Survey on Permutohedra and Associahedra

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Cited Works

- Kira Adaricheva, *Stasheff polytope as a sublattice of a permutohedron*, in arxiv.
- Kira Adaricheva, *Almost Distributive Lattices In Connection to Permutation Lattices*, unpublished notes, June 11 2011
- R. Freese, J. Ježek, and J. B. Nation, *Free Lattices*, Mathematical Surveys and Monographs **42**, Amer. Math. Soc., Providence, RI 1995.
- N. Reading, *Cambrian lattices*, Advances in Mathematics **205** (2006) 313-353
- L. Santocanale, F. Wehrung, *Sublattices of associahedra and permutohedra*, in arxiv.

Outline

- Introduction and Review of Bounded Lattices
- Permutohedra, Associahedra, and Cambrian Lattices
- The "French" Paper
- Open Questions

- A lattice L is called *lower bounded (upper bounded)* iff for every finitely generated lattice K and every lattice homomorphism $f\colon K \to L$ the set $f^{-1}(a)$ contains a least (greatest) element for every $a \in L$. If L is both upper bounded and lower bounded, it is called *bounded*.
- $a\ D\ b$ iff $a \neq b$, b is join irreducible, and there is a $p \in L$ with $a \leq b \vee p$ and $a \not\leq c \vee p$ for $c < b$.

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Theorem

A finite lattice L is lower bounded if and only if it contains no D cycle. [Freese, Jezek, and Nation, 2.39, p. 42]

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A finite lattice is bounded if and only if it can be obtained from a one-element lattice by a sequence of applications of Day's doubling applied to intervals. [Freese, Jezek, and Nation, 2.44, p. 44]

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- Let $[n] := \{1, 2, \ldots, n\}$.
- Consider the symmetric group S_n, denote the elements with "one line" notation, i.e. the string $\sigma(1)\sigma(2)\ldots\sigma(n)$ denotes the group element that acts on $[n]$ via $i \mapsto \sigma(i)$.
- $$I(\sigma) := \{(i,j) \in [n] \times [n] : i < j, \sigma(i) > \sigma(j)\}.$$
- Define a partial order on S_n by
$$\sigma \leq \gamma \Leftrightarrow I(\sigma) \subseteq I(\gamma).$$

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- This partial order makes S_n into a lattice (Guildbad and Rosentiehl) that is semidistributive (Duquenne and Cherfouh) and bounded (Caspard) [Adaricheva 1, p. 2].
- These lattices are referred to as *permutation lattices* or *permutohedra*.
- In [Santocanale and Wehrung], S_n with this lattice structure is denoted $P(n)$, we adopt this notation.
- Exercise: Compute $3124 \vee 1342$ and $3124 \wedge 1342$ in the lattice $P(4)$.

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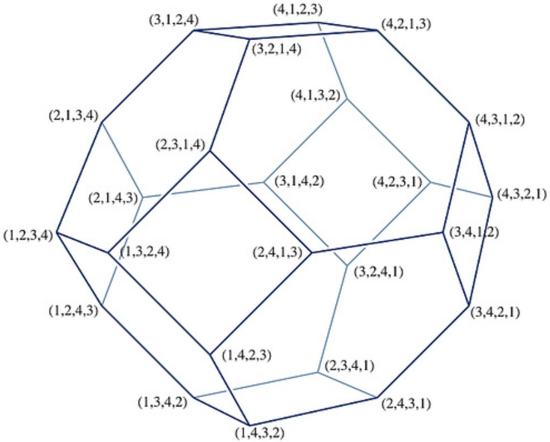
The Permutohedra
P(4)



http://www.flickr.com/photos/ethanhein/2281698129/

- Consider the set of all associative bracketings of n letters, denoted $A(n)$ as in [Santocanale and Wehrung].
- E.g., $A(4) := \{((ab)c)d, (ab)(cd), (a(bc))d, a((bc)d), a(b(cd))\}$.
- Define $\prec$ on $A(n)$ by $\alpha \prec \beta$ if and only if β can be obtained from α by moving exactly one set of brackets to the right.
- The transitive closure of $\prec$ is a partial order that turns $A(n)$ into a lattice for all n (Freedman and Tamari) [Adaricheva 1, p.2].This is called a *Tamari lattice* or an *associahedron*.
- Exercise: Compute $a(((bc)d)e) \vee (a(b(cd)))e$ and $a(((bc)d)e) \wedge (a(b(cd)))e$ in A(5).

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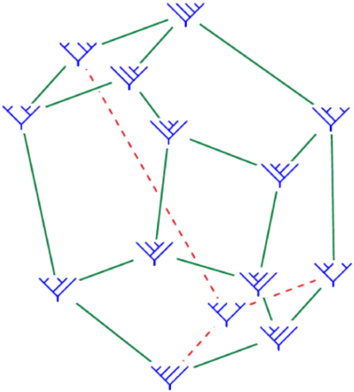

http://golem.ph.utexas.edu/category/2009/08/this_weeks_finds_in_mathematic_39.html

- Every $A(n)$ is semidistributive and bounded (Geyer) [Adaricheva 1, p. 2].
- Every distributive lattice with n join-irreducible elements can be embedded into $A(n+1)$ (Markowsky) [Santocanale and Wehrung, p. 2].
- Geyer conjectured that every finite, bounded lattice could be embedded into $A(n)$ for some n [Santocanale and Wehrung, p. 2]. This conjecture was recently settled in the negative [Santocanale and Wehrung, Theorem 10.7, p. 20].

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- Despite their similarities, the fundamental connection between $P(n)$ and $A(n)$ was not established until recently.

Theorem

For every n, $A(n) \leq P(n)$ (Borner and Wachs). Indeed, $A(n)$ is a retract of $P(n)$ (Reading). Moreover, for every n there is an embedding that preserves the height of elements [Adaricheva 1, Theorem 1, p.3]

- Can every finite bounded lattice be embedded into $P(n)$ for some n? This has been settled in the negative [Santocanale and Wehrung, Theorem 11.1, p. 22].

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- A *Coxeter system* (W, S) is a *Coxeter group* W presented as a generating set S with the following relations:
  $s^2 = 1 \forall s \in S$,
  for every $s, t \in S$ such that $s \neq t$ there exists $m(s, t)$ which is the least integer $2 \leq m(s, t) < \infty$ for which $(st)^{m(s,t)} = 1$.
- A Coxeter system can be encoded in a graph called a *Dynkin diagram* with vertices labeled by S and the edges (s, t) for $m(s, t) \geq 3$ labelled by $m(s, t)$.
- By convention, the edge (u, v) is unlabelled when $m(u, v) = 3$.

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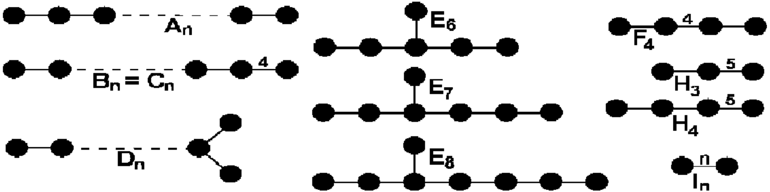

http://en.wikipedia.org/wiki/File:Finite_coxeter.png

- Let $w \in W$ be a reduced group word in the alphabet S. Define the *right inversion set* of w as $I(w) := \{s \in S \colon l(w) < l(ws)\}$.
- Define the *right weak order* on W as the relation

$$v \leq w \Leftrightarrow I(v) \subseteq I(w) \forall v, w \in W.$$

- For any Coxeter group W the weak order makes W a meet-semilattice. If W is finite, the weak order makes W a lattice [Reading, p. 320]

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- Let G be the Dynkin diagram of a Coxeter system (W, S). An *orientation of* G, denoted $\vec{G}$ is a directed graph with vertex set S and a single directed edge for every undirected edge in G.
- Given an orientation $\vec{G}$ define the *Cambrian congruence* associated to $\vec{G}$, denoted $\Theta(\vec{G})$, to be the smallest congruence on the lattice of W with the weak order such that for each directed edge $s \to t$ in $\vec{G}$, $t \equiv tsts \cdots$ with $m(s, t) - 1$ letters.

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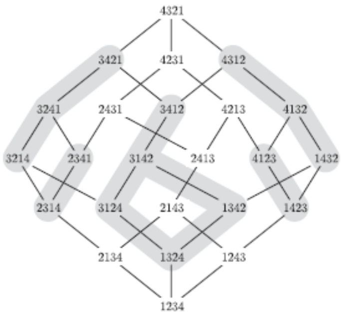
- Define the *Cambrian lattice* $C(\overrightarrow{G}) := W/\Theta(\overrightarrow{G})$.
- Cambrian lattices of the form $C(\overrightarrow{S_n})$ are called *Cambrian lattices of type A*.



Reading, Fig. 2, p. 315

- We will abuse notation and let $\overrightarrow{S_n}$ denote an orientation of the Dynkin diagram for the Coxeter group S_n.

Theorem

For every orientation $\overrightarrow{S_n}$ the Cambrian lattice $C(\overrightarrow{S_n})$ is a sublattice of the weak order on S_n [Reading, Theorem 6.5, p. 336].

- We now have three related classes of finite lattices: associahedra, permutohedra, and Cambrian lattices of type A.

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Theorem

The associahedron $A(n)$ is a Cambrian lattice of type A. [Santocanale and Wehrung, Proposition 5.2, p. 8]

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$C(\overrightarrow{S_n})$ is a lattice theoretical retract of $P(n)$ for every orientation $\overrightarrow{S_n}$. [Santocanale and Wehrung, Proposition 6.4]

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Theorem

Let $O_1, O_2, \ldots, O_k$ be the orientations of S_n. Define $\pi_k : P(n) \twoheadrightarrow C(O_k)$ to be the canonical projection. Every lattice $C(O_k)$ is subdirectly irreducible, and the diagonal map

$$\pi : P(n) \to \prod_{1,2,\ldots,k} O_K, x \mapsto (x/\Theta(O_k))$$

is a subdirect product decomposition of of $P(n)$.
[Santocanale and Wehrung, Proposition 6.7, p 11]

- Define the $B(m, n)$ to be the lattice obtained from the Boolean lattice with $m + n$ atoms by doubling the join of m atoms.

Theorem

The lattice $B(3, 3)$ cannot be embedded into any permutohedron. [Santocanale and Wehrung, Theorem 11.1, p. 22]

- But hope springs eternal...

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The lattice $B(3, 3)$ is the homomorphic image of a sublattice of a permutohedron. [Santocanale and Wehrung, Theorem 12.1, p. 26]

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Open Questions

- Can we prove that every almost distributive lattice is in HS(P(n)) for some n? [Adaricheva 2, p. 2]
- Even better, can we show that every finite lattice obtained by doubling intervals (i.e., every finite bounded lattice) is in $HS(P(n))$ for some n?
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Thank You!