ROADMAP TO ABELIAN ALGEBRA IN CM VARIETIES

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Remak's Principle. If A and B are normal subgroups of a group and $A \cap B = \{e\}$, then $ab = ba$ for all $a \in A$ and $b \in B$ (just note $a^{-1}b^{-1}ab$ is in both A and B). Using this it is easy to prove that if $V < A, B, C < U$ are normal subgroups forming an $\mathbf{M}_3$, then U/V is abelian. This is *Remak's principle*. Roughly it says the nontrivial parts of the lattice of normal subgroups correspond to the trivial parts of the group.

We will show that if $\mathbf{A}$ lies in a CM variety and $\mathbf{Con}\ (\mathbf{A})$ contains a spanning $\mathbf{M}_3$, then $\mathbf{A}$ is polynomial equivalent to a module.

A preview of the commutator theory. Coming soon.

Now you should read section 4 of chapter 10 of the second volume of Algebras, Lattices, and Varieties, ALVII, which is on the web page, up to and including Cor 10.10.

Now we follow section 4.13 of the first volume, ALVI. We define the center $\zeta_{\mathbf{A}}$ of an arbitrary $\mathbf{A}$, by $(a, b) \in \zeta_{\mathbf{A}}$ if for all $n \geq 1$, for all term operations $t \in \mathrm{Clo}_{n+1}\mathbf{A}$ for all $c_1, \ldots, c_n$, $d_1, \ldots, d_n$

$$t(a, \mathbf{c}) = t(a, \mathbf{d}) \leftrightarrow t(b, \mathbf{c}) = t(b, \mathbf{d})$$

$\mathbf{A}$ is *abelian* if $\zeta_{\mathbf{A}} = 1_{\mathbf{A}}$. We prove $\zeta_{\mathbf{A}}$ is a congruence (Thm 4.147 of ALVI).

Next we defined $\Delta(\mathbf{A}) = \mathrm{Cg}^{\mathbf{A}^2}(\{(x, x), (y, y)) : x, y \in A\})$ and proved that if $\mathbf{A}$ is abelian then $((x, x), (y, z)) \in \Delta(\mathbf{A})$ iff $y = z$, for all x, y, $z \in A$. This is part of Thm 4.152.

Now we went back to section 10.4 of ALVII and proved Cor 10.11:

Corollary 1. *If $\mathbf{A}$ is abelian then $\mathbf{Con}\ (\mathbf{A}^2)$ has a spanning $\mathbf{M}_3$. So If $\mathbf{A}$ is abelian and lies in a CM variety, then $\mathbf{V}(\mathbf{A})$ is CP.*

Now we are ready for the main theorem:

Theorem 2. *TFAE for $\mathbf{A}$ in a CM variety.*

(1) *$\mathbf{A}$ is abelian.*
(2) *$\boldsymbol{V}(\mathbf{A})$ is CP and if $p(x, y, z)$ is a Maltsev term for $\mathbf{A}$, then $p^{\mathbf{A}} : \mathbf{A}^3 \to \mathbf{A}$ is a homomorphism.*
(3) *$\mathbf{A}$ is polynomially equivalent to a module over a ring.*

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