ON n-PERMUTABLE CONGRUENCES

J. HAGEMANN and A. MITSCHKE

In this note we prove a theorem equivalent to the well-known Mal'cev-type-theorem for n-permutable equational classes, but simpler in form.

The result which is stated in [2], [5] and [8] is the following one.

THEOREM 1. *For any equational class* $\mathfrak{A}$ *the following statements are equivalent:*

(a) *The congruence relations of every algebra of* $\mathfrak{A}$ *are n-permutable.*

(b) *There exist* $(n+1)$*-ary algebraic operations* $\bar{p}_0, \dots, \bar{p}_n$ *of* $\mathfrak{A}$ *satisfying the following identities*

$$\begin{aligned}
&\bar{p}_0(x_0, \dots, x_n) = x_0, \\
&\bar{p}_{i-1}(x_0, x_0, x_2, x_2, \dots) = \bar{p}_i(x_0, x_0, x_2, x_2, \dots) \qquad i \text{ even}, \\
&\bar{p}_{i-1}(x_0, x_1, x_1, x_3, x_3, \dots) = \bar{p}_i(x_0, x_1, x_1, x_3, x_3, \dots) \quad i \text{ odd}, \\
&\bar{p}_n(x_0, \dots, x_n) = x_n.
\end{aligned}$$

THEOREM 2. *For any equational class* $\mathfrak{A}$ *the following statements are equivalent:*

(i) *The congruence relations of every algebra of* $\mathfrak{A}$ *are n-permutable.*

(ii) *There exist ternary algebraic operations* $\bar{q}_1, \dots, \bar{q}_{n-1}$ *of* $\mathfrak{A}$ *such that*

$$\begin{aligned}
&\bar{q}_1(x, z, z) = x, \\
&\bar{q}_{i-1}(x, x, z) = \bar{q}_i(x, z, z), \\
&\bar{q}_{n-1}(x, x, z) = z.
\end{aligned}$$

Remark. These algebraic operations are the natural generalization of the well-known Mal'cev condition for permutable classes. H. Werner proved in [7] that the following statements are equivalent for any equational class $\mathfrak{A}$:

(1) *The congruences of each* $A \notin \mathfrak{A}$ *are permutable.*

(2) *For any* $A \in \mathfrak{A}$ *each reflexive subalgebra of* A^2 *is symmetric.*

(3) *For any* $A \in \mathfrak{A}$ *each reflexive subalgebra of* A^2 *is transitive.*

Generalizing this J. Hagemann proved in [3]: For any equational class $\mathfrak{A}$ the following statements are equivalent:

(1) *The statements of Theorem* 2.

(2) *For any* $A \in \mathfrak{A}$ *and each reflexive subalgebra* R *of* A^2

$$R^{-1} \subset R \circ \cdots \circ R \qquad (n-1)\text{-times } R.$$

Presented by G. Grätzer. Received November 1, 1972. Accepted for publication in final form December 18, 1972.

(3) *For any $A \in \mathfrak{A}$ and each reflexive subalgebra R of A^2*

$$\underbrace{R \circ \cdots \circ R}_{n\text{-times}} \subset \underbrace{R \circ \cdots \circ R}_{(n-1)\text{-times}}$$

Proof. We shall prove the equivalence of (ii) and theorem 1 (b).

(ii) $\Rightarrow$ (b). If we define the operations $\bar{p}_0, \ldots, \bar{p}_n$ by

$$\begin{aligned} &\bar{p}_0(x_0, \ldots, x_n) := x_0 \\ &\bar{p}_i(x_0, \ldots, x_n) := \bar{q}_i(x_{i-1}, x_i, x_{i+1}) \qquad 1 \leqslant i \leqslant n-1 \\ &\bar{p}_n(x_0, \ldots, x_n) := x_n, \end{aligned}$$

we get for $2 \leqslant i \leqslant n-1$

$$\begin{aligned} \bar{p}_{i-1}(x_0, x_0, x_2, x_2, \ldots) &= \bar{q}_{i-1}(x_{i-1}, x_{i-1}, x_i) \\ &= \bar{q}_i(x_{i-1}, x_i, x_i) \qquad i \text{ even} \\ &= \bar{p}_i(x_0, x_0, x_2, x_2, \ldots) \\ \bar{p}_{i-1}(x_0, x_1, x_1, x_3, x_3, \ldots) &= \bar{q}_{i-1}(x_{i-1}, x_{i-1}, x_i) \\ &= \bar{q}_i(x_{i-1}, x_i, x_i) \qquad i \text{ odd} \\ &= \bar{p}_i(x_0, x_1, x_1, x_3, x_3, \ldots) \end{aligned}$$

because in both cases $x_{i-2} = x_{i-1}$ and $x_i = x_{i+1}$ and condition (ii) can be applied.

Moreover we have

$$\bar{p}_0(x_0, x_1, x_1, x_3, x_3, \ldots) = x_0$$

by definition and

$$\bar{p}_1(x_0, x_1, x_1, x_3, x_3, \ldots) = \bar{q}_1(x_0, x_1, x_1) = x_0 \quad \text{by} \quad \text{(ii)}.$$

From the above formulae we see that it does not matter if n is odd or even. In both cases we get by (ii) $\bar{q}_{n-1}(x_{n-1}, x_{n-1}, x_n) = x_n$.

(b) $\Rightarrow$ (ii). We define for $1 \leqslant i \leqslant n-1$

$$\bar{q}_i(x, y, z) := \bar{p}_i(\underbrace{x, \ldots, x}_{i\text{-times}}, y, \underbrace{z, \ldots, z}_{(n-i)\text{-times}})$$

Then we get

$$\begin{aligned} \bar{q}_1(x, z, z) &= \bar{p}_1(x, z, z, \ldots, z) \\ &= \bar{p}_0(x, z, z, \ldots, z) \qquad \text{by (b)} \\ &= x \end{aligned}$$

and for $2 \leqslant i \leqslant n-1$

$$\begin{aligned}\bar{q}_{i-1}(x, x, z) &= \bar{p}_{i-1}(\underbrace{x, \ldots, x}_{i\text{-times}}, \underbrace{z, \ldots, z}_{(n+1-i)\text{-times}}) \\ &= \bar{p}_i(\underbrace{x, \ldots, x}_{i\text{-times}}, \underbrace{z, \ldots, z}_{(n+1-i)\text{-times}}) \\ &= \bar{q}_i(x, z, z)\end{aligned}$$

because in both cases i even or odd the formula of (b) can be applied. Moreover we get by the above argument

$$\begin{aligned}\bar{q}_{n-1}(x, x, z) &= \bar{p}_{n-1}(x, \ldots, x, z) \\ &= \bar{p}_n(x, \ldots, x, z) \\ &= z\end{aligned}$$

which completes the proof.

Now we investigate the few known concrete examples. Using theorem 2 (ii) the operations can be defined more symmetrically.

One example for $(n+1)$-permutable equational classes has been given by E. T. Schmidt [6]. He defines an *n-Boolean algebra* $\underline{B} = (B, \vee, \wedge, f_1, \ldots, f_n, u_0, \ldots, u_n)$ of type $(2, 2, \underbrace{1, \ldots, 1}_{n\text{-times}}, \underbrace{0, \ldots, 0}_{(n+1)\text{-times}})$ by the following conditions:

$(B, \vee, \wedge)$ is a distributive lattice and the equations

$$\begin{aligned}&x \vee u_0 = x, \\ &x \vee u_n = u_n, \\ &[(x \vee u_{i-1}) \wedge u_i] \vee f_i(x) = u_i, \\ &[(x \vee u_{i-1}) \wedge u_i] \wedge f_i(x) = u_{i-1}\end{aligned}$$

are valid for $\underline{B}$.

One easily verifies the equations $f_i(x) \vee x = u_i \vee x$ and $f_i(x) \wedge x = u_{i-1} \wedge x$.

We now define for $1 \leqslant i \leqslant n$ the operations

$$\bar{p}_i(x, y, z) := [x \wedge (f_{n-i+1}(y) \vee z)] \vee [z \wedge (f_i(y) \vee x)]$$

and get

$$\begin{aligned}\bar{p}_1(x, z, z) &= [x \wedge (f_n(z) \vee z)] \vee [z \wedge (f_1(z) \vee x)] \\ &= (x \wedge u_n) \vee (z \wedge u_0) \vee (z \wedge x) \\ &= x, \\ \bar{p}_{i-1}(x, x, z) &= [x \wedge (f_{n-i+2}(x) \vee z)] \vee [z \wedge (f_{i-1}(x) \vee x)] \\ &= (x \wedge u_{n-i+1}) \vee (x \wedge z) \vee [z \wedge (u_{i-1} \vee x)] \\ &= [x \wedge (u_{n-i+1} \vee z)] \vee (z \wedge u_{i-1}) \vee (z \wedge x) \\ &= [x \wedge (f_{n-i+1}(z) \vee z)] \vee (z \wedge f_i(z)) \vee (z \wedge x) \\ &= [x \wedge (f_{n-i+1}(z) \vee z)] \vee [z \wedge (f_i(z) \vee x)] \\ &= \bar{p}_i(x, z, z)\end{aligned}$$

and

$$\begin{aligned}\bar{p}_n(x, x, z) &= [x \wedge (f_1(x) \vee z)] \vee [z \wedge (f_n(x) \vee x)] \\ &= (x \wedge u_0) \vee (x \wedge z) \vee (z \wedge u_n) \\ &= z\end{aligned}$$

which are the conditions of theorem 2 (ii) for $n+1$.

There are examples of 3-permutable equational classes.

EXAMPLE 1 ([4]). *Implication algebras.*

An implication algebra is an algebra $(I, \cdot)$ of type (2) which satisfies the following equations

$$\begin{aligned}(xy)\,x &= x\,, \\ (xy)\,y &= (yx)\,x, \\ x\,(yz) &= y\,(xz)\,.\end{aligned}$$

Here we write xy instead of $x \cdot y$.

From the definition one can conclude the existence of a unique element 1 satisfying $x \cdot x = 1$ and $1 \cdot x = x$.

So, if we define ternary operations p, q by

$$\bar{p}(x, y, z) := (zy)\,x \quad \text{and} \quad \bar{q}(x, y, z) := (xy)\,z\,,$$

we get

$$\begin{aligned}\bar{p}(x, z, z) &= (zz)\,x = 1x = x\,, \\ \bar{p}(x, x, z) &= (zx)\,x = (xz)\,z = \bar{q}(x, z, z)\end{aligned}$$

and

$$\bar{q}(x, x, z) = (xx)\,z = z\,.$$

EXAMPLE 2 ([1]). *Right-complemented semigroups.*

A right-complemented semigroup is an algebra $(S, \cdot, *)$ of type (2, 2) satisfying the equations

$$\begin{aligned}x \cdot (x * y) &= y \cdot (y * x)\,, \\ xy * z &= y * (x * z)\,, \\ x \cdot (y * y) &= x\,.\end{aligned}$$

Right-complemented semigroups are 3-permutable as was shown by B. Bosbach according to theorem 1 (b).

Defining $\bar{p}(x, y, z) := x(y*z)$ and $\bar{q}(x, y, z) := z(y*x)$ we get

$$\begin{aligned} \bar{p}(x, z, z) &= x(z*z) = x, \\ \bar{p}(x, x, z) &= x(x*z) \\ &= z(z*x) \\ &= \bar{q}(x, z, z) \end{aligned}$$

and

$$\bar{q}(x, x, z) = z(x*x) = z.$$

This proof even shows that we have a larger class of algebras which is 3-permutable, because we used only two of the three independent axioms.

REFERENCES

[1] B. Bosbach, *Komplementäre Halbgruppen Axiomatik und Arithmetik*, Fund. Math. *64* (1969), 257–287.
[2] G. Grötzer, *Two Mal'cev type theorems in universal algebra*, J. Comb. Theory *8* (1970), 334–342.
[3] J. Hagemann, *Grundlagen der allgemeinen topologischen Algebra*, in preparation.
[4] A Mitschke, *Implication algebras are 3-permutable and 3-distributive*, Algebra Universalis *1* (1971), 182–186.
[5] E. T. Schmidt, *Kongruenzrelationen algebraischer Strukturen*, Math. Forschungsberichte 25, Berlin, 1969.
[6] E. T. Schmidt, *On n-permutable equational classes*, to appear in Acta Math. Hungaricae.
[7] H. Werner, *A Mal'cev condition for admissible relations*, mimeographed note.
[8] R. Wille, *Kongruenzklassengeometrien*, Lecture Notes in Mathematics *113* (1970). Springer-Verlag.

Technische Hochschule Darmstadt
Darmstadt,
West Germany