Mal'tsev conditions

1. The idea

Based on a theorem of Mal'tsev discussed below, a "Mal'tsev condition" is any condition on a variety that is can be characterized using the existence of terms obeying laws of some sort[1]. Some typical examples are

- V is *congruence-permutable.* In other words, for any $\mathcal{A} \in V$ and any $\theta, \psi \in \mathrm{Con}(A)$, we have $\theta\psi = \psi\theta$.

  Examples: The variety of all groups; the variety of all rings.

- V is *congruence-distributive.* In other words, for any $\mathcal{A} \in V$, $\mathrm{Con}(A)$ is a distributive lattice.

  Example: The variety of all lattices; the variety of all Boolean algebras.

- V is *congruence-modular.* In other words, for any $\mathcal{A} \in V$, $\mathrm{Con}(A)$ is a modular lattice.

  Since the distributive law implies the modular law, any congruence-distributive variety is also congruence-modular. Also, we have:

  Proposition. Any congruence-permutable variety is congruence-modular.

- V is *arithmetic* ("arithmet′ic"). This means that V is both congruence-permutable and congruence-distributive.

  Example: The variety of rings generated by a finite field.

A relevant kind of term: A ternary term $m(x, y, z)$ is said to be a *majority term* for a variety V if V has the laws

$m(x, x, y) = x$, $m(x, y, x) = x$, $m(y, x, x) = x$.

2. Some theorems showing Mal'tsev conditions

2.1 Theorem (Mal'tsev) For a variety V, the following are equivalent:

(a) V is congruence-permutable (i.e., $\theta\phi = \phi\theta$ in congruence lattices of algebras in V);

(b) there is a term $p(x, y, z)$ such that in V these laws hold:

$p(x, x, z) = z$,

$p(x, z, z) = x$.

[1] Mal'tsev, also transliterated Mal'cev, was a famous Russian algebraist.

2.2 Theorem (Pixley) For a variety V, the following are equivalent:

(a) V is arithmetic;

(b) there are terms $p(x,y,z)$ and $m(x,y,z)$ such that in V, p obeys Mal'tsev's laws of (1b) and m is a majority term;

(c) there is a term $q(x,y,z)$ such that in V,

$q(x,x,z) = z$ (minority),

$q(x,z,z) = x$ (minority),

$q(x,y,x) = x$ (majority).

2.3 Theorem (Jónsson) For a variety V, the following are equivalent:

(a) V is congruence-distributive;

(b) for some $n \geq 2$, there are terms $t_0, \ldots, t_n$ in x, y, z such that in V,

(i) $t_0(x,y,z) = x$, $t_n(x,y,z) = z$;

(ii) $t_i(x,y,x) = x$, for all i;

(iii) $t_i(x,x,z) = t_{i+1}(x,x,z)$ for i even, $t_i(x,z,z) = t_{i+1}(x,z,z)$ for i odd.

(Notice that the case $n = 2$ is equivalent to the existence of a majority term.)

2.4 Theorem (Day, Gumm) For a variety V, the following are equivalent:

(a) V is congruence-modular;

(b) for some $n \geq 0$, there are terms $t_0, \ldots, t_n$ and p in x, y, z such that in V,

(i) $t_0(x,y,z) = x$

(ii) $t_i(x,y,x) = x$, for all i;

(iii) $t_i(x,z,z) = t_{i+1}(x,z,z)$ for i even, $t_i(x,x,z) = t_{i+1}(x,x,z)$ for i odd.

(iv) $t_n(x,z,z) = p(x,z,z)$,

(v) $p(x,x,z) = z$.

3. Problems

Problem F-1. Prove Mal'tsev's theorem.

Problem F-2. Prove Pixley's theorem.

Problem F-3. (a) Another Mal'tsev condition: Show that the following are equivalent for a variety V:

- V has a majority term;
- meets of congruences distribute over composition: $\alpha \cap (\beta\gamma) = (\alpha \cap \beta)(\alpha \cap \gamma)$.

(b) Use (a) to show that a variety with a majority term is congruence-distributive (the case $n = 2$ of Jónsson's theorem).