Notes on Concrete Representations

Ralph Freese

July 6, 2009

These lectures will start with preliminaries (mostly below), then cover the Pálfy-Pudlák [3] reduction of the finite (abstract) congruence representation problem to the problem of representing finite lattices as intervals in finite groups. Finally we will consider McKenzie's [1] results which put restrictions on $\mathrm{Pol}(\mathbf{A})$ on algebras with certain congruence lattices.

We begin with a simple lemma.

Lemma 1. *If F is a set of operations on A, then*

$$\mathbf{Con}(\langle A, F\rangle) = \mathbf{Con}(\langle A, F'\rangle)$$

where F' is any of $\mathrm{Pol}(\mathbf{A})$, $\mathrm{Pol}_1(\mathbf{A})$ or the set of basic translations (operations in $\mathrm{Pol}_1(\mathbf{A})$ obtained from F by fixing all but one coordinate).

The students should review Bill's *Notes on* **G***-sets*, available on my 619 webpage.

Lemma 2. *If $\mathbf{A}$ is a unary algebra such that the nonconstant operations form a group $\mathbf{G}$ that acts transitively on A and $a \in A$, then $\mathbf{Con}(\mathbf{A})$ is isomorphic to the interval $[\mathbf{G}_a, \mathbf{G}]$ of the subgroup lattice of $\mathbf{G}$, where $\mathbf{G}_a = \{g \in G : g(a) = a\}$ (the stabilizer of a).*

If $\theta \in \mathbf{Con}(\mathbf{A})$, the isomorphism is $\theta \mapsto \mathbf{G}_\theta = \{g \in G : g(a)\ \theta\ a\}$. The student should verify this.

The problem of characterizing $\mathbf{Con}(\langle A, G\rangle)$ when $\mathbf{G}$ is intransitive seems open. You might use the Universal Algebra Calculator to find the congruence lattice of the algebra on a 5 element set whose only operation is the permutation f whose cycle decomposition is $(0\,1)(2\,3\,4)$ and compare it with the congrence lattice of the six element algebra with the operation g with cycle deomposition $(0\,1\,2)(3\,4\,5)$.

If $B \subseteq A$, we define an algebra $\mathbf{B} = \mathbf{A}|_B$ on B with operations $\mathrm{Pol}(\mathbf{A})|_B$ which consists of those operations f of $\mathrm{Pol}(\mathbf{A})$ which map $B^k \to B$, where k is the arity of f.

If $e \in \mathrm{Pol}_1(\mathbf{A})$ is idempotent, that is, $e(e(x)) = e(x)$, and $B = e(A)$, then the operations of $\mathbf{A}|_B$ are $\hat{f}(\mathbf{b}) = e(f(\mathbf{b}))$, where $f \in \mathrm{Pol}(\mathbf{A})$.

The next lemma is used in both papers.

Lemma 3. *Let $e \in \mathrm{Pol}_1(\mathbf{A})$ be idempotent, $B = e(A)$, and $\mathbf{B} = \mathbf{A}|_B$ as defined above. Then the map*

$$\theta \mapsto \theta|_B = \theta \cap B \times B$$

defines a (complete) lattice homomorphism of $\mathbf{Con}(\mathbf{A})$ onto $\mathbf{Con}(\mathbf{B})$.

The proof is outlined in [3]. The students should fill in the details.

References

[1] R. McKenzie, *Finite forbidden lattices*, Universal Algebra and Lattice Theory (R. Freese and O. Garcia, eds.), Springer-Verlag, New York, 1983, Lecture notes in Mathematics, vol. **1004**, pp. 176–205.

[2] P. P. Pálfy, *Unary polynomials in algebras I*, Algebra Universalis **18** (1984), 262–273.

[3] P. P. Pálfy and P. Pudlák, *Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups*, Algebra Universalis **11** (1980), 22–27.