THE RAMSEY NUMBER $R(3,4) = 9$

Let K_n be the complete graph on n vertices; that is, n points with each pair of points connected by an edge. Suppose every edge is colored red or blue. Let x be one of the points and let

$$B_x = \{y : (x, y) \text{ is blue}\} \quad \text{and} \quad R_x = \{y : (x, y) \text{ is red}\}$$

Lemma 1. *If $|B_x| \geq 6$, then the graph either has a red triangle or a blue quadruple (4 points with all connecting lines blue).*

Proof. Since B_x is a graph with at least 6 points and every edge is colored red or blue, it contains either a red or a blue triangle because $R(3, 3) = 6$. We are done if it has a red triangle. If it has a blue triangle, the three points of the triangle together with x form a blue quadruple since all the lines from anything in B_x to x are blue. □

Lemma 2. *If $|R_x| \geq 4$, then the graph either has a red triangle or a blue quadruple.*

Proof. Exercise. □

Corollary 3. $R(3, 4) \leq 10$.

Theorem 4. $R(3, 4) = 9$

Proof. By the homework exercise, $R(3, 5) > 8$.

Suppose the edges of K_9 are colored red or blue. We need to show that there is either a red triangle or a blue quadruple. Let x be a vertex. By the first lemma there can be at most 5 blue lines coming out of x. By the second lemma, there are at most 3 red lines coming out of x. But there are 8 lines coming out of x so *every point has exactly 3 red lines and 5 blues coming out of it*. So if we count the blue lines we seem to get $9 \cdot 5 = 45$ lines. But actually each line got counted twice, once from each end. So there $45/2 = 22.5$ lines. Oops. This is a contradiction which shows it is impossible to color the lines of K_9 so that there are exactly 5 blues coming from each point. □

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