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Every finite lattice can be embedded in a finite partition lattice

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We give here a proof of the theorem stated in the title. The theorem was conjectured by P. M. Whitman in [11]. The proof, mostly of combinatorial character, is based on "the regraph power technique" making use of special edge-colored graphs, called regraphs, as construction schemes. We use the proof of Combinatorial Lemma 6.1 by B. Wolk and B. Sands, which is shorter and more elegant than the original one. The list of references is a random collection of papers related to Whitman's conjecture.

1. The class of finite lattices

Here we present a simple method of generating all lattices. We shall use the notion of join-homomorphism, meet-homomorphism, order preserving mapping and lattice embedding in the sense of [2]. The greatest and the least elements of a lattice L will be denoted by 1_L, 0_L.

DEFINITION. Let L be a lattice, $u, v \in L$, $u \leq v$. Define a subset $L_{u,v}$ of L by

$$L_{u,v} = \{x \in L : v \leq x \quad \text{or} \quad u \nleq x\}.$$

LEMMA 1.1. *$L_{u,v}$ with an ordering induced by the ordering of L forms a lattice. The operation of meet in $L_{u,v}$ is the meet of L, join in $L_{u,v}$ is described by*

$$x \vee_{u,v} y = x \vee y \quad \textit{if} \quad x \vee y \ngeq u,$$
$$x \vee_{u,v} y = x \vee y \vee v \quad \textit{if} \quad x \vee y \geq u.$$

Proof. It is an easy assertion that the meet in L of any subset of $L_{u,v}$ is in $L_{u,v}$. This implies that $L_{u,v}$ is a lattice and the meet in $L_{u,v}$ is the restriction of the meet of L to the set $L_{u,v}$. The description of the join in $L_{u,v}$ is also evident.

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Let us define a mapping $\sigma_{u,v}: L \to L_{u,v}$:

$$\sigma_{u,v}(x) = x \quad \text{for} \quad x \not\geq u,$$
$$\sigma_{u,v}(x) = x \vee v \quad \text{for} \quad x \geq u.$$

LEMMA 1.2. (a) *$\sigma_{u,v}$ is a surjective join-homomorphism.* (b) *Every join-homomorphism $\varphi: L \to K$ such that $\varphi(u) = \varphi(v)$ can be decomposed into $\sigma_{u,v}: L \to L_{u,v}$ and a join-homomorphism $\psi: L_{u,v} \to K$.*

Proof. (a) If $x \vee y \geq u$, then

$$\sigma_{u,v}(x) \vee_{u,v} \sigma_{u,v}(y) = \sigma_{u,v}(x) \vee \sigma_{u,v}(y) \vee v = x \vee y \vee v = \sigma_{u,v}(x \vee y).$$

If $x \vee y \not\geq u$, then

$$\sigma_{u,v}(x) \vee_{u,v} \sigma_{u,v}(y) = x \vee_{u,v} y = x \vee y = \sigma_{u,v}(x \vee y).$$

(b) If $\sigma_{u,v}(x) = \sigma_{u,v}(y)$ and $x \neq y$, then $x \vee v = y \vee v$ and $x, y \geq u$. Hence

$$\varphi(x) = \varphi(x \vee u) = \varphi(x) \vee \varphi(u) = \varphi(x) \vee \varphi(v) = \varphi(x \vee v)$$
$$= \varphi(y \vee v) = \varphi(y) \vee \varphi(v) = \varphi(y) \vee \varphi(u) = \varphi(y \vee u) = \varphi(y).$$

Now, we can define $\psi(x) = \psi(\sigma_{u,v}(x)) = \varphi(x)$, for $x \in L_{u,v}$.

THEOREM 1.3. *Let $\mathcal{L}$ be a class of lattices closed under isomorphisms and such that*

(1) *every Boolean lattice belongs to $\mathcal{L}$,*

(2) *if $L \in \mathcal{L}$, u, $v \in L$ and $u < v$, then $L_{u,v} \in \mathcal{L}$.*

Then $\mathcal{L}$ is the class of all lattices.

Proof. Let K be a lattice. Denote by $P(K)$ the Boolean lattice of all subsets of K. Then the mapping $\sigma: P(K) \to K: X \mapsto \bigvee X$ is a surjective join-homomorphism. Observe that for every non-injective join-homomorphism there is a pair of *comparable* elements, which is collapsed. Therefore we can use Lemma 1.2(b) to decompose σ to a finite chain of surjective join-homomorphisms, everyone of them being $\sigma_{u,v}$, for some lattice L and some u, $v \in L$, $u < v$, and the last one being an isomorphism. Now $P(K) \in \mathcal{L}$ and repeated application of the second condition and closure of $\mathcal{L}$ under isomorphisms gives $K \in \mathcal{L}$.

2. A plan of the proof

This is to introduce a necessary notation and to prove Lemma 2.1, which can be considered a plan of the proof.

The lattice of all equivalences over a set A will be denoted by Eq. (A). We shall use equivalence lattices rather than partition lattices, because equivalence lattices are ordered by a quite simple relation of inclusion. The identity relation on A will be denoted by Id_A, the partition corresponding to an equivalence E over A, that is the set of blocks of E, will be denoted by A/E.

A lattice L will be called *embeddable*, if there exists a set A and a lattice embedding $\varphi : L \to \mathrm{Eq}\,(A)$.

It is known that every Boolean lattice is embeddable. This follows, for example, from the fact that the congruence lattice of a Boolean lattice L is isomorphic to L. By Theorem 1.3, it remains only to investigate the operation $L \mapsto L_{u,v}$ in the class of embeddable lattices. To this end the following lemma is a useful tool.

LEMMA 2.1. *Let L, K be lattices, u, $v \in L$, $u \leq v$, and $\varphi : L \to K$ a mapping with the following properties*

(1) *φ is a join-homomorphism,*

(2) *$\varphi_{u,v} : L_{u,v} \to K$ (the restriction of φ to $L_{u,v}$) is an injective meet-homomorphism,*

(3) *$\varphi(u) = \varphi(v)$.*

Then $\varphi_{u,v}$ is a lattice embedding.

Proof. Let the assumptions of the lemma be satisfied. By Lemma 1.2(b), there is a join-homomorphism $\psi : L_{u,v} \to K$ such that $\psi \circ \sigma_{u,v} = \varphi$. Since the restriction of $\sigma_{u,v}$ to the set $L_{u,v}$ is the identity mapping on $L_{u,v}$, we have $\psi = \varphi_{u,v}$. This proves that $\varphi_{u,v}$ is a join-homomorphism.

Now we can formulate our method more precisely. For a given embeddable lattice L and u, $v \in L$, $u < v$, we modify the original embedding in three steps – Lemma 3.2, Lemma 6.2, and the regraph power by the regraph $\mathbf{G}$ in Construction 7.3. For the new embedding $\psi : L \to \mathrm{Eq}\,(B)$, both (1) and (2) of Lemma 2.1 are satisfied, while (3) is not. Then we change slightly ψ to ψ^* (using $\mathbf{G}^*$ instead of $\mathbf{G}$) so that (1) and (2) are still preserved and, in addition, $\psi^*(u) = \psi^*(v)$. Doing so we get an embedding $\psi^*_{u,v}$ of $L_{u,v}$ into the same equivalence lattice. In §§3–6 we investigate methods, which allow us to construct an embedding ψ and to show that changing ψ to ψ^* does not damage the properties (1) and (2).

3. Lattice metrics

For a given lattice L, we define a lattice L^∞ by adjoining a new element ∞ to L, which is greater than all elements of L. If a mapping $\varphi: L \to \mathrm{Eq}(A)$ is given, a φ-distance $\varphi^-: A \times A \to L$ is defined by

$$\varphi^-(a, b) = \wedge\{x \in L^\infty : (a, b) \in \varphi(x)\}.$$

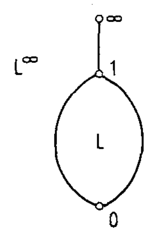

Figure 1

LEMMA 3.1. *Let* $\varphi: L \to \mathrm{Eq}(A)$ *be a meet-homomorphism. Then*

(a) *for* $x \in L$, $\varphi^-(a, b) \leq x$ *iff* $(a, b) \in \varphi(x)$,

(b) *the* φ*-distance satisfies the following "triangle inequality,"* $\varphi^-(a, c) \leq \varphi^-(a, b) \vee \varphi^-(b, c)$, *for every* $a, b, c \in A$.

Proof. (a) If $\varphi^-(a, b) \leq x$, $x \in L$, then the set $M = \{x \in L : (a, b) \in \varphi(x)\}$ is non-empty and

$$(a, b) \in \cap\{\varphi(x) : x \in M\} = \varphi(\wedge M) = \varphi(\varphi^-(a, b)) \subseteq \varphi(x).$$

The converse implication is trivial.

(b) The triangle inequality is trivial if $\varphi^-(a, b)$ or $\varphi^-(b, c)$ is equal to ∞. If this is not the case, then

$$(a, b) \in \varphi(\varphi^-(a, b)) \quad \text{and} \quad (b, c) \in \varphi(\varphi^-(b, c)).$$

Since every meet-homomorphism is order-preserving,

$$(a, b), (b, c) \in \varphi(\varphi^-(a, b) \vee \varphi^-(b, c)),$$

thus

$$\varphi^-(a, c) \leq \varphi^-(a, b) \vee \varphi^-(b, c).$$

The class of embeddings of a given embeddable lattice into equivalence lattices appears to be very rich. We shall make ample use of this fact. Here is our first lemma in this spirit.

LEMMA 3.2. *Let L be an embeddable lattice. Then there are a set B and a lattice embedding $\psi : L \to \text{Eq}(B)$ such that, for every $x \in L$, $a \in B$, there is a $b \in B$ with $\psi^{-}(a, b) = x$.*

Proof. Let us denote by Sym (A) the group of all permutations of a set A, by Sub (G) the lattice of subgroups of a group G. For a set A, define a mapping $Q_A : \text{Eq}\,(A) \to \text{Sub}\,(\text{Sym}\,(A))$ by

$$Q_A(E) = \{\pi \in \text{Sym}\,(A) : (a, \pi(a)) \in E \quad \text{for all} \quad a \in A\}.$$

For a given group G, define a mapping $P_G : \text{Sub}\,(G) \to \text{Eq}\,(G)$ as follows: for every $H \in \text{Sub}\,(G)$ and a, $b \in G$ put

$$(a, b) \in P_G(H) \quad \text{iff} \quad ba^{-1} \in H.$$

As it is observed in [1] (see also [11]) these two mappings are lattice embeddings. For P_G, it it quite trivial, for Q_A, one needs only to realize that every subgroup $Q_A(E_1 \vee E_2)$ is generated by the system of transpositions lying in $Q_A(E_1) \cup Q_A(E_2)$.

Let $\varphi : L \to \text{Eq}\,(A)$ be a lattice embedding. Then the mapping $\psi = P_B \circ Q_A \circ \varphi : L \to \text{Eq}\,(B)$, where $B = \text{Sym}\,(A)$, is again an embedding. Let $\pi \in B$ be an arbitrary permutation on A, $x \in L$. Choose a permutation $\varphi \in B$ such that the restriction of φ to any block of $\varphi(x)$ is a cycle, and put $\rho = \gamma \circ \pi$. Then for $y \in L$, $(\pi, \rho) \in \psi(y)$ iff $\rho \circ \pi^{-1} = \gamma \in Q_A(\varphi(y))$ iff $\varphi(x) \subseteq \varphi(y)$. Hence $\psi^{-}(\pi, \rho) = x$.

We shall prove one more lemma of this kind in §6. To this end, however, we need the regraph constructions.

4. Regraph constructions

DEFINITION. *A regraph valued by a set* A is a triple $(G, R, \sigma) = \mathbf{G}$, where G is a non-empty set, R is a symmetric antireflexive relation on G, and σ is a mapping of R into A.

If, moreover, a mapping $\varphi : L \to \text{Eq}\,(A)$ is given, we define a new mapping $\psi : L \to \text{Eq}\,(A \times G)$, called $\mathbf{G}$*-power of* φ as follows: for $x \in L$, $\psi(x)$ is the least equivalence on $A\times G$ containing the following two relations

$$S_x=\{[(a,g),(b,g)]:g\in G \quad \text{and} \quad (a,b)\in\varphi(x)\},$$
$$S=\{[(\sigma(g,h),g),(\sigma(h,g),h)]:(g,h)\in R\}.$$

Intuitively, $\psi(x)$ is constructed in the following way:

(1) for every $g\in G$, we take one copy of $\varphi(x)$, say $\varphi_g(x)$,

(2) if $\varphi_g(x)$, $\varphi_h(x)$ are copies for $g,h\in G$ and $(g,h)\in R$, then we join the block of $\varphi_g(x)$ containing an occurrence of $\sigma(g,h)$ and the block of $\varphi_h(x)$ containing an occurrence of $\sigma(h,g)$.

LEMMA 4.1. *Let* $\mathbf{G}=(G,R,\sigma)$ *be a regraph valued by* A, *let* L *be a lattice.*

(a) *If* $\varphi:L\to\mathrm{Eq}(A)$ *is an order-preserving mapping, then its* $\mathbf{G}$ *-power* $\psi:L\to\mathrm{Eq}(A\times G)$ *is also order-preserving.*

(b) *If* $\varphi:L\to\mathrm{Eq}(A)$ *is a join-homomorphism, then its* $\mathbf{G}$ *-power* ψ *is also a join-homomorphism.*

Proof. The assertion (a) is trivial. Let φ be a join-homomorphism. Define, for every $g\in G$ and $x\in L$,

$$\psi_g(x)=\{[(a,g),(b,g)]:(a,b)\in\varphi(x)\}\cup I_{A\times G}.$$

Then $\psi_g:L\to\mathrm{Eq}(A\times G)$ is a join-homomorphism. Now the following formula shows that ψ is a join-homomorphism:

$$\psi(x)=\psi(0_L)\vee\bigvee_{g\in G}\psi_g(x), \quad \text{for all} \quad x\in L.$$

An analogy of Lemma 4.1 for meet-homomorphisms or for injective mappings is not true. We are going to present sufficient conditions for these properties.

Let a regraph $\mathbf{G}=(G,R,\sigma)$ valued by A be given. An *R-chain* is any sequence $g_0,g_1,\dots,g_k$ of points from G such that $k>0$ and $(g_{i-1},g_i)\in R$, for $i=1,2,\dots,k$. If moreover a mapping $\varphi:L\to\mathrm{Eq}(A)$ is given, we define the *φ, σ-value of an R-chain* $g_0,\dots,g_k$:

$$\text{val-}\varphi,\sigma(g_0,g_1)=0,$$
$$\text{val-}\varphi,\sigma(g_0,g_1,g_2)=\varphi^{-}(\sigma(g_1,g_0),\sigma(g_1,g_2)),$$
$$\text{val-}\varphi,\sigma(g_0,g_1,\dots,g_k)=\bigvee_{i=1}^{k-1}\text{val-}\varphi,\sigma(g_{i-1},g_i,g_{i+1}), \quad \text{for} \quad k>2.$$

We can say that "val-$\varphi,\sigma(g_0,g_1,g_2)$ is the φ-distance between g_0 and g_2 from the point of view of g_1."

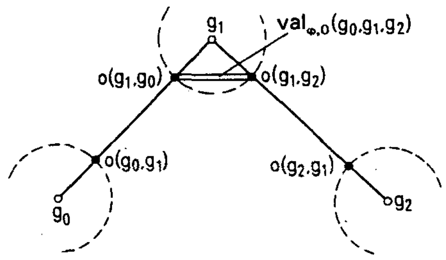

Figure 2

LEMMA 4.2. *Let $\varphi: L \to \mathrm{Eq}(A)$ be a meet-homomorphism,* $\mathbf{G} = (G, R, \sigma)$ *be a regraph valued by A and $\psi: L \to \mathrm{Eq}(A \times G)$ be the* $\mathbf{G}$ *-power of φ. Then, for $x \in L$ and $(a, g), (b, h) \in A \times G$, $[(a, g), (b, h)] \in \psi(x)$ if and only if*

(a) *either $g = h$ and $\varphi^{-}(a, b) \leq x$,*

(b) *or there exists an R-chain $g = g_0, \ldots, g_k = h$ such that*

$$\varphi^{-}(a, \sigma(g, g_1)) \vee \text{val-}\varphi, \sigma(g_0, g_1, \ldots, g_k) \vee \varphi^{-}(b, \sigma(h, g_{k-1})) \leq x.$$

We shall use val-$\varphi, \sigma(a, g_0, g_1, \ldots, g_k; b)$, instead of the join on the left side of the last inequality.

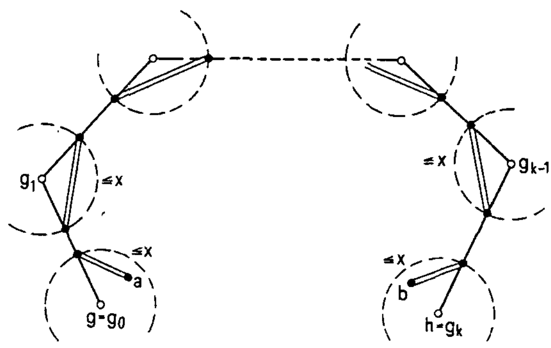

Figure 3

Proof. (1) Let $[(a, g), (b, h)] \in \psi(x)$. Then, by definition, there is a sequence $s: s_0, s_1, \ldots, s_k$ in $A \times G$ such that any pair of consecutive elements of it is in $S \cup S_x$. Observe that S_x is an equivalence. Since S_x is reflexive, we can duplicate each element of s. Since S_x is transitive, we can replace any segment of s, with the property that each pair of consecutive elements is in S_x, by a pair in S_x. Thus we can suppose that we have a sequence of the form

$$(a, g)=(a_0, g_0),\ (b_0, g_0),\ (a_1, g_1),\ (b_1, g_1), \ldots, (b_k, g_k)=(b, h),$$

and $(a_i, b_i)\in\varphi(x)$, for all $i=0,\ldots,k$. Now we distinguish two cases: if $k=0$, then $g=h$ and $(a, b)\in\varphi(x)$, hence $\varphi^{-}(a, b)\leqslant x$, if $k>0$, then $(g_i, g_{i+1})\in R$, $a_{i+1}=\sigma(g_{i+1}, g_i)$, $b_i=\sigma(g_i, g_{i+1})$, for $i=0,\ldots,k-1$. But, in this case, $(a, \sigma(g, g_1))\in\varphi(x)$, $(b, \sigma(h, g_{k-1}))\in\varphi(x)$, and, if $k\geqslant 2$, then also

$$(\sigma(g_i, g_{i-1}), \sigma(g_i, g_{i+1}))=(a_i, b_i)\in\varphi(x), \quad \text{for} \quad i=1,\ldots,k-1.$$

Hence val-$\varphi, \sigma(a; g_0, g_1, \ldots, g_k; b)\leqslant x$.

(2) Let $x\in L$ and (a, g), $(b, h)\in A\times G$. If $g=h$ and $\varphi^{-}(a, b)\leqslant x$, then by Lemma 3.1(a) $(a, b)\in\varphi(x)$. Hence $[(a, g), (b, h)]\in\psi(x)$. If $g=g_0,\ldots,g_k=h$ is an R-chain required in (b), we can form the same sequence as in (1):

$$s:(a, g)=(a_0, g_0),\ (b_0, g_0),\ (a_1, g_1),\ (b_1, g_1),\ldots,(b_k, g_k)=(b, h),$$

where $a_{i+1}=\sigma(g_{i+1}, g_i)$, $b_i=\sigma(g_i, g_{i+1})$, for all $i=0,\ldots,k-1$. Since

$$\text{val-}\varphi, \sigma(a; g_0, g_1, \ldots, g_k; b)\leqslant x.$$

we have $\varphi^{-}(a_i, b_i)=\text{val-}\varphi, \sigma(g_{i-1}, g_i, g_{i+1})\leqslant x$, for $i=1,\ldots,k-1$, and $\varphi^{-}(a_0, b_0)\leqslant x$, $\varphi^{-}(a_k, b_k)\leqslant x$. By Lemma 3.1(a) we can again conclude that $(a_i, b_i)\in\varphi(x)$, for all $i=0,\ldots,k$. Thus any pair of subsequent elements of the sequence s lies in $S\cup S_x$, which proves $[(a, g), (b, h)]\in\psi(x)$.

The importance of the following concepts is derived from the fact that they depend only on φ, σ-values of chains and not on the concrete form of σ. This enables us to change σ in §7 in such a way that some properties (injectivity, meet-homomorphism) of the **G**-power are still preserved.

DEFINITION. Let $\mathbf{G}=(G, R, \sigma)$ be a regraph valued by A, $\varphi: L\to \mathrm{Eq}(A)$ be an arbitrary mapping. Then we say:

(a) The couple **G**, φ satisfies *home-is-best property* (see [12]), if, for every R-chain $g=g_0, g_1,\ldots,g_k=g$,

$$\text{val-}\varphi, \sigma(g_1, g, g_{k-1})\leqslant\text{val-}\varphi, \sigma(g_0, g_1, \ldots, g_k).$$

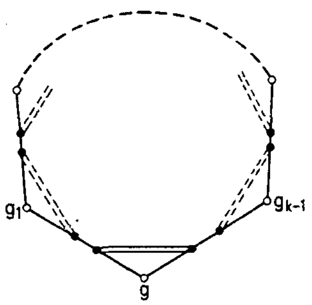

Figure 4

(b) A sequence $g=g_0, g_1, \ldots, g_k=h$ is a *φ, σ-high-way*, if it is an R-chain and, for every R-chain $g=h_0, \ldots, h_n=h$,

$$\text{val-}\varphi, \sigma(h_1, g_0, g_1, \ldots, g_k, h_{n-1}) \leq \text{val-}\varphi, \sigma(h_0, \ldots, h_n).$$

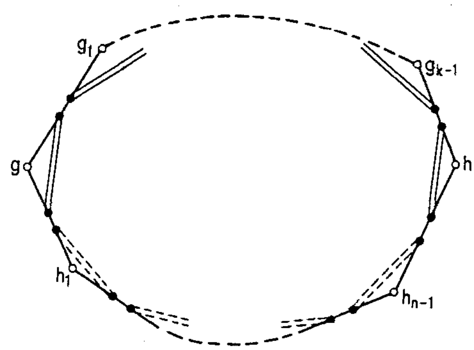

Figure 5

(c) The couple $\mathbf{G}$, φ satisfies *high-way property*, if for any two different elements $g, h \in G$, there exists a high-way $g=g_0, \ldots, g_k=h$.

LEMMA 4.3. *Let $\mathbf{G}=(G, R, \sigma)$ be a regraph valued by A and $\varphi: L \to \mathrm{Eq}(A)$ a meet-homomorphism such that the couple $\mathbf{G}$, φ satisfies home-is-best property. If ψ is $\mathbf{G}$-power of φ, then*

(a) *for $x \in L$, $a, b \in A$ and $g \in G$,*

$$[(a, g), (b, g)] \in \psi(x) \quad \textit{iff} \quad (a, b) \in \varphi(x),$$

(b) *ψ is injective, whenever φ is injective.*

Proof. (a) By definition, if $(a, b) \in \varphi(x)$, then $[(a, g), (b, g)] \in \psi(x)$, for any $g \in G$ and $x \in L$. Conversely, let $[(a, g), (b, g)] \in \psi(x)$. Then, by Lemma 4.2, either

$\varphi^{-}(a, b) \leq x$ or there exists an R-chain $g = g_0, \ldots, g_k = g$ such that

$$\text{val-}\varphi, \sigma(a; g_0, \ldots, g_k; b) \leq x,$$

In the first case, by Lemma 3.1(a), $(a, b) \in \varphi(x)$. In the second case, using home-is-best property, we have

$$\text{val-}\varphi, \sigma(g_1, g, g_{k-1}) = \varphi^{-}(\sigma(g, g_1), \sigma(g, g_{k-1})) \leq \text{val-}\varphi, \sigma(g_0, g_1, \ldots, g_k) \leq x.$$

Now, by triangle inequality,

$$\varphi^{-}(a, b) \leq \varphi^{-}(a, \sigma(g, g_1)) \vee \varphi^{-}(\sigma(g, g_1), \sigma(g, g_{k-1})) \vee \varphi^{-}(\sigma(g, g_{k-1}), b) \leq x$$

and again $(a, b) \in \varphi(x)$.

(b) Let $x \not\leq y$. Assuming φ injective, there is $(a, b) \in \varphi(x) - \varphi(y)$. For $g \in G$ we have $[(a, g), (b, g)] \in \psi(x) - \psi(y)$, thus ψ is injective.

LEMMA 4.4. *Let* $\mathbf{G} = (G, R, \sigma)$ *be a regraph valued by* A, $\varphi : L \to \mathrm{Eq}\,(A)$ *a meet-homomorphism and* $\psi : L \to \mathrm{Eq}\,(A \times G)$ *the* $\mathbf{G}$*-power of* φ. *Let* $\mathbf{G}$, φ *satisfy home-is-best and high-way properties. Then* ψ *is a meet-homomorphism.*

Proof. φ is order preserving being a meet-homomorphism, thus by Lemma 4.1(a) ψ is also order-preserving. So it remains only to show the inclusion $\psi(x) \cap \psi(y) \subseteq \psi(x \wedge y)$ for any two $x, y \in L$. Let $[(a, g), (b, h)] \in \psi(x) \cap \psi(y)$. If $g = h$, by Lemma 4.3 $(a, b) \in \varphi(x) \cap \varphi(x) \cap \varphi(y)$. Since φ is a meet-homomorphism, we have $(a, b) \in \varphi(x \wedge y)$, hence $[(a, g), (b, h)] \in \psi(x \wedge y)$. If $g \neq h$, there is (by Lemma 4.2) an R-chain $g = h_0, h_1, \ldots, h_n = h$ with

$$\text{val-}\varphi, \sigma(a; h_0, \ldots, h_n; b) \leq x.$$

Now, let us take a φ, σ-high-way $g = g_0, \ldots, g_k = h$.

$$\text{val-}\varphi, \sigma(h_1, g_0, \ldots, g_k, h_{n-1}) \leq \text{val-}\varphi, \sigma(h_0, \ldots, h_n) \leq x,$$

in particular

$$\varphi^{-}(\sigma(g, h_1), \sigma(g, g_1)) = \text{val-}\varphi, \sigma(h_1, g_0, g_1) \leq \text{val-}\varphi, \sigma(h_1, g_0, \ldots, g_k, h_{n-1}) \leq x,$$

$$\varphi^{-}(\sigma(h, h_{n-1}), \sigma(h, g_{k-1})) = \text{val-}\varphi, \sigma(h_{n-1}, g_k, g_{k-1})$$
$$\leq \text{val-}\varphi, \sigma(h_1, g_0, \ldots, g_k, h_{n-1}) \leq x.$$

Using triangle inequality we obtain

$$\varphi^{-}(a, \sigma(g, g_1)) \leqslant \varphi^{-}(a, \sigma(g, h_1)) \vee \varphi^{-}(\sigma(g, h_1), \sigma(g, g_1)) \leqslant x$$

and

$$\varphi^{-}(b, \sigma(h, g_{k-1})) \leqslant \varphi^{-}(b, \sigma(h, h_{n-1})) \vee \varphi^{-}(\sigma(h, h_{n-1}), \sigma(h, g_{k-1})) \leqslant x.$$

Then, for the R-chain $g = g_0, \ldots, g_k = h$, we have

$$\text{val-}\varphi, \sigma(a; g_0, \ldots, g_k; b) \leqslant x.$$

For the same reason

$$\text{val-}\varphi, \sigma(a; g_0, \ldots, g_k; b) \leqslant y,$$

and so

$$\text{val-}\varphi, \sigma(a; g_0, \ldots, g_k; b) \leqslant x \wedge y.$$

Using Lemma 4.2 we can conclude that $[(a, g), (b, h)] \in \psi(x \wedge y)$.

5. Perfect regraphs and products of regraphs

Let $\mathbf{G} = (G, R, \sigma)$ be a regraph valued by A. Then $\mathbf{G}$ is called *symmetric* if σ is symmetric, i.e. if $\sigma(g, h) = \sigma(h, g)$ for any $(g, h) \in R$. For a symmetric regraph $\mathbf{G}$ we define *the set-value of an R-chain* $g_0, \ldots, g_k$ by the formula set-$\sigma(g_0, \ldots, g_k) = \{\sigma(g_0, g_1), \sigma(g_1, g_2), \ldots, \sigma(g_{k-1}, g_k)\}$. If, for a given R-chain $g = g_0, \ldots, g_k = h$, set-$\sigma(g_0, \ldots, g_k) \subseteq$ set-$\sigma(h_0, \ldots, h_n)$, whenever $g = h_0, \ldots, h_n = h$ is an R-chain, then we say that $g_0, \ldots, g_k$ is an *σ-shortest path.*

DEFINITION. A regraph $\mathbf{G} = (G, R, \sigma)$ values by A is called *perfect* if it is symmetric and for every two different points $g, h \in G$, there is an σ-shortest path $g = g_0, \ldots, g_k = h$.

LEMMA 5.1. *Let $\mathbf{G} = (G, R, \sigma)$ be a symmetric regraph valued by A, $\varphi: L \to \operatorname{Eq}(A)$ a meet-homomorphism. Then*

(a) *for any R-chain* $g_0, \ldots, g_k$

$$\text{val-}\varphi, \sigma(g_0, \ldots, g_k) = \bigvee_{i,j=0}^{k-1} \varphi^{-}(\sigma(g_i, g_{i+1}), \sigma(g_j, g_{j+1})) =$$

$$= \bigvee \{\varphi^{-}(a, b) : a, b \in \text{set-}\sigma(g_0, \ldots, g_k)\},$$

(b) *the couple* **G**, φ *satisfies home-is-best property,*
(c) *any* σ*-shortest path is a* φ, σ*-high-way.*

Proof. Let $g_0, \ldots, g_k$ be an R-chain. If $k = 1$, then assertion (a) is trivial. If $k \geq 2$, then since **G** is symmetric,

$$\text{val-}\varphi, \sigma(g_0, \ldots, g_k) = \bigvee_{i=1}^{k-1} \varphi^{-}(\sigma(g_{i-1}, g_i), \sigma(g_i, \sigma(g_i, g_{i+1})).$$

Applying the triangle inequality we obtain (a).

The assertion (b) is a consequence of (a), since

$$\text{set-}\sigma(g_1, g_0, g_{k-1}) \subseteq \text{set-}\sigma(g_0, \ldots, g_k)$$

whenever $g_0, \ldots, g_k$ is an R-chain and $g_0 = g_k$.

Let $g_0, \ldots, g_k$ be an σ-shortest path and let $g_0 = h_0, \ldots, h_n = g_k$ be any R-chain. Then we have set-$\sigma(g_0, \ldots, g_k) \subseteq$ set-$\sigma(h_0, \ldots, h_n)$, $\sigma(h_1, g_0) = \sigma(h_0, h_1)$, $\sigma(g_k, h_{n-1}) = \sigma(h_{n-1}, h_n)$, whence set-$\sigma(h_1, g_0, \ldots, g_k, h_{n-1}) \subseteq$ set-$\sigma(h_0, \ldots, h_n)$. Thus using (a), $g_0, \ldots, g_k$ is a φ, σ-high-way.

COROLLARY 5.2. *If* $\mathbf{G} = (G, R, \sigma)$ *is a perfect regraph valued by* A *and* $\varphi : L \to \mathrm{Eq}(A)$ *is a meet-homomorphism, then* **G**, φ *satisfies home-is-best and high-way properties, the* **G***-power of* φ *is a meet-homomorphism and it is injective if* φ *is injective. In particular, if* φ *is an embedding, then the* **G***-power of* φ *is also an embedding.*

Our main tool will be products of cyclic two-valued regraphs. A cyclic two-valued regraph (G, R, σ) consists of an (unoriented) cycle of an even length ≥ 4 and a symmetric two-valued function σ, which assigns different values to any two adjacent edges. (We consider $(g, h) \in R$ and $(h, g) \in R$ being a single unoriented edge.) Such a regraph is determined up to isomorphism by an even number $n \geq 4$ and a pair a, b. We shall denote it by Cyk (n, a, b). It is a trivial assertion that Cyk (n, a, b) is a perfect regraph.

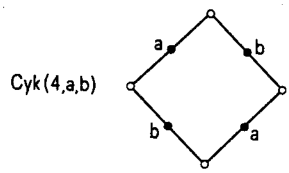

Figure 6

DEFINITION. Let $\mathbf{G}_i=(G_i, R_i, \sigma_i)$ be a regraph valued by A, $i=1,\ldots,n$. Then we define product: $\mathbf{G}_1\times\mathbf{G}_2\times\cdots\times\mathbf{G}_n=\Pi_{i=1}^n \mathbf{G}_i=(G, R, \sigma)$ – a regraph valued by A – as follows

(a) $G=G_1\times G_2\times\cdots\times G_n$ (G is the cartesian product of G_i's),

(b) $[(g^1,\ldots,g^n), (h^1,\ldots,h^n)]\in R$ iff there is a $j\in\{1,\ldots,n\}$ such that $(g^j, h^j)\in R_j$ and $g^i=h^i$ for all $i\neq j$,

(c) in this case $\sigma((g^1,\ldots,g^n), (h^1,\ldots,h^n))=\sigma_j(g^j, h^j)$. Thus the product of symmetric regraphs is symmetric. As usual, we shall assume associativity of the operation of product.

LEMMA 5.3. *Let* $\mathbf{G}=(G, R, \sigma)=\Pi_{i=1}^n \mathbf{G}_i$ *be a product of symmetric regraphs* $\mathbf{G}_i=(G_i, R_i, \sigma_i)$ *valued by A. Let* $g_0,\ldots,g_k$ *be an R-chain,* $g_i=(g_i^1,\ldots,g_i^n)$ *for* $i=0,\ldots,k$. *If* $g_0^j\neq g_k^j$ *and* $g_0^j=h_0^j, h_1^j,\ldots,h_m^j=g_k^j$ *is an* σ_j*-shortest path, then* set-$\sigma_j(h_0^j, h_1^j,\ldots,h_m^j)\subseteq$ set-$\sigma(g_0,\ldots,g_k)$.

Proof. We can choose a subsequence $g_{i_0}^j, g_{i_1}^j,\ldots,g_{i_p}^j$ of the R-chain $g_0^j, g_1^j,\ldots,g_k^j$ in this way: $g_{i_0}^j=g_0^j$, $g_{i_p}^j=g_k^j$ and

$$g_{i_q}^j=g_{i_q+1}^j=\cdots=g_{i_{q+1}-1}^j\neq g_{i_{q+1}}^j \quad \text{for} \quad q=0,\ldots,p-1.$$

From the definition of product we can conclude that $(g_{i_q}^j, g_{i_{q+1}}^j)\in R_j$ and $\sigma(g_{i_{q+1}-1}, g_{i_{q+1}})=\sigma_j(g_{i_q}^j, g_{i_{q+1}}^j)$ for all $q=0,\ldots,p-1$. Thus $g_{i_0}^j, g_{i_1}^j,\ldots,g_{i_p}^j$ is an R_j-chain, and set-$\sigma_j(g_{i_0}^j,\ldots,g_{i_p}^j)\subseteq$ set-$\sigma(g_0,\ldots,g_k)$. The assertion now follows from the fact that $g_0^j=h_0^j,\ldots,h_m^j=g_k^j$ is an σ_j-shortest path.

The following consequence, though easy, is of an essential character.

THEOREM 5.4. *Product of perfect regraphs is a perfect regraph.*

Proof. It is enough to prove the theorem for two regraphs. Let $(G, R, \sigma)=(G_1, R_1, \sigma_1)\times(G_2, R_2, \sigma_2)$ be product of perfect regraphs. Let $(g^1, g^2), (h^1, h^2)\in G$, $g^1\neq h^1$, $g^2\neq h^2$. By the assumption, there are an σ_1-shortest path $\mathscr{C}_1: g^1=g_0, g_1,\ldots,g_k=h^1$, and an σ_2-shortest path $\mathscr{C}_2: g^2=h_0, h_1,\ldots,h_n=h^2$. Let us form the R-chain

$$\mathscr{C}:(g^1, g^2)=(g_0, g^2),\ (g_1, g^2), \ldots,$$
$$(g_k, g^2)=(h^1, g^2)=(h^1, h_0), (h^1, h_1), \ldots, (h^1, h_n)=(h^1, h^2).$$

Then set-$\sigma(\mathscr{C})$ = set-$\sigma_1(\mathscr{C}_1) \cup$ set-$\sigma_2(\mathscr{C}_2)$. If $\mathscr{D}$ is an arbitrary R-chain with end-points (g^1, g^2), (h^1, h^2), then by Lemma 5.3 we have set-$\sigma(\mathscr{C})$ = set-$\sigma_1(\mathscr{C}_1) \cup$ set-$\sigma_2(\mathscr{C}_2) \subseteq$ set-$\sigma(\mathscr{D})$ and so $\mathscr{C}$ is the σ-shortest path. In the case $g^1 = h^1$ or $g^2 = h^2$ we should use only the chain $\mathscr{C}_2$ or $\mathscr{C}_1$.

COROLLARY 5.5. *Any product of cyclic two-valued regraphs is a perfect regraph.*

6. The basic lemma

An important concept of this paragraph is the concept of a column in a product of regraphs. However, we can define it for products of sets, since it does not depend on R or σ.

DEFINITION. A column in the ith coordinate in the product $G_1 \times \cdots \times G_n$ is any set of the form

$$\{g^1\} \times \{g^2\} \times \cdots \times \{g^{i-1}\} \times G_i \times \{g^{i+1}\} \times \cdots \times \{g^n\},$$

where $1 \leq i \leq n$ and $g^j \in G_j$ for $j = 1, \ldots, i-1,\ i+1, \ldots, n$.

The set of all columns in $G_1 \times \cdots \times G_n$ will be denoted by Clm.

Let Cyk (m) denote an unoriented cycle of length m. We define a metric ρ on Cyk(m) by: $\rho(g, h)$ is the least r such that there exists a chain in Cyk(m) $g = g_0, \ldots, g_r = h$.

COMBINATORIAL LEMMA 6.1. *For every $n \geq 1$, there is an even number $m \geq 4$ and a function $v = (v^1, \ldots, v^n): \{0, 1\} \times \mathrm{Clm} \to [\mathrm{Cyk}(m)]^n$ such that*

(1) *$v(i, C) \in C$ for every $C \in \mathrm{Clm}$, $i = 0, 1$,*

(2) *for every two different couples (i, C), $(j, D) \in \{0, 1\} \times \mathrm{Clm}$ there is a coordinate k, $1 \leq k \leq n$, with $\rho(v^k(i, C), v^k(j, D)) \geq 2$.*

Proof by B. Sands and B. Wolk. Let $n \geq 1$ be given. Put $m = 2^n(2^n + 1)$ and let the sequence $0, 1, \ldots, m-1, 0$ form the cycle in Cyk(m). Let Clm be the set of columns of $[\mathrm{Cyk}(m)]^n$. Define a mapping $v: \{0, 1\} \times \mathrm{Clm} \to [\mathrm{Cyk}(m)]^n$ as follows:

for $i \in \{0, 1\}$,

$$C = \{x_1\} \times \cdots \times \{x_{j-1}\} \times \{0, \ldots, m-1\} \times \{x_{j+1}\} \times \cdots \times \{x_n\} \in \text{Clm}$$

put $v(i, C) = (x_1, \ldots, x_n)$, where $0 \leqslant x_j < m$ and

$$\sum_{k=1}^{n} x_k 2^{k-1} \equiv 0 \mod (2^n + 1), \tag{1}$$

$$\sum_{k=1}^{n} x_k \equiv i2^{n-1} + j \mod (2^n). \tag{2}$$

(1) Since both 2^n, $2^n + 1$ and 2^{j-1}, $2^n + 1$ are mutually prime, the congruences above have a unique simultaneous solution x_j modulo $m = 2^n(2^n + 1)$ for any $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$. Hence the definition of v is correct.

(2) By the congruence (2), we have $j \equiv \sum_{k=1}^{n} x_k \mod (2^{n-1})$. Since $1 \leqslant j \leqslant n \leqslant 2^{n-1}$, the coordinate j, in which the column ranges, is uniquely determined, hence C is uniquely determined by $(x_1, \ldots, x_n)$. Congruence (2) also determines i. Thus v is injective.

(3) Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be distinct elements in the range of v. Suppose by way of contradiction that the distance of each pair x_k, y_k is less than 2, that is $x_k - y_k \equiv a_k \mod (m)$, $|a_k| \leqslant 1$ for $k = 1, \ldots, n$. Since $\mathbf{x}$, $\mathbf{y}$ satisfy congruence (1) and $2^n + 1$ divides m, we have

$$\sum_{k=1}^{n} a_k 2^{k-1} \equiv \sum_{k=1}^{n} (x_k - y_k) 2^{k-1} \equiv \sum_{k=1}^{n} x_k 2^{k-1} - \sum_{k=1}^{n} y_k 2^{k-1} \equiv 0 \mod (2^n + 1).$$

On the other hand

$$\left| \sum_{k=1}^{n} a_k 2^{k-1} \right| \leqslant \sum_{k=1}^{n} 2^{k-1} = 2^n - 1 < 2^n + 1,$$

whence we obtain

$$\sum_{k=1}^{n} a_k 2^k = 0.$$

Letting k be minimal such that $a_k \neq 0$, we see that a_k is even contrary to assumption.

BASIC LEMMA 6.2. *Let L be an embeddable lattice, $u \in L$, $u \neq 0_L$. Then there is a set B, a lattice embedding $\psi: L \to \mathrm{Eq}(B)$ and a set $V \subseteq B$ such that*

(1) $|e \cap V| \geq 2$, *for every block e of the equivalence $\psi(u)$,*

(2) $\psi^-(a, b) \geq u$, *for every $a, b \in V$, $a \neq b$.*

Explanation. The set V chooses two elements from every block of $\psi(u)$ and the ψ-distance between any two chosen points is at least u.

Proof. (a) By Lemma 3.2 we can assume that we have an embedding $\varphi: L \to \mathrm{Eq}(A)$ and in any block $e \in A/\varphi(u)$ a couple a_e, b_e such that $\varphi^-(a_e, b_e) = u$. Let n be the number of blocks of the equivalence $\varphi(u)$ and let $m \geq 4$ be an even number satisfying conditions of the Combinatorial Lemma 6.1. Then, let us take a regraph $\mathbf{G} = (G, R, \sigma) = \Pi_{e \in A/\varphi(u)} \mathbf{G}_e$, where $\mathbf{G}_e = (G_e, R_e, \sigma_e) = \mathrm{Cyk}(m, a_e, b_e)$. Using Corollaries 5.2 and 5.5 we have that the $\mathbf{G}$-power of φ $\psi: L \to \mathrm{Eq}(A \times G)$ is a lattice embedding.

(b) Let $v: \{0, 1\} \times \mathrm{Clm} \to \Pi_{e \in A/\varphi(u)} \mathbf{G}_e$ be a choice function given by the Combinatorial Lemma. We put in the choice set $V \subseteq A \times G$ every element of the form $(a_e, v(i, C))$, where C is a column in the eth coordinate in $\Pi_{e \in A/\varphi(u)} \mathbf{G}_e$ and $i = 0, 1$.

(c) To prove the condition (1) it is enough to show that for every element $(a, g) \in A \times G$ there are two elements in V, which are equivalent to (a, g) in $\psi(u)$.

Let (a, g) be given, $g = (g^f)_{f \in A/\varphi(u)}$ and let $a \in e \in A/\varphi(u)$. Define a column in the eth coordinate containing $g: C = \{(h^f) \in G : h^f = g^f \text{ for } f \neq e\}$. Put $h_i = (h_i^f) = v(i, C)$ for $i = 0, 1$. Then $(a_e, h_i) \in V$ and we shall prove that

$$[(a, g), (a_e, h_i)] \in \psi(u).$$

If $h_i^e = g^e$, then $h_i = g$. Since $a_e \in e$ and $a \in e$, we have $(a, a_e) \in \varphi(u)$, hence $[(a, g), (a_e, h_i)] \in \psi(u)$.

If $h_i^e \neq g^e$, let $g^e = g_0^e, g_1^e, \ldots, g_k^e = h_i^e$ be an R_e-chain connecting g^e and h_i^e. Such a chain exists, because $\mathbf{G}_e$ is a cycle. The set-value of this chain is a subset of $\{a_e, b_e\}$. Now we define an R-chain $g = g_0, g_1, \ldots, g_k = h_i$ by the formula

$$g_j^f = g_j^e \quad \text{for} \quad f = e,$$
$$g_j^f = g^f = h_i^f \quad \text{for} \quad f \neq e, j = 0, \ldots, k.$$

Hence the set-value of this chain is again a subset of $\{a_e, b_e\}$. By Lemma 5.1(a) val-φ, $\sigma(g_0, \ldots, g_k) \leq u$. Moreover $\varphi^-(a, \sigma(g_0, g_1)) \leq u$ and $\varphi^-(a_e, \sigma(g_{k-1}, h_i)) \leq u$, since a, $\sigma(g_0, g_1)$, a_e, $\sigma(g_{k-1}, h_i)$ lie in the same block $e \in A/\varphi(u)$. Applying

Lemma 4.2 we obtain $[(a, g), (a_e, h_i)] \in \psi(u)$ for $i = 0, 1$. The couples (a_e, h_0) and (a_e, h_1) are different, because $h_0 = v(0, C)$ and $h_1 = v(1, C)$ differ in the eth coordinate.

(d) Let (a_e, g), (a_f, h) be two distinct elements of V, $g = v(i, C)$, $h = v(j, D)$ Since a_e, a_f are determined by the columns C, D, we have $(i, C) \neq (j, D)$. If $g = (g^d)_{d \in A/\varphi(u)}$ and $h = (h^d)_{d \in A/\varphi(u)}$, using property (2) of Combinatorial Lemma, there is a $d \in A/\varphi(u)$ such that $\rho(g^d, h^d) \geq 2$.

Let $\mathscr{C}: g = g_0, g_1, \ldots, g_k = h$ be an arbitrary R-chain, and $g^d = h_0^d, h_1^d, \ldots, h_p^d = h^d$ an arbitrary σ_d-shortest path. Since $\rho(g^d, h^d) \geq 2$, we have set-$\sigma_d(h_0^d, h_1^d, \ldots, h_p^d) = \{a_d, b_d\}$. By Lemma 5.3 $\{a_d, b_d\} \subseteq$ set-$\sigma(g_0, \ldots, g_k)$. Hence, using Lemma 5.1(a) val-φ, $\sigma(\mathscr{C}) \geq \varphi^{-}(a_d, b_d) = u$. Applying Lemma 4.2 we obtain that, if $[(a_e, g), (a_f, h)] \in \psi(x)$, for some $x \in L$, then $u \leq x$, and so $\psi^{-}((a_e, g), (a_f, h)) \geq u$. Thus we have proved condition (2).

7. The final construction

We are going to prove that $L_{u,v}$ is embeddable whenever L is embeddable, for $u, v \in L$, $u < v$.

CONSTRUCTION 7.1. Let L be an embeddable lattice, $u, v \in L$, $O_L < u < v$. Let us take an embedding $\varphi: L \to \mathrm{Eq}\,(A)$ and a set $V \subseteq A$ given by Basic Lemma 6.2. We can assume that in any block e of the equivalence $\varphi(u)$ there are exactly two different elements a_e, b_e of V. Then for every $f \in A/\varphi(v)$ we define a regraph

$$\mathbf{D}_f = (D_f, R_f, \sigma_f) = \Pi\{\mathrm{Cyk}(4, a_e, b_e) : e \in A/\varphi(u), e \subseteq f\}.$$

Since the graph (D_f, R_f) is symmetric, we can investigate it as an unoriented graph. The index of any vertex of D_f, that is the number of unoriented edges incident with it, is an even number $2\,|\{e \in A/\varphi(u) : e \subseteq f\}|$. As proved in graph theory, this implies that there is an Euler cycle in (D_f, R_f), which means: there is a sequence $g_0, g_1, \ldots, g_n = g_0$ such that

(1) $(g_i, g_{i+1}) \in R_f$, for $i = 0, \ldots, n-1$,

(2) for any couple $(g, h) \in R_f$, there is exactly one $i \in \{0, \ldots, n-1\}$, such that $(g_i, g_{i+1}) = (g, h)$ or $(g_{i+1}, g_i) = (g, h)$.

For the given Euler cycle, we define a new valuation $\sigma_f^*: R_f \to A$ as follows: every edge of R_f occurs exactly once and in only one direction in the Euler cycle, therefore it suffices to define $\sigma_f^*(g_i, g_{i+1})$ and $\sigma_f^*(g_{i+1}, g_i)$ for $i = 0, 1, \ldots, n-1$. Let us denote moreover $g_{n+1} = g_1$. Then we can take an $e \in A/\varphi(u)$, $e \subseteq f$, for every $i = 1, \ldots, n$, and define $\sigma_f^*(g_i, g_{i-1}) = a_e$, $\sigma_f^*(g_i, g_{i+1}) = b_e$. As it has been noted above, this is a correct definition of $\sigma_f^*: R_f \to A$ for any choice of e's.

The number of occurrences of any $g \in D_f$ in the Euler cycle is one half of its index, that is $|\{e \in A/\varphi(u): e \subseteq f\}|$. Hence, given $g \in D_f$ and defining $\sigma_f^*(g_i, g_{i-1})$, $\sigma_f^*(g_i, g_{i+1})$ for $g_i = g$, we can take different e's, $e \subseteq f$, for different occurrences of g in the cycle. Then, moreover, any e, $e \subseteq f$ is used for some occurrence $g = g_i$.

The main properties of such an σ_f^* are summarized below.

CLAIM 7.2. (a) $\varphi^-(\sigma_f^*(g_i, g_{i-1}), \sigma_f^*(g_i, g_{i+1})) = u$ for $i = 1, \ldots, n$,

(b) for every $g \in D_f$, $e \in A/\varphi(u)$, $e \subseteq f$ there are h_1, $h_2 \in D_f$ such that $\sigma_f^*(g, h_1) = a_e$ and $\sigma_f^*(g, h_2) = b_e$.

(c) if, for some g, h_1, $h_2 \in D_f$, $\sigma_f^*(g, h_1) = \sigma_f^*(g, h_2)$, then $h_1 = h_2$.

Proof. The assertion (a) follows immediately from the fact that $\varphi^-(a_e, b_e) = u$ for all $e \in A/\varphi(u)$. The assertions (b) and (c) are consequences of bijection between the occurrences of g in the Euler cycle and the set $\{e \in A/\varphi(u): e \subseteq f\}$.

CONSTRUCTION 7.3. For every $f \in A/\varphi(v)$ we fix an Euler cycle in (D_f, R_f) and define a regraph $\mathbf{D}_f^* = (D_f, R_f, \sigma_f^*)$. Finally, we define regraphs

$$\mathbf{G} = (G, R, \sigma) = \prod_{f \in A/\varphi(v)} \mathbf{D}_f \quad \text{and} \quad \mathbf{G}^* = (G, R, \sigma^*) = \prod_{f \in A/\varphi(v)} \mathbf{D}_f^*.$$

Then $\mathbf{G}$ is a perfect regraph, while $\mathbf{G}^*$ is even not symmetric.

The following Claim formalizes the sentence "σ is slightly changed to σ^*."

CLAIM 7.4. $\varphi^-(\sigma(g, h), \sigma^*(g, h)) \leq v$ for every $(g, h) \in R$.

Proof. This is true for σ_f, σ_f^* and $(g^f, h^f) \in R_f$, because the values $\sigma_f(g^f, h^f)$, $\sigma_f^*(g^f, h^f)$ are elements of the same block $f \in A/\varphi(v)$. The assertion now follows from the definition of product.

The next claim states that the injectivity property of 7.2(c) extends to the regraphs $\mathbf{G}$ and $\mathbf{G}^*$.

CLAIM 7.5. For an R-chain h_1, g, h_2, if $h_1 \neq h_2$, then $\sigma(g, h_1) \neq \sigma(g, h_2)$ and $\sigma^*(g, h_1) \neq \sigma^*(g, h_2)$.

Proof. Let $\sigma^*(g, h_1) = \sigma^*(g, h_2)$ and $h_1 \neq h_2$. Since the ranges of the valuations σ_f are disjoint, for different f's, $f \in A/\varphi(v)$, both g, h_1 and g, h_2 have to differ in the same, say, dth coordinate. Hence h_1^d, g^d, h_2^d is an R_d-chain, and $\sigma_d^*(g^d, h_1^d) = \sigma^*(g, h_1) = \sigma^*(g, h_2) = \sigma_d^*(g^d, h_2^d)$. Using Claim 7.2(c) we have $h_1^d = h_2^d$, which contradicts the assumption.

If $\sigma(g, h_1)=\sigma(g, h_2)$ and $h_1 \neq h_2$, then by the same argument we reduce the question to $\mathrm{Cyk}(4, a_e, b_e)$ for some $e \in A/\varphi(u)$. Recall that in $\mathrm{Cyk}(4, a_e, b_e)$ two adjacent edges have different values, which is the injectivity property.

Denote by ψ^* the $\mathbf{G}^*$-power of φ and let $\varphi_{u,v}$, $\psi^*_{u,v}$ be the restrictions of φ, ψ^* to $L_{u,v}$ respectively. Then $\psi^*_{u,v}$ is the $\mathbf{G}^*$-power of $\varphi_{u,v}$.

CLAIM 7.6. For an R-chain $g_0, \ldots, g_k$,

$$\text{val-}\varphi_{u,v}, \sigma(g_0, \ldots, g_k) = \text{val-}\varphi_{u,v}, \sigma^*(g_0, \ldots, g_k).$$

Proof. If $k=1$ or, $k=2$ and $g_0=g_2$, then both sides of the equality are $0_{L_{u,v}}$.

Let $k=2$ and $g_0 \neq g_2$. Then by Claim 7.5 $\sigma(g_1, g_0) \neq \sigma(g_1, g_2)$, and $\sigma^*(g_1, g_0) \neq \sigma^*(g_1, g_2)$. Both $\sigma(g_1, g_0)$, $\sigma(g_1, g_2)$ and $\sigma^*(g_1, g_0)$, $\sigma^*(g_1, g_2)$ are elements of the set V, hence the φ-distance of any of these two pairs is at least u. Since $\wedge\{x \in L^\infty_{u,v} : u \leq x\} = v$, the $\varphi_{u,v}$-distance of any of these two pairs is at least v. Now using Claim 7.4 and the triangle inequality we have

$$\begin{aligned}
&\varphi^-_{u,v}(\sigma(g_1, g_0), \sigma(g_1, g_2))\\
&\leq \varphi^-_{u,v}(\sigma(g_1, g_0), \sigma^*(g_1, g_0)) \vee \varphi^-_{u,v}(\sigma^*(g_1, g_2)) \vee \varphi^-_{u,v}(\sigma^*(g_1, g_2), \sigma(g_1, g_2))\\
&\leq \varphi^-_{u,v}(\sigma^*(g_1, g_0), \sigma^*(g_1, g_2)).
\end{aligned}$$

The converse inequality can be proved by the same argument. Since

$$\text{val-}\varphi_{u,v}, \sigma(g_0, \ldots, g_k) = \bigvee_{i=1}^{k-1} \text{val-}\varphi_{u,v}, \sigma(g_{i-1}, g_i, g_{i+1}), \quad \text{for} \quad k>2,$$

and the same holds for σ^*, the claim is proved.

CLAIM 7.7. $\psi^*_{u,v}$ is an injective meet-homomorphism.

Proof. Recall that $\mathbf{G}$ is a perfect regraph, $\varphi_{u,v}$ a meet-homomorphism and that this assures home-is-best and high-way properties for the couple $\mathbf{G}$, $\varphi_{u,v}$. Applying the previous claim, we have that the couple $\mathbf{G}^*$, $\varphi_{u,v}$ also satisfies home-is-best and high-way properties. Since $\varphi_{u,v}$ is moreover injective, we have that $\psi^*_{u,v}$ is an injective meet-homomorphism.

CLAIM 7.8. ψ^* is a join-homomorphism.

Proof. Any regraph power of a join-homomorphism is a join-homomorphism.

CLAIM 7.9. $\psi^*(u)=\psi^*(v)$.

Proof. As a consequence of the above claim we have $\psi^*(u)\subseteq\psi^*(v)$. The equivalence $\psi^*(v)$ is defined as the least equivalence containing relations S_v and S. In order to prove $\psi^*(u)\supseteq\psi^*(v)$ it is enough to show that $S_v\subseteq\psi^*(u)$, since $S\subseteq\psi^*(x)$ for any $x\in L$. In fact, it suffices to prove the following implication: if $(a,b)\in\varphi(v)-\varphi(u)$, $g\in G$, then $[(a,g),(b,g)]\in\psi^*(u)$.

Let $(a,b)\in\varphi(v)-\varphi(u)$, $g=(g^f)_{f\in A/\varphi(u)}\in G$. Then there are blocks $d\in A/\varphi(v)$, $e_1, e_2\in A/\varphi(u)$ such that $a\in e_1$, $b\in e_2$, $e_1, e_2\subseteq d$ and $e_1\neq e_2$. Further, let us take the Euler cycle $g_0^d, g_1^d,\dots,g_n^d=g_0^d$ used in the construction of σ_d^*. By Claim 7.2(b), there are $i, j\in\{0,\dots,n-1\}$, $i\neq j$, such that $g^d=g_i^d=g_j^d$ and $\sigma_d^*(g_i^d, g_{i+1}^d)=b_{e_1}$, $\sigma_d^*(g_j^d, g_{j-1}^d)=a_{e_2}$. Suppose, without loss of generality, $i<j$. Define an R-chain $g_i, g_{i+1},\dots,g_j$, where $g_k=(g_k^f)_{f\in A/\varphi(v)}$, and

$$g_k^f=g^f \quad \text{if} \quad f\neq d,$$
$$g_k^f=g_k^d \quad \text{if} \quad f=d.$$

(This is a chain, of which $g_i^d, g_{i+1}^d,\dots,g_j^d$ is the projection on the dth coordinate.)

By definition of product and Claim 7.2(a) we have

$$\text{val-}\varphi,\sigma^*(g_i, g_{i+1},\dots,g_j)=\text{val-}\varphi,\sigma_d^*(g_i^d, g_{i+1}^d,\dots,g_j^d)=u.$$

Moreover, $a, b_{e_1}\in e_1$ and $b, a_{e_2}\in e_2$, so $\varphi^-(a,\sigma^*(g_i, g_{i+1}))=\varphi^-(a,b_{e_1})\leqslant u$ and $\varphi^-(b,\sigma^*(g_j,g_{j-1}))=\varphi^-(b,a_{e_2})\leqslant u$. Therefore

$$\text{val-}\varphi,\sigma^*(a;g_i,g_{i+1},\dots,g_j;b)\leqslant u,$$

whence $[(a,g),(b,g)]\in\psi^*(u)$.

THEOREM 7.10. *Every lattice is embeddable.*

Proof. If L is embeddable, $u, v\in L$ and $u=0_L<v$, then $L_{u,v}=\{x\in L: x\geqslant v\}$ and it is embeddable being a sublattice of an embeddable lattice L. If $0_L<u$, then $\psi^*: L\to \mathrm{Eq}(A\times G)$, constructed in this paragraph, satisfies conditions (1)–(3) of Lemma 2.1, hence $\psi_{u,v}^*$ is embedding of $L_{u,v}$ into $\mathrm{Eq}(A\times G)$. Thus the class of embeddable lattices is closed under the operation $L\mapsto L_{u,v}$. Since it contains all Boolean lattices, it is by Theorem 1.3 the class of all lattices.

Conclusions

There are many questions, which arise in connection with the theorem presented. In general, we would like to know more about the class of embeddings of a given lattice in the lattices of all equivalences over finite sets. Some of these problems are studied in [4]. In this paper, an embedding is called normal, if it preserves 0 and 1. Using regraphs, our result can be easily improved as follows:

THEOREM. *For every lattice L, there exists a positive integer* n_0*, such that for every* $n \geq n_0$*, there is a normal embedding* $\varphi : L \to \mathrm{Eq}(A)$*, where* $|A| = n$.

Embeddings satisfying special properties are shown in Lemma 3.2 and Basic Lemma 6.2. We hope that our method of regraph powers will produce other interesting results.

There is also a question about the effectiveness of finding an embedding of a given lattice. In particular, the proof presented here cannot be directly used to solve the following

Problem. Can the dual of Eq (4) be embedded into Eq (2^{1000})?

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