Algebra Universalis, 18 (1984) 262–273 0003–5240/84/030262–12$01.50+0.20/0

Unary polynomials in algebras, I

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Introduction

The unary algebraic functions of an algebra already determine the congruences. Therefore, investigations of algebras with some prescribed properties of congruences can be divided into two major steps. First, one characterizes the monoid of unary algebraic functions of the algebra. Secondly, one tries to build up the algebra, knowing its unary algebraic functions. Of course we may only aim at the determination of the possible clones F of operations with given monoid M of unary operations contained in F. However, in general, this task is hopeless. Because, for a finite base set A, there are only finitely many monoids operating on A but infinitely many clones (more exactly, the number of clones is $\aleph_0$ if $|A|=2$, and $2^{\aleph_0}$ if $|A|\geqq 3$, see [11], p. 79). Fortunately, for some special choices of M we can obtain a complete list of the corresponding clones. The best known example for such a choice is G. A. Burle's [1] determination of the clones containing all unary operations.

We shall deal with those algebras in which any unary algebraic function is either a permutation or a constant. In other words we deal with clones, the unary members of which are all the constants and some permutations. These algebras (clones) will be called *permutational algebras* (resp., clones).

Permutational algebras naturally arise in some investigations of the above mentioned feature (see, e.g., T. Ihringer [6], [7], E. Fried, L. Szabó and Á. Szendrei [3], P. P. Pálfy and P. Pudlák [10], H. Länger and R. Pöschel [8]). In the last section we shall indicate how our results can be applied in these situations.

Besides the trivial examples of permutational algebras, namely, the unary permutational algebras and the vector spaces, we present a more sophisticated clone defined on the set of real numbers. This example emphasizes the need for an additional condition for the infinite permutational algebras to be still under

Presented by V. Trnková. Received December 31, 1981. Accepted for publication in final form October 8, 1982.

control. (The extension of our investigations to infinite algebras was inspired by the work of E. Fried, L. Szabó, and Á. Szendrei [3]. We shall, also, use some of the ideas of this work.)

It turns out, that under a certain finiteness condition, the permutational algebras are either essentially unary algebras or vector spaces. Notice that this conclusion holds without any restriction for finite permutational algebras.

Incidentally, some recent papers are closely related to the present work. T. Ihringer [7, Theorem 3.8] obtains the same conclusion for finite algebras having a linear congruence class geometry (see Application 2). Some parts of R. McKenzie's profound paper [9] (especially Theorem 2.6) have also great resemblance to our investigations.

Our terminology is quite standard, except that speaking of an n-ary operation we always suppose that it really depends on all of its n variables.

Examples for permutational algebras

1. *Unary algebras.* The most trivial examples are the unary permutational algebras.

2. *Vector spaces.* The clone of algebraic functions of a vector space A over a field K is obviously permutational. This clone consists of the operations of the form

$$\sum_{i=1}^{n} \lambda_i x_i + a \quad (\lambda_i \in K, a \in A).$$

The subclones containing all constants are also permutational.

3. *Monotone continuous functions.* Let $A = \mathbb{R}$, the set of real numbers, the clone F consist of the constant functions, the functions $f(x_1, \ldots, x_n)$ such that fixing arbitrarily $n-1$ variables the resulting function is an order-preserving bijection (i.e., monotone, unbounded and continuous), and the functions obtained from the previous ones by introducing fictitious variables. As it is easily seen, in order to show that F is a clone, it is enough to prove the following: If $f \in F$ is an n-ary operation $(n \geqq 1)$ and $g_1, \ldots, g_n \in F$ are unary operations, then $f(g_1, \ldots, g_n)$ also belongs to F, i.e., this function is a monotone bijection. This function is obviously monotone and unbounded. Concerning the continuity, we

have

$$\lim_{x\to x_0+0} f(g_1(x),\ldots)=\inf_{x>x_0} f(g_1(x),\ldots)$$
$$=\inf\{f(y_1,\ldots)\mid y_1>g_1(x),\ldots\}$$
$$=\inf\{f(y_1,y_2,\ldots)\mid y_1\geqq g_1(x),\ y_2>g_2(x),\ldots\}$$
$$=\cdots=\inf\{f(y_1,\ldots,y_n)\mid y_1\geqq g_1(x),\ldots,y_n\geqq g_n(x)\}$$
$$=f(g_1(x_0),\ldots)$$

for any fixed $x_0\in\mathbb{R}$, and similarly

$$\lim_{x\to x_0-0} f(g_1(x),\ldots)=f(g_1(x_0),\ldots)$$

hence $f(g_1,\ldots)$ is continuous.

4. *Extensions with* ∞. Let F be a permutational clone on A. Define $A^*=A\cup\{\infty\}$, and for any n-ary ($n\geqq 1$) operation $f\in F$

$$f^*(x_1,\ldots,x_n)=\begin{cases} f(x_1,\ldots,x_n) & \text{if } \forall x_i\in A\\ \infty & \text{otherwise.}\end{cases}$$

Then the constant operations, the operations of the form f^* ($f\in F$) and the operations obtained from them by introducing fictitious variables form a permutational clone on A^*, provided for every n-ary operation $f\in F$ and arbitrary $g_1,\ldots,g_n\in F$ we have

$f(g_1,\ldots,g_n)$ is constant if and only if every g_i is constant. (1)

Such are, e.g., the clone defined in Example 3, and the following subclone of algebraic functions of a vector space over an ordered field K:

$$\left\{\sum_{i=1}^{n}\lambda_i x_i+a\mid \lambda_i\in K,\ \lambda_i\geqq 0,\ a\in A\right\}.$$

This Example serves as the converse of Lemma 3.

5. *The permutational clones on the 2-element set.* From E. Post's [12] determination of the clones on the 2-element set we obtain the following list of

permutational clones (with the original notation):

$O_8 = \langle 0, 1 \rangle$

$S_6 = \langle 0, 1, \vee \rangle$

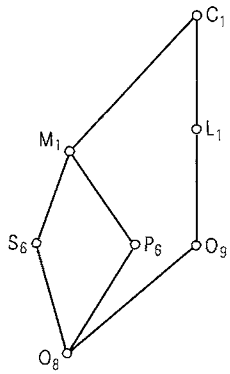
$P_6 = \langle 0, 1, \wedge \rangle$

$M_1 = \{0, 1, \vee, \wedge \rangle$

$O_9 = \langle 0, 1, \neg \rangle$

$L_1 = \langle 0, 1, \neg, + \rangle$

$C_1 = \langle 0, 1, \neg, \vee, \wedge \rangle$



The corresponding group consists of the identity only in the cases O_8, S_6, P_6, M_1 and it has two elements in the cases O_9, L_1, C_1.

The main result

The finiteness condition we shall use is the following. A monoid G of permutations on the set A is said to satisfy the *Bounded Blocks Condition* (briefly BBC), iff there exists a natural number m such that for every $a, b \in A$ there is a block B of G containing a and b with $|B| \leqq m$. (As usual $B \subseteq A$ is a *block* of G iff for any $g \in G$ either $g(B) = B$ or $g(B) \cap B = \emptyset$ holds.)

In other words, the congruence classes of the principal congruences of $\langle A; G \rangle$ are bounded by m.

For finite A, G obviously satisfies the BBC. Lemma 4 will show that a monoid satisfying the BBC is automatically a group. The permutations in the clone of algebraic functions of a vector space over a field K satisfy the BBC iff K is finite. The monoid of permutations in the clone defined in Example 3 or in its nontrivial subclones does not satisfy the BBC.

Our main result is the following.

THEOREM. *Let F be a permutational clone on the set A, $|A| \geqq 3$. Suppose that the monoid G of permutations contained in F satisfies the* BBC. *Then F is either a unary clone or the clone of algebraic functions of a vector space on A over a finite field.*

Notice that the clone of algebraic functions of a vector space is uniquely determined by its unary members. For having picked out the 0, $a + b$ $(a \neq 0)$ is the image of b by $a + x$, where $a + x$ is the unique permutation without fixed point sending 0 to a. The multiplications by the nonzero field elements are just those permutations which fix 0. If we pick out another element $c \in A$ for zero, we obtain the operations $x \oplus y = x + y - c$ and $\lambda \odot x = \lambda x + (1-\lambda)c$ which give rise to the same clone.

Therefore, given M consisting of a monoid of permutations satisfying the BBC and all constants, there are at most two clones which contain M as the monoid of their unary operations.

The proof of the theorem is almost self-contained, we need only a result of Frobenius from group theory. Frobenius' theorem states, that in a finite transitive permutation group, in which only the identity fixes more than one point, the elements without fixed point together with the identity form a (normal) subgroup.

Preliminary lemmas

LEMMA 1. *If a clone F contains all constants and a non-unary operation then it contains a binary operation.*

Proof. Let $f \in F$ be an n-ary ($n \geqq 2$) operation. We prove the assertion by induction on n. For $n = 2$, there is nothing to prove. Let $n \geqq 3$. Since f depends on x_3 there exists an $a \in A$ such that $f(a, x_2, x_3, \ldots, x_n)$ depends on x_3. If it depends on x_2, too, we are done by induction. Now suppose that it does not depend on x_2. However, f itself depends on x_2, hence there exist $b_1, b_3, \ldots, b_n \in A$ such that $f(b_1, x_2, b_3, \ldots, b_n)$ is not constant. Then $f(x_1, x_2, b_3, \ldots, b_n)$ belongs to F and depends on both x_1 and x_2 since $f(b_1, x_2, b_3, \ldots, b_n)$ is not constant but $f(a, x_2, b_3, \ldots, b_n)$ is.

LEMMA 2. *Let F be a permutational clone on the set A, $|A| \geqq 3$, and G be the monoid of permutations contained in F. If f is a binary operation in F then either*

(a) *$\langle A; f\rangle$ is a quasigroup and G is transitive on A; or*

(b) *there is a unique element $\infty \in A$ such that $\forall a \in A$: $f(a, \infty) = f(\infty, a) = \infty$, $A' = A \setminus \{\infty\}$ is a subalgebra of $\langle A; f\rangle$, $\langle A'; f\rangle$ is a quasigroup, $\forall g \in G$: $g(\infty) = \infty$ and G is transitive on A'.*

Proof. The operations $f(a, x)$, $f(x, a)$ ($a \in A$) are permutations or constants. If all of them are permutations then $\langle A; f\rangle$ is obviously a quasigroup. Given $a, b \in A$, the permutation $f(a, x)$ maps an element $c \in A$ to b. Then the permutation $f(x, c)$ maps a to b, hence G is transitive.

Now let us suppose that $\langle A; f\rangle$ is not a quasigroup. This means that there is a constant row (or column) in the "multiplication table" of f. Denote this constant by ∞. Not all rows are constant, because f depends on the second variable. So there is a row which is a permutation, therefore all columns but one are permutations, the remaining one being constant ∞. Hence we have elements $a, b, \infty \in A$ such that

$$f(x, y) = \infty \quad \text{iff} \quad x = a \quad \text{or} \quad y = b.$$

Take any $g \in G$, then $f(x, g(x))$ takes on ∞ at one or two places (namely when $x = a$ or $g(x) = b$). Since $|A| \geqq 3$, $f(x, g(x))$ is a permutation, so takes on ∞ exactly once, therefore $g(a) = b$ holds for every $g \in G$. Applying this for the identity permutation, we get $a = b$, and for a permutation $f(x, c)$ $(c \neq b)$, $\infty = f(a, c) = b$. Now every assertion of (b) follows easily.

According to the previous lemma we may speak of transitive (case a) and non-transitive (case b) non-unary permutational algebras. We shall show that the non-transitive algebras are extensions of certain transitive ones. We use the notations of Lemma 2.

LEMMA 3. *Let F be a non-unary permutational clone with non-transitive G, $h \in F$ an n-ary $(n \geqq 1)$ operation. Then*
(i) *$h(x_1, \ldots, x_n) = \infty$ iff at least one $x_i = \infty$; and*
(ii) *for $h_1, \ldots, h_n \in F$, $h(h_1, \ldots, h_n)$ is a constant $\neq \infty$ iff every h_i is constant $(\neq \infty)$.*

Proof. First observe that h takes on ∞. Indeed, $h(x, a_2, \ldots, a_n)$ is not constant for some $a_2, \ldots, a_n \in A$, hence by Lemma 2(b) $h(\infty, a_2, \ldots, a_n) = \infty$. Putting ∞ in place of a_i $(2 \leqq i \leqq n)$ one by one, the value of h must remain ∞, so $h(\infty, \ldots, \infty) = \infty$.

For $n = 1$ (and even for $n = 2$) (i) follows from Lemma 2(b). For $n > 1$, by the previous remark, we may restrict ourselves to the case when one of the x_i is not ∞. Put the constant $a \neq \infty$ in place of x_i. We claim that $h(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n)$ still depends on its $n - 1$ variables. Without loss of generality take $i = 1$, and suppose by way of contradiction that $h(a, x_2, \ldots, x_n)$ does not depend on x_2. But h itself does depend on x_2, which forces the existence of $b, b_3, \ldots, b_n \in A$ such that $h(b, x_2, b_3, \ldots, b_n)$ is a permutation. Then $f(x_1, x_2) = h(x_1, x_2, b_3, \ldots, b_n)$ depends on both of its variables, so we can invoke again Lemma 2(b), which yields $a = \infty$, a contradiction. Now (i) follows obviously by induction.

If $h(h_1, \ldots, h_n)$ is a constant $\neq \infty$, i.e., it does not take on ∞, then by (i) no one of the h_i takes on ∞, hence they are also constants.

By Lemma 3, if F is a non-transitive permutational clone on A, then restricting all operations in F except the constant ∞ onto A', we obtain a transitive permutational clone on A'. This restriction fulfils the condition (1) of Example 4, so this condition is necessary and sufficient for a permutational clone to have an extension with ∞.

We shall need some properties of permutation monoids with BBC.

LEMMA 4. *Let G be a monoid of permutations of the set A, satisfying the* BBC *with bound m. Then any $g \in G$ has finite order dividing $m!$, consequently G is a group.*

Proof. Choose arbitrary $g \in G$, $a \in A$ and put $b = g(a)$. The BBC yields a block b of G with $|B| \leqq m$, $a, b \in B$. Since $b \in B \cap g(B)$, we have $g(B) = B$, i.e., B is a union of cycles of g. Hence any cycle of g has length $\leqq m$.

Proof of the theorem

Let F be a non-unary permutational clone on the set A, and denote by G the monoid of permutations belonging to F. Suppose that G satisfies the BBC, then by Lemma 4, G is a permutation group on A. Lemma 2 allows two possibilities: (a) G is transitive; (b) G fixes one element of A and permutes transitively the others. First we deal with the transitive case. We prove through Claims 1–5 that F is the clone of algebraic functions of a vector space over a finite field.

CLAIM 1. There exist a binary operation $+$, a unary operation $-$, and a constant 0 in F such that $+$ is a quasigroup operation, $0 + x = x + 0 = x$ and $(-x) + x = 0$.

Proof. Since F is non-unary, there exists a binary operation $f \in F$ (Lemma 1) which is a quasigroup operation, since we have assumed G to be transitive (Lemma 2). Consider $f(x, x)$. If it is constant, then take $0 = f(x, x)$, $g_0(x) = x$. If $f(x, x)$ is a permutation, then let 0 be an arbitrary constant function and put $g_0(x) = f(x, x)$. In any case $g_0 \in G$. Moreover, define $g_1(x) = g_0^{-1}(f(x, 0))$ and $g_2(y) = g_0^{-1}(f(0, y))$. Obviously, $g_1, g_2 \in G$ and $g_1(0) = g_2(0) = 0$. Now let

$$x + y = g_0^{-1}(f(g_1^{-1}(x), g_2^{-1}(y))).$$

Then $x + 0 = g_0^{-1}(f(g_1^{-1}(x), 0)) = x$, and similarly, $0 + y = y$. Since f is a quasigroup operation, $+$ is also such.

Now define nx ($n \in \mathbb{N}$) recurrently as follows: $1x = x$ and $nx = (n-1)x + x$ for

$n>1$. For every $a\in A$ the unary operation $x+a$ belongs to G, hence its order divides $m!$ by Lemma 4. Therefore we have $(m!+1)a=a=0+a$. Cancelling a from the right we get $(m!-1)a+a=0$. Thus defining

$$-x=(m!-1)x$$

we obtain $-\in F$ and $(-x)+x=0$.

From now on we fix these operations $+$, $-$, 0. We do not distinguish the constant operation $0\in F$ and the corresponding element $0\in A$.

CLAIM 2. Only the identity $1\in G$ has more than one fixed point. The permutations without fixed point together with the identity form a regular normal subgroup N of G. Moreover, $N=\{x+a \mid a\in A\}=\{a+x \mid a\in A\}$.

Proof. Suppose that $g\in G$ fixes at least two points. Then $(-x)+g(x)$ takes on the value 0 at least twice, hence $(-x)+g(x)=0=(-x)+x$, so after cancellation we get $g(x)=x$.

If G contains no permutation with exactly one fixed point, then $N=G$ is a regular permutation group. If G contains permutations with exactly one fixed point, then G is a so called Frobenius group. Now we prove that N is a subgroup. For, let $g, h\in N$ be different from 1 and suppose that gh has a fixed point a. Set $b=h(a)\neq a$, then $g(b)=a$. By the BBC there is a finite block B containing a and b. Consider $G_B=\{f\in G \mid f(B)=B\}$. Obviously, $g, h\in G_B$. G_B induces a transitive permutation group on B, which is itself a Frobenius group. From the finiteness of B we infer that those elements of G_B which have no fixed point in B together with those which act identically on B form a (normal) subgroup of G_B (see [5], p. 495). Hence $gh=1$ on B. But $|B|\geqq 2$, so by the first assertion of the present claim we have $gh=1$. This shows that N is a subgroup of G. The normality of N is a trivial fact.

If $x+a$ $(a\in A)$ has a fixed point, say b, then $b+a=b=b+0$, so $a=0$ and $x+a=x+0$, the identity permutation. This shows that $N\supseteq\{x+a \mid a\in A\}$. This latter set of permutations is transitive (cf. the proof of Lemma 2(a)), so N is transitive. Now N is clearly regular, therefore we have $N=\{x+a \mid a\in A\}$, and also $N=\{a+x \mid a\in A\}$.

CLAIM 3. $\langle A;+\rangle\cong N$ is a commutative group.

Proof. Since N is regular, the correspondence $g\mapsto g(0)$ between N and A is one-to-one. Let $g, h \in N$. By Claim 2, $g(0)+x$ is a permutation belonging to N. It maps 0 to $g(0)+0=g(0)$ hence this permutation is g, i.e., $g(0)+x=g(x)$. Therefore,

$$g(0)+h(0)=g(h(0))=gh(0),$$

establishing the isomorphy of N and $\langle A;+\rangle$. Moreover, reasoning in the same way, we get $x+h(0)=h(x)$, so

$$g(0)+h(0)=h(g(0))=hg(0)$$

holds also, proving the commutativity.

CLAIM 4. $K=\{0\}\cup\{k\in G \mid k(0)=0\}$ is a finite field with the operations $(k_1+k_2)(x)=k_1(x)+k_2(x)$, $(k_1k_2)(x)=k_1(k_2(x))$; $\langle A;+,K\rangle$ is a vector space.

Proof. Let $a, b\in A$ be arbitrary, then there exist (unique) $g, h\in N$ with $g(0)=a$, $h(0)=b$. Take a $k\in K\cap G$. Then we have

$$\begin{aligned}k(a+b)&=k(g(0)+h(0))=kgh(0)\\&=(kgk^{-1})(khk^{-1})k(0)\end{aligned}$$

Now k fixes 0 and N is a normal subgroup of G, hence

$$\begin{aligned}&=kgk^{-1}(0)+khk^{-1}(0)=kg(0)+kh(0)\\&=k(a)+k(b).\end{aligned}$$

Obviously this holds for $k=0$, too, hence we have

$$k(a+b)=k(a)+k(b) \quad \text{for all} \quad k\in K,\ a, b\in A. \tag{2}$$

Take any $a\in A$, $a\neq 0$. Then from Claim 2 it follows that $k(a)$ uniquely determines $k\in K$. The BBC provides a finite block B containing 0 and a. For $k\in K$ we have $0\in B\cap k(B)$, hence $k(a)\in B$, so $|K|\leqq|B|$, K is finite.

Now it is straightforward to check the field and vector space axioms using Claim 3 and (2). The commutativity of K follows from its finiteness.

CLAIM 5. $F=\{\sum_{i=1}^{n} k_i(x_i)+a \mid k_i\in K, a\in A\}$, i.e., the clone of algebraic functions of the vector space $\langle A;+,K\rangle$.

Proof. F obviously contains every function of the form $\sum k_i(x_i)+a$. To prove the converse take an n-ary operation $f \in F$. Set $a=f(0,\ldots,0)$, $k_i(x_i)=f(0,\ldots,x_i,\ldots,0)-a$, $\bar{f}(x_1,\ldots,x_n)=f(x_1,\ldots,x_n)-\sum_{i=1}^n k_i(x_i)-a$. Then $a\in A$, $k_i \in K$, $\bar{f}\in F$, and $\bar{f}(x_1,\ldots,x_n)=0$ if at most one x_i is not 0. Suppose by way of contradiction that $\bar{f}$ is not constant 0, and choose elements $a_1,\ldots,a_n$ with minimal number of nonzeroes among them such that $\bar{f}(a_1,\ldots,a_n)\neq 0$. Without loss of generality assume that $a_1\neq 0$, $a_2\neq 0$ and define $\tilde{f}(x_1,x_2)=\bar{f}(x_1,x_2,a_3,\ldots,a_n)$. Now we have $\tilde{f}\in F$ and $\tilde{f}(0,x)=\tilde{f}(x,0)=0$, by our minimal choice. By assumption $f(a_1,a_2)\neq 0$, hence f is essentially binary. Then Lemma 2 (a) is applicable (since G is transitive now) and it yields that $\tilde{f}$ is a quasi-group operation, which is certainly not the case. This contradiction shows $\bar{f}=0$, therefore

$$f(x_1,\ldots,x_n)=\sum_{i=1}^n k_i(x_i)+a,$$

as we have claimed.

So we have proved our theorem when G is transitive. If G were not transitive on A with $|A|\geqq 3$, then we would have a permutational clone on $A'=A\setminus\{\infty\}$ satisfying (1) (see Lemma 3). The permutations in this clone form a transitive monoid on A' and this monoid obviously inherits the BBC. According to what we have already proved, this clone would be the clone of algebraic functions of a vector space on A'. But in a vector space the non-constant operations $+$, $-$, and the identity produce a constant operation: $(-x)+x=0$. Hence these clones do not satisfy (1), therefore they cannot have extensions with an ∞ element.

Applications

1. *Algebras in which every equivalence relation is a congruence.* As it is well-known, such algebras on an at least 3-element set are trivial in the sense that their operations are constants or projections. One proves this statement first establishing that the unary algebraic functions are the constants and the identity. So this is the simplest special case of our theory. By Lemmas 1 and 2, the algebra is essentially unary, which finishes the proof. (Cf. [8].)

2. *Algebras in which every principal congruence is minimal.* This generalizes the situation of the preceding example. The condition has relevance only if the algebra is not simple. In this case one can prove immediately that the non-constant unary algebraic functions of the algebra are injective (see [3]). If we make the additional assumption that the congruence classes of principal congruences are finite and uniformly bounded, then we see that these functions are permutations and the monoid formed by them satisfies the BBC, so our theorem

yields that the algebra is either unary or a vector space over a finite field. Under a stronger assumption, namely, that each congruence class of every principal congruence consists of m elements E. Fired, L. Szabó, and Á. Szendrei [3] establish more properties of the algebra even in the unary case.

Making use of the results of R. Wille [14, Chapter 4] one can easily show that the principal congruences of a non-simple algebra are all minimal congruences, if and only if the congruence class geometry of the algebra is linear (in the terminology of R. Wille [14]: "Kongruenzklassengeometrie mit eindeutigen Verbindungsgeraden"). So our result can be considered as a generalization of T. Ihringer's [7] characterization of finite algebras with linear congruence class geometry.

3. *Congruence lattices of finite algebras.* It is still an open problem whether every finite lattice L is isomorphic to the congruence lattice of some finite algebra $\mathfrak{A}$. If L is representable in this way, then we may search for the minimal algebra of this kind. In analysing the structure of $\mathfrak{A}$, we may confine ourselves to unary algebras, but in constructions it is sometimes more convenient to use also binary, ternary, etc. operations. P. P. Pálfy and P. Pudlák [10] proved that for certain lattices this minimal unary algebra must be a transitive permutation group (i.e., a minimal algebra with such a congruence lattice is permutational in our terminology). This result was used to prove the equivalence of the following two assertions:

(i) every finite lattice is isomorphic to the congruence lattice of a finite algebra;

(ii) every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.

The present theorem states that, except for the vector spaces, finite permutational algebras are essentially unary. This means that the only way of constructing minimal algebras for example with congruence lattice M_7 (the lattice of length two with 7 atoms, cf. [4]) goes through finding intervals isomorphic to M_7 in subgroup lattices of finite groups. And, in fact, W. Feit [2] has recently found a representation of M_7 in this way.

4. *Subclones of the clone of algebraic functions of a vector space over a finite field.* L. Szabó has called my attention that the theorem can be used for determining those subclones of the clone of algebraic functions of a vector space on A over a finite field K which contain all constants. Let F be such a subclone. Then either F is a unary clone or F is the clone of algebraic functions of a certain vector space on A over some finite field. In the first case F is defined by specifying a subgroup of the permutation group $G=\{\lambda x+a \mid \lambda \in K, \lambda \neq 0, a \in A\}$. Each subgroup of G has the form

$$\{\lambda x+(1-\lambda)c+a \mid \lambda \in L, a \in B\},$$

where L is a subgroup of the multiplicative group of K, B is a subspace of A regarded as a vector space over the subfield of K generated by L, and $c \in A$; L, B, and c modulo B are uniquely determined by F.

In the second case, the addition $\oplus$ in F has the form $x \oplus y = \lambda x + \mu y + c$, λ, $\mu \in K$, $c \in A$. Since there is a zero element for $\oplus$, it follows that $\lambda = \mu = 1$. Then $x + y = x \oplus y \oplus (-2c)$, so $+ \in F$, too. Now it is easy to show that

$$F = \left\{ \sum_{i=1}^{n} \lambda_i x_i + a \mid \lambda_i \in K', a \in A \right\},$$

for a certain subfield K' of K. (Cf. [13], p. 705.)

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