Free algebras

1. The concept

Definition. Let V be a variety. The algebra F is *free* in V on $g_1, \ldots, g_n$ if

(i) $F \in V$,

(ii) F is generated by $g_1, \ldots, g_n$, and

(iii) the *only* term relations holding between $g_1, \ldots, g_n$ are those that hold for *all* n-tuples in *all* algebras in V, i.e., are the laws holding in V.

(In examples generators may also be labeled g, h, k or a, b, c, etc.)

2. Examples

#1. In a diagram of the free distributive lattice FDL(3) (Figure 1), if the generators are g_1, g_2, g_3 you can see that

$$(g_1 \vee g_2) \wedge (g_1 \vee g_3) \wedge (g_2 \vee g_3) = (g_1 \wedge g_2) \vee (g_1 \wedge g_3) \vee (g_2 \wedge g_3).$$

Once it is known that this lattice is indeed a free distributive lattice on three generators, then it follows that this law holds in all distributive lattices:

$$(x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$$

#2. The free Boolean algebra FBA(3), corresponding to a Venn diagram with three circles. It has 8 atoms and 256 elements.

#3. The free modular lattice FML(3) shown in Figure 2. It has 28 elements.

#4. The free lattice FL(3) shown in Figure 3. It is infinite. Dashed lines represent infinitely many elements not shown.

#5. The free abelian group on n generators is $\mathbf{Z}^n$.

#6. The free group FG(2) consists of all finite expressions such as $g^2 h^{-3} g h^2$, with appropriate equalities.

#7. Every vector space is free, with generators being any basis.

#8. For a given type τ, the *term algebra* $T_\tau(n)$ is the set of all n-ary terms of type τ, with operations being formal compositions. The generators are the variable symbols $x_1, \ldots, x_n$.

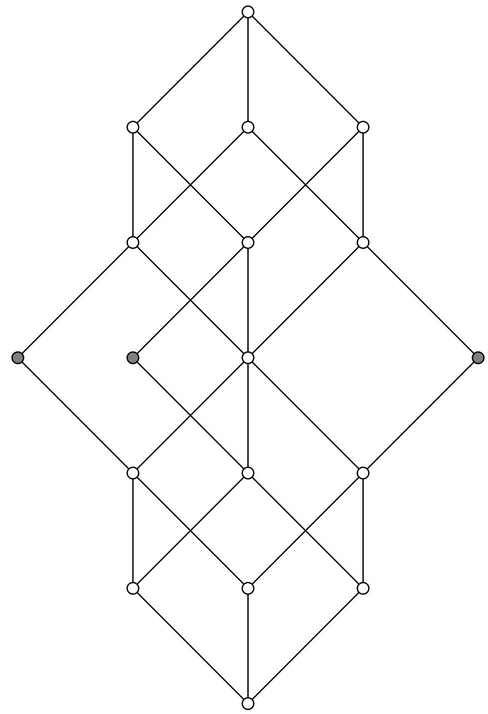

Figure 1: FDL(3)

3. The universal mapping property

Proposition. If F is free in V on $g_1, \ldots, g_n$ and A is any algebra in V and $a_1, \ldots, a_n \in A$, then there is a unique homomorphism $\phi : F \to A$ with $f(g_i) = a_i$ for each i. (In other words, you can aim the generators of F at any elements of any algebra in V and find a homomorphism that takes the generators there.)

Corollary 1. Up to isomorphism, there is only one free algebra in V on n generators.

Let us call this algebra $F_V(n)$.

Corollary 2. Every n-generated algebra of V is a homomorphic image of $F_V(n)$.

Corollary 3. If $F_V(n)$ is finite, then it is the largest n-generated algebra in V, and the only one of its size (up to isomorphism).

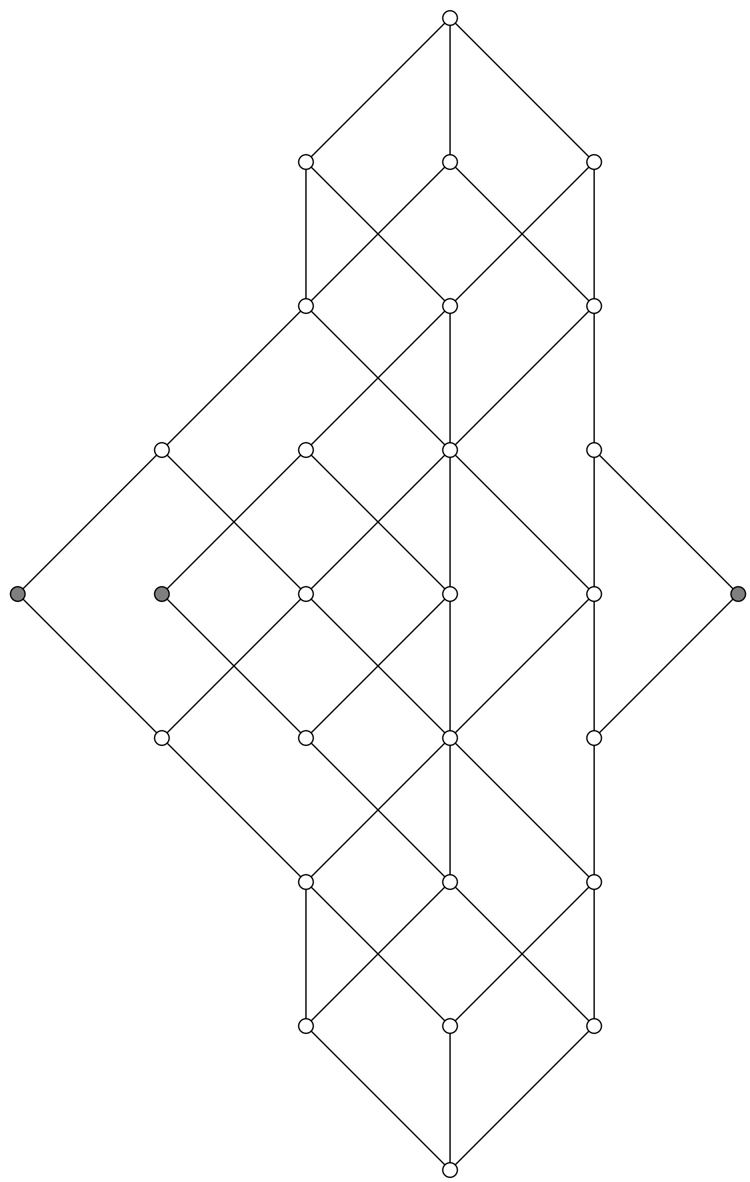

Figure 2: FML(3)

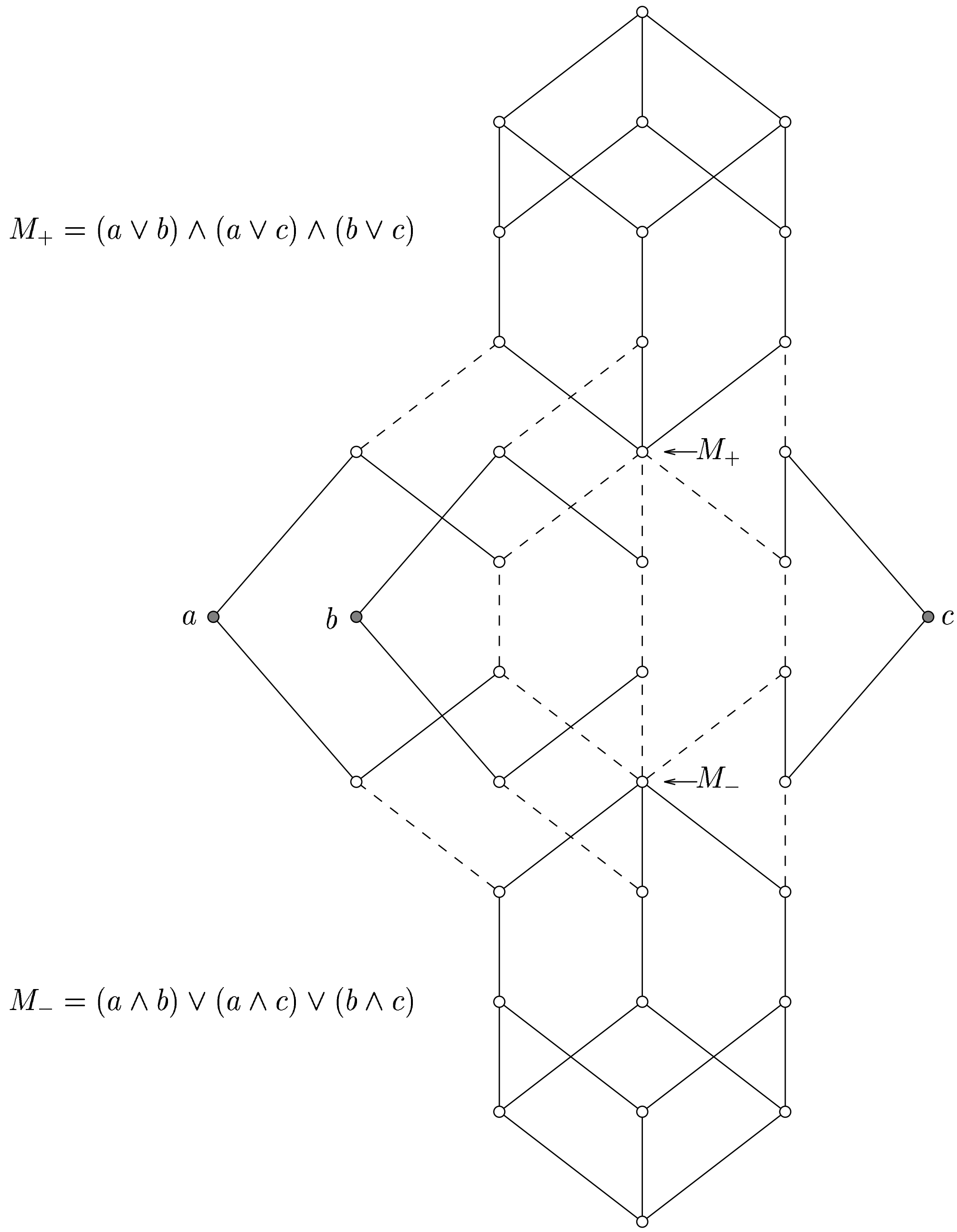
$M_+ = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$
$\leftarrow M_+$
a
b
c
$\leftarrow M_-$
$M_- = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$

Figure 3: FL(3)

4. Existence of free algebras in $V = \mathbf{Var}(A)$

Let the free algebra on n generators in Var(A) be denoted $F_A(n)$.

Theorem (Birkhoff) $F_A(n)$ can be constructed as follows:

Let Δ be the set of all functions $\delta : \{1, \dots n\} \to A$, and let $P = A^\Delta$.

For $i = 1, \dots, n$ let $g_i \in P$ be the element whose δ-th coordinate is $\delta(i)$.

Let F be the subalgebra of P generated by $g_1, \dots, g_n$.

Then $F = F_A(n)$.

Example. To generate $F_{\mathbf{2}}(3)$ (= FDL(3)), where **2** is the 2-element lattice, proceed as shown in Figure 4.

Row	coordinate values								using	expression
1:	0	1	0	1	0	1	0	1	gen	g
2:	0	0	1	1	0	0	1	1	gen	h
3:	0	0	0	0	1	1	1	1	gen	k
4:	0	0	0	1	0	0	0	1	$2 \wedge 1$	$g \wedge h$
5:	0	1	1	1	0	1	1	1	$2 \vee 1$	$g \vee h$
6:	0	0	0	0	0	1	0	1	$3 \wedge 1$	$g \wedge k$
7:	0	1	0	1	1	1	1	1	$3 \vee 1$	$g \vee k$
8:	0	0	0	0	0	0	1	1	$3 \wedge 2$	$h \wedge k$
9:	0	0	1	1	1	1	1	1	$3 \vee 2$	$h \vee k$
10:	0	0	0	0	0	0	0	1	$4 \wedge 3$	$g \wedge h \wedge k$
11:	0	0	0	1	1	1	1	1	$4 \vee 3$	$(g \wedge h) \vee k$
12:	0	0	0	0	0	1	1	1	$5 \wedge 3$	$(g \vee h) \wedge k$
13:	0	1	1	1	1	1	1	1	$5 \vee 3$	$g \vee h \vee k$
14:	0	0	1	1	0	1	1	1	$6 \vee 2$	$(g \wedge k) \vee h$
15:	0	0	0	1	0	1	0	1	$6 \vee 4$	$(g \wedge h) \vee (g \wedge k)$
16:	0	0	0	1	0	0	1	1	$7 \wedge 2$	$(g \vee k) \wedge h$
17:	0	1	0	1	0	1	1	1	$7 \wedge 5$	$(g \vee h) \wedge (g \vee k)$
18:	0	0	0	1	0	1	1	1	$11 \wedge 5$	$((g \wedge h) \vee k) \wedge (g \vee h)$

Figure 4: Construction of FDL(3) as $F_{\mathbf{2}}(3)$

As another example, Figure 5 shows the table obtain for $A = \mathbf{Z}_3$ under subtraction and for $n = 2$:

The rows form the free algebra $F_A(2)$ inside A^9. Of course, this example is really a disguised version of an additive group.

row	9-tuple	from?	expr
R1	0 1 2 0 1 2 0 1 2	gen	g
R2	0 0 0 1 1 1 2 2 2	gen	h
R3	0 0 0 0 0 0 0 0 0	R1–R1	$g-g$
R4	0 1 2 2 0 1 1 2 0	R1–R2	$g-h$
R5	0 2 1 1 0 2 2 1 0	R2–R1	$h-g$
R6	0 2 1 2 1 0 1 0 2	R1–R5	$g-(h-g)$
R7	0 2 1 0 2 1 0 2 1	R3–R1	$(g-g)-g$
R8	0 0 0 2 2 2 1 1 1	R3–R2	$(g-g)-h$
R9	0 1 2 1 2 0 2 0 1	R4–R2	$(g-h)-h$

Figure 5: Construction of $F_{\mathbf{Z}_3}(2)$ under subtraction

5. Existence of free algebras in arbitrary varieties

Proposition. For every variety V and every n there exists a free algebra in V with n generators. In other words, $F_V(n)$ always exists.

Outline of proof #1: The method of saving term relations in common.

This is a generalization of the "table" method (above) for a single algebra: We start by considering all functions $\delta : \{1,\ldots,n\} \to A$ where A runs through all algebras in V. Since V is too large to be a set, there are also too many δ's, so we restrict our attention to cases where the image of δ generates A, and we remark that up to isomorphism there is only a set (rather than a class) of ways in which an image of such a δ can sit inside the A it generates. Let Δ consist of one δ from each isomorphism class. Then inside A^Δ, for $i = 1,\ldots n$ let g_i be the element whose δ-th coordinate is $\delta(i)$, and let F be the subalgebra of A^Δ generated by $g_1,\ldots,g_n$. Then we remark that F has the Universal Mapping Property (UMP), so is free. I call this the method of "saving relations in common", because the only relations $t = u$ between the g_i are those true in every factor, and the factors account for all ways that n elements of an algebra in V can be related. As you see, there are two elements in this proof: choosing the isomorphism types and taking the subalgebra of a product.

Outline of proof #2: The method of overshooting.

For $T = T_\tau(n)$ (the algebra of all terms in n variables), let $F_V(n) = T/\theta_0$, where $\theta_0 = \cap\{\theta \in \mathrm{Con}(T) : T/\theta \in V\}$. Here θ_0 is the least congruence relation θ on t such that $T/\theta \in V$. One can show that $F_V(n)$ inherits the UMP from T, which is free in the variety of all algebras of type τ. I call this the "overshooting" method: Since T is free but much too big, you have overshot, and you must trim T down to where it fits in V, by taking T modulo a congruence relation.

6. Infinite generating sets

Everything discussed above works for the case of infinite generating sets $g_i, i \in \alpha$, where α represents any cardinal number. For example, we can make $F_V(\aleph_0)$. Even for infinitely many generators, though, every term t still involves only finitely many of the variables.

7. Application to construction of varieties

For a class $\mathcal{K}$ of similar algebras, let $\mathbf{S}(\mathcal{K})$, $\mathbf{P}(\mathcal{K})$, and $\mathbf{H}(\mathcal{K})$ denote the classes constructed from $\mathcal{K}$ by taking respectively subalgebras, products, and homomorphic images of members of $\mathcal{K}$.

Theorem (G. Birkhoff) A class $\mathcal{V}$ of similar algebras is a variety if and only if $\mathcal{V}$ is closed under $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{H}$.

Corollary (Birkhoff-Tarski) For any class $\mathcal{K}$ of similar algebras, $\mathrm{Var}(\mathcal{K})$ (the smallest variety containing $\mathcal{K}$) is obtainable as $\mathrm{Var}(\mathcal{K}) = \mathbf{HSP}(\mathcal{K})$, meaning $\mathbf{H}(\mathbf{S}(\mathbf{P}(\mathcal{K})))$.

8. The free 2-generated group in the quaternion group variety (to be discussed in lecture)

Let $F = F_V(2)$ for $V = \mathrm{Var}(D_8) = \mathrm{Var}(Q_8)$.

Laws determining V are $x^4 = e$, $x^2y = yx^2$.

Let a, b be generators of F and let $c = (ab)^{-1}$.

Every element of F has the form $a^ib^ja^{2k}b^{2\ell}c^{2m}$, where $0 \le i, j, k, \ell, m \le 1$.

F is the semidirect product of $\mathbf{Z}_2 \times \mathbf{Z}_4$ by $\mathbf{Z}_4$ via powers of $\sigma(u, v) = (u+v, v)$. See Figure 6.

9. Problems

(Some of these problems depend on additional material from lectures.)

Problem C-1. Describe (a) the free 1-unary algebra on n generators;

(b) $F_V(2)$, where V is the variety of 1-unary algebras with $f^3(x) = f^5(x)$;

(c) the free 2-unary algebra on 1 generator;

(d) $F_S(3)$, where $S = \langle \mathbf{2}, \vee \rangle$, using the table method. (Here S is a *semilattice*—a set with a single binary operation that is associative, commutative, and idempotent. A semilattice can also be defined as a set with a partial order such that any two elements have a least upper bound. Thus one way to obtain a semilattice is to take a lattice and ignore the meet operation, as has been done to make S.)

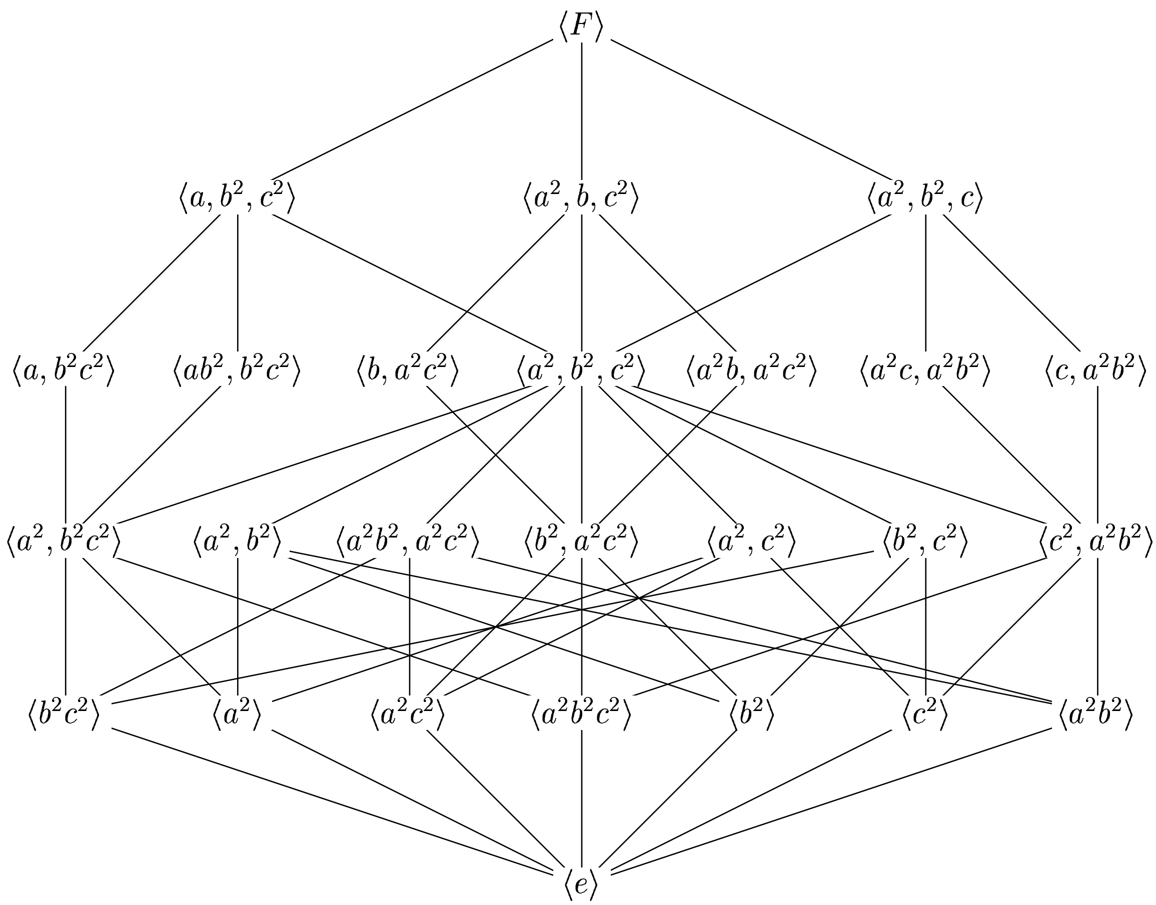

Figure 6: Con(F), the lattice of normal subgroups of F

Problem C-2. *Theorem.* In a group G, every commutator $a^{-1}b^{-1}ab$ is a product of squares.

Proof #1. Let $S = \{\text{products of squares}\}$. Observe that S is a normal subgroup. Moreover, G/S satisfies $x^2 = e$ and so is abelian. Then in G/S, $\overline{a}^{-1}\overline{b}^{-1}\overline{a}\overline{b} = \overline{e}$. This is the same as saying that in G, $a^{-1}b^{-1}ab \in S$.

This proof was indirect. A more direct proof would be to exhibit a law $x^{-1}y^{-1}xy = (\dots)^2(\dots)^2\dots(\dots)^2$ true in all groups, where each $(\dots)$ contains some expression in x, y.

(a) Before attempting to *give* such a proof, explain why there *must* exist a direct proof of this form.

(b) Somehow or other, find the direct proof.

Problem C-3. For Murskii's algebra M, suppose you want to compute $F_M(2)$, using the table method. (a) Show what generating rows you would use. (b) Compute new rows in some reasonable order, labeling each row with the expression in the generators that produced it, until you generate a row that is already there. What law have you found? (c) If your law was in one variable, continue further until you get a law involving two variables. (d) Actually, $F_M(2)$ has 11 elements. How many multiplications of rows would be involved in computing the whole free algebra and verifying that you are done?

Problem C-4. Two proofs of the existence of the free algebra $F_V(n)$ are described in §6 above. They sound very different. Nevertheless, they are essentially the same. The problem: Explain why, by analyzing how the two elements of the first one are really present in the second.

Problem C-5. (a) Suppose that an algebra F has a given set of generators $g_1, \dots, g_n$. Show that if F has the universal mapping property for maps into *itself*, then F is free in *some* variety V. (Thus being free is in effect an absolute property of an algebra, without having to name a variety containing it.)

(b) An achievement of recent years was the solution of the restricted Burnside problem: For any k and n, there is a largest finite group with n generators that obeys $x^k = 1$. (There could also be infinite groups fitting this description; it's just that there is a largest finite one.) Is this largest finite group necessarily free? (Discuss.)

Problem C-6. Let V be the variety of *idempotent semigroups*: 1-binary algebras whose operation is associative and obeys the law $x^2 = x$.

By experimenting with expressions, make a conjecture as to whether $F_V(3)$ is finite or infinite. Explain briefly how you arrived at your answer.

Problem C-7. The term algebra $T_\tau(n)$ is described in §3 above; in §6 it is used in the second proof of the existence of free algebras in a variety.

For the variety V of 1-unary algebras obeying the law $f^3(x) = f^5(x)$ and for $n = 1$, explicitly describe $T_\tau(1)$ and all $\theta \in \mathrm{Con}(T_\tau(1))$ giving a quotient in V. (Here $\tau = \langle 1 \rangle$.)

Problem C-8. Consider the "constructions" $\mathbf{H}, \mathbf{S}, \mathbf{P}$ on classes of algebras.

(a) Say which containment relations between pairs of constructions must hold, e.g., $\mathbf{SH}(\mathcal{K}) \subseteq \mathbf{HS}(\mathcal{K})$. (All the valid relations have easy proofs, but it is not required to write them down. Interpret $\mathbf{H}, \mathbf{S}, \mathbf{P}$ up to isomorphism.)

(b) For one such potential relation that does *not* hold, find a counterexample, with brief proof.

Problem C-9. Let $F = F_{Q_8}(2)$. Refer to Figure 6. (a) Find a normal subgroup N of F such that $F/N \cong \mathbf{Z}_2 \times \mathbf{Z}_4$. (b) Find a normal subgroup N such that $F/N \cong D_8$. Find a normal subgroup N such that $F/N \cong Q_8$. Find the commutator subgroup F' of F. (Determine the order of each subgroup. Recall the Correspondence Theorem, which says that the subgroups of F that contain N form the same diagram as the subgroups of F/N; the same is true if just normal subgroups are considered. From the previous problem you know that for abelian 2-groups (groups whose order is a power of 2), the group can be identified from the subgroup diagram. Recall that F' is contained in every N for which F/N is abelian.)

Problem C-10. Figure 7 shows homomorphisms of FML(3) onto FDL(3) and M_3, determined by mapping generators to generators.

On a copy of Figure 7, indicate ker α and ker β. (You will need to decide which elements go to which, but you need not write this information down. A congruence relation on a finite lattice is best diagrammed simply by darkening the coverings that are "collapsed", i.e., coverings between elements in the same block. Use different coloring or markings for the two congruence relations involved.)

Note. If there are surjections $A \to B$ and $A \to C$ whose kernels have intersection 0, then A is embeddable in $B \times C$, as we'll discuss in class. Since this is the case in Figure 7, you have shown the interesting fact that FML(3) is embeddable in the direct product of FDL(3) and a single copy of M_3.

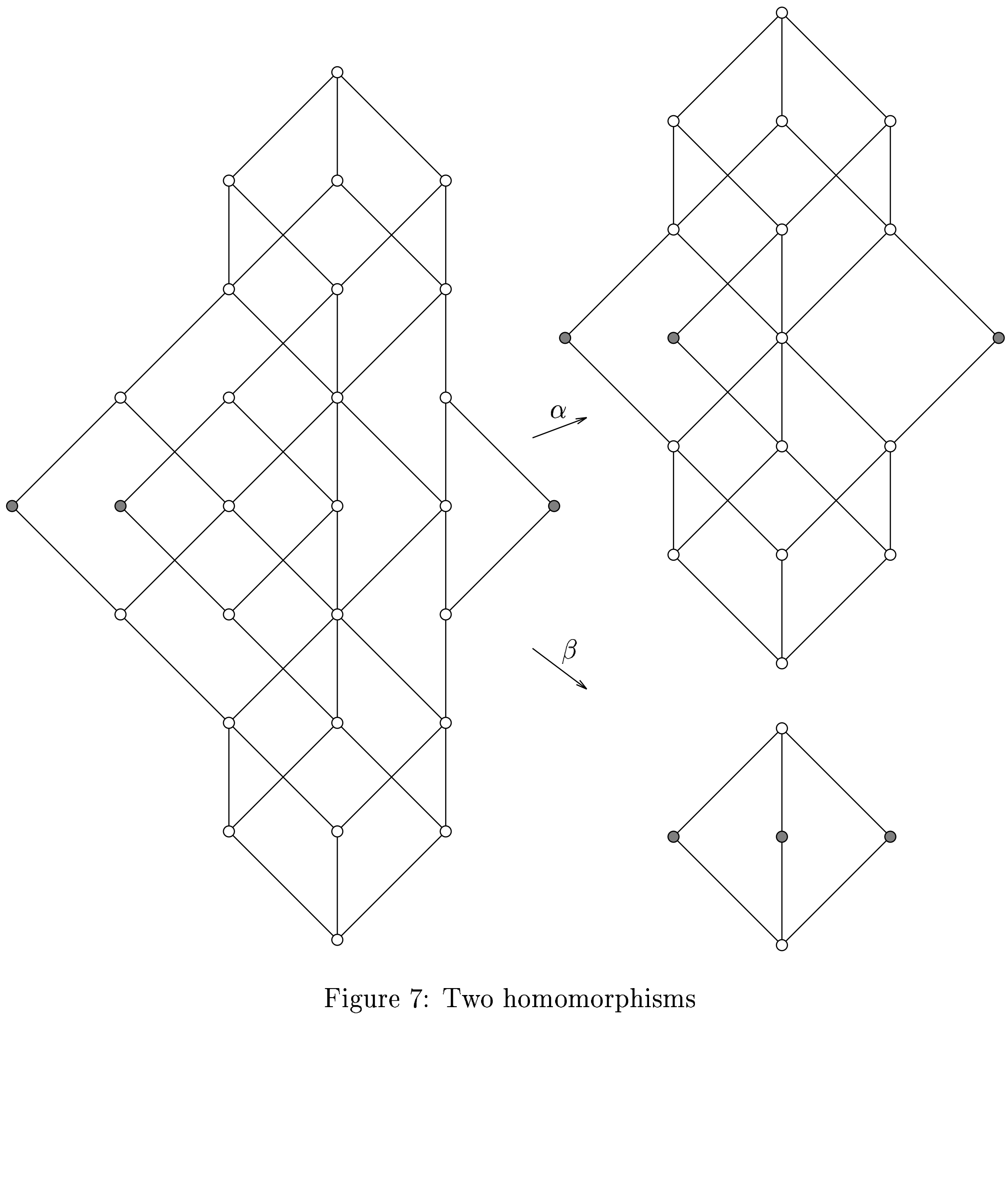

Figure 7: Two homomorphisms

Problem C-11. For the free algebra from the table shown in Figure 5:

(a) Whenever we subtract two rows we get a relation between generators, which is then a law, usually nontrivial. What relation between generators, and so what law, comes from the computation R8-R9 $= 0\ 2\ 1\ 1\ 0\ 2\ 2\ 1\ 0 =$ R5, where R8 means row 8, etc.?

(b) Suppose we want to use the universal mapping property to map F to A with $g \mapsto 2$, $h \mapsto 1$. Which column of the table gives the projection that achieves this, and what is the homomorphism on F?