X; that is, for x, y, and $z \in X$

(1) $x \leq x$ (reflexivity)
(2) $x \leq y$ and $y \leq x$ imply $x = y$ (anti-symmetry)
(3) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity)

If we define $x < y$ by $x \leq y$ and $x \neq y$, then $<$ satisfies the axioms

(1) $x \not< x$ (anti-reflexivity)
(2) $x < y$ and $y < x$ never holds (strict anti-symmetry)
(3) $x < y$ and $y < z$ imply $x < z$ (transitivity)

A relation satisfying these axioms is called a *strict paritial order*. Given a strict partial order $<$, you can get back $\leq$ be defining $x \leq y$ if $x < y$ or $x = y$, as you can easily verify.

Lemma 1. *If $(X, \leq)$ is a finite partially ordered set, then X has a maximal and a minimal element.*

An element $x \in X$ is *maximal* if $x \leq y$ implies $x = y$. Note there can be more than one maximal element. Minimal elments are defined in an obvious way.

Proof. Pick $x_1 \in X$. If x_1 is not maximal, there is another element $x_2 \in X$ with $x_1 < x_2$. Continuing in this way for r steps, we get $x_1 < x_2 < \cdots < x_r$. Since $<$ is transitive, $x_i < x_j$ whenever $i < j$. In particular they are distinct. Since X is finite this process must stop and the last element is a maximal element. $\square$

Directed Graphs. A *directed graph* consists of a set V of vertices and a set E of directed edges or arrows between some of the pairs of elements of V. More formally, E is a subset of $V \times V$. An arrow from a to itself is called a *loop*; a graph without loops is called *loopless*. A *path* from a to b is a sequence of edges

$$a \to a_1 \to a_2 \to \cdots \to b$$

A *cycle* is a path from a to itself. A graph is *acyclic* if there are no nontrivial cycles. (A cycle from a to a is trivial if all the a_i's are equal to a.) The *transitive closure* of the relation of E (or the transitive closure of $\to$) is the set of all $(a, b) \in V \times V$ such that there is a path from a to b.

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We say that b *covers* a and a is *covered by* b if $a < b$ and there is no c with $a < c < b$. This is written as $a \prec b$. Define

$$\begin{aligned} E_{\leq} &= \{\langle a, b\rangle : a \leq b\} \qquad & e_{\leq} &= |E_{\leq}| \\ E_{<} &= \{\langle a, b\rangle : a < b\} \qquad & e_{<} &= |E_{<}| \\ E_{\prec} &= \{\langle a, b\rangle : a \prec b\} \qquad & e_{\prec} &= |E_{\prec}|. \end{aligned}$$

Note $E_{\prec} \subseteq E_{<} \subseteq E_{\leq}$. We showed in class that the transitive closure of $E_{\prec}$ is $E_{<}$. A similar argument shows that if $E_{\prec} \subseteq E \subseteq E_{<}$ then the transitive closure of E is $E_{<}$. (Also the transitive, reflexive closure of E is $E_{\leq}$, but we won't worry about that.) The graph on V with edges $E_{\prec}$ is called the *Hasse diagram.*

An order $\leq$ on X is a *linear order* (or a *total order* or a *chain*) if for x, $y \in X$, either $x \leq y$ or $y \leq x$. A linear order $\leq'$ is a *linear extension* of $\leq$ if $E_{\leq} \subseteq E_{\leq'}$.

Theorem 2 (Szpilrajn)**.** *Every ordered set has a linear extension.*

Lemma 3. *Let $(X, \leq)$ be a partially ordered set with elements a and b satisfying $b \not\leq a$. Define $\leq'$ by $x \leq' y$ if $x \leq y$ or $x \leq a$ and $b \leq y$. Then $\leq'$ is a partial order containing $\leq$ and $a \leq' b$.*

This lemma can be used to prove Szpilrajn's theorem for finite posets: if a and b are incomparable elements ($a \not\leq b$ and $b \not\leq a$) we can extend $\leq$ to a bigger relation $\leq'$ with $a \leq' b$. Continuing we get a partial order $\leq''$ extending $\leq$, with no incomparabilities, which means it is a chain.

But if a and b are incomparable, we could have defined $\leq'$ so that $b \leq' a$. The intersection of a relation $\leq'$ with $b \leq' a$ and another $\leq''$ with $a \leq' b$ has a and b incomparable. So we get the next corollary.

Corollary 4. *Every partial order is the intersection of linear orders that extend it.*

[This corollary and Szpilrajn's theorem hold for infinite partially ordered sets. Szpilrajn's theorem can be proved with a straightforward Zorn's lemma argument using the lemma.]

The *dimension* or *order dimension* of a partially ordered set is the minimal number of linear extensions that intersect to it.

Exercises

1. Let $E \subseteq E_{<}$. Show that E is acyclic and show that if the transitive closure of E is $E_{<}$ then $E_{\prec} \subseteq E$.

2. Prove the lemma.

3. Let S be a set with n elements and let $\mathcal{P}(S)$ be all subsets of S ordered by set inclusion. Show that the dimension of $\mathcal{P}(S)$ is n. (This is hard—I'll give some hints later.)

4. Show the dimension of the partially ordered set in the figure is the Catalan number: $\frac{1}{n+1}\binom{2n}{n}$. (I will give the figure and give hints.)

Lattices

If $a \leq c$, $b \leq c$ in a partially ordered set $\mathbf{P} = (X, \leq)$, we say that c is an *upper bound* of a and b. If $d \leq a$, $d \leq b$ we say d is a lower bound of a and b. We say an upper bound c of a and b is the *least upper bound* if $c \leq c'$ for every upper bound c' of a and b. It is denoted $a \vee b$ and called the *join* of a and b. The greatest lower bound is defined similarly and denoted $a \wedge b$ and called the *meet* of a and b. A *lattice* is a partially ordered set in which every pair of elements has a join and a meet.

Examples. The set $\mathcal{P}(S)$ of subsets of a set S is a lattice with $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. Notice that $\wedge$ looks like $\cap$ and $\vee$ looks like $\cup$.

The set of all equivalence relations on a set S is a lattice. The meet of two equivalence relations is just their intersection but join is not just the union.