THE McKENZIE-BURRIS CONJECTURE - A SURVEY AND AN APPROACH

by

Alan Day*

§1 THE CONJECTURE

Since Jónsson, [10], characterized distributivity and the author, [1], characterized modularity, no new Mal'cev type characterizations for non-trivial lattice identities have been discovered. This statement is of course not completely true as stated. Gedeonová, [9], obtained such a characterization for p-modularity but this was shown to be equivalent to congruence modularity in [3]. Also, Mederly, [12], characterized n-distributivity, ℓ-modularity and dual ℓ-modularity but showed in the same paper that these implies congruence modularity. At the same time, Nation, [14], showed that if the congruence variety satisfied non-trivial lattice identities of a certain form then the variety was already congruence modular. Nation's general result (and more specific ones for unary algebrqs (Nation [14]) semigroups and semi-lattices (Freese and Nation [8])) led Burris to make a conjecture which McKenzie formulated at the 1973 Lattice Theory conference at Houston, viz:

McKenzie-Burris conjecture: If the congruence variety of a variety of algebras satisfies any non-trivial lattice identity, it is already congruence modular.

Several results have been obtained since that formulation; these will be discussed later along with a possible way to attack this conjecture. It is noteworthy to state now that even congruence modularity is not sacred. Freese has announced in [7] that if a variety of

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algebras is congruence modular it already satisfies more special lattice laws. In fact, Jónsson, in a private communication, has stated that Freese's argument can be modified to produce even the standard Arguesian law.

§2 PROVING CONGRUENCE MODULARITY

As noted originally by Nation, the methods used so far to force congruence modularity involve a reformulation of the author's result in [1].

Theorem: Let K be a variety of algebras. Then the following are equivalent:

(CM1) K is congruence modular

(CM2) $\mathrm{Con}(F_K(4))$ is congruence modular

(CM3) $(a,b) \in \mathrm{con}(c,d) \vee [\mathrm{con}(a,c;b,d) \wedge \mathrm{con}(a,b;c,d)]$

(CM4) There exist quaternary polynomials $a = m_0, m_1, \ldots, m_n = b$ satisfying:

(M1) $m_i(x,x,yy) = x$ (all i)

(M2) $m_i(x,y,z,z) = m_{i+1}(x,y,z,z)$ (i even)

(M3) $m_i(x,y,x,y) = m_{i+1}(x,y,x,y)$ (i odd)

As Nation observed, statement (CM3) of this result is the most useful version in proving congruence modularity. For in all cases known to the author, the proof that some weaker hypothesis implies congruence modularity involves essentially two steps.

Step (1): Derive some polynomial identities in many (say n) variables from the (weaker) hypothesis.

Step (2): Find the correct substitutions from n variables to four variables that will allow one to deduce (CM3).

A cursory look through the literature will perhaps convince one that the second step is not that difficult. However, the literature presents a fait accompli and not the time and ingenuity required to come up with the proper substitutions. A result that might provide more latitude in performing the second step (but so far hasn't) is the following:

Theorem ([4]): Let K be a vareity of algebras. Then K is congruence modular iff (CM5) for any set X, $|X| \geq 5$, and any equivalence relations $\rho,\sigma,\tau \in Eq(X)$, if $\rho \vee \sigma = \tau \vee \sigma$, $\rho \wedge \sigma = \tau \wedge \sigma$ and $\rho < \tau$ then in $F_K(X)$:

$$\tau \subseteq con(\rho) \vee [con(\sigma) \wedge con(\tau)].$$

§3 TACKLING THE CONJECTURE

If the conjecture is true then we need a theorem statement like the following:

(1) For any lattice equation, ϵ, and any variety of algebras, K, if Con $K \models \epsilon$ then K is congruence modular.

The above format has been used in [3], [11], [12] and [14] for special forms of ϵ. To use this format for a general proof however would seem to require a set N of "nice" equations for which two things would hold. Firstly, congruence ϵ would have to imply congruence modularity for all $\epsilon \in N$, and secondly, (perhaps most importantly) one would need that any lattice identity implies an identity in N. The author is unaware of any such set N that might fill these two roles (with the exception of a possibility to be described later).

The aforementioned procedure is essentially a syntactical one. A rephrasing of the conjecture might supply a semantical approach, viz:

(2) If a variety of algebras, K, is not congruence modular, then Con(K) = L.

This approach was attempted in [4] and Freese apparently has some results in this direction also. The problem here though is what "nice" set (or class) of lattices does the job? We must be able to find them in Con(K) and also know that they generate L. One might try the set of all finite partition lattices, and with respect to them Jónsson has asked the following initial question.

<u>Problem 1</u>: If K is not congruence modular, does $\Pi(3)$ ($=M_3$) belong to Con(K)? *

An affirmative solution here would provide more support to the conjecture. One must however keep in mind that even though $M_3 \notin$ Con(S,$\wedge$)

for any $\wedge$-semilattice S (Papert [15]), Freese and Nation, [18], showed that $\mathrm{Con}(S_{\wedge}) = L$.

The author, at present, feels that a strictly syntactical or semantical approach would be very difficult at best and a combination of both is probably necessary. This feeling is of course purely subjective but has some basis in some recent results.

§4 SPLITTING LATTICES

The approach to the McKenzie-Burris conjecture that we present here is based on McKenzie's concept of a splitting lattice (see [13]). Essentially a splitting lattice is a finite subdirectly irreducible lattice, S, which can be paired with a (conjugate) equation, ϵ, such that for any variety of lattices, V, either $V \models \epsilon$ or $S \in V$ but not both.

The relevance of splitting lattices is three fold. Firstly, all of the results known so far can be obtained by using conjugate equations. Secondly, in general the pairing of a lattice with an equation allows a combination syntactic-semantic approach which can utilize both methods to their respective maxima. Thirdly, they satisfy a main requirement of a semantical approach in that:

(4.1) Theorem ([5]): Splitting lattices generate the variety of all lattices.

The following statements are corollaries whose equivalence with the theorem was noted by Kostinsky (see [13]).

(4.2) Corollary 1: In a finitely generated free lattice, every proper quotient contains a prime quotient.

(4.3) Corollary 2: If V is a proper subvariety of lattices then there exists a splitting lattice S such that V satisfies its conjugate equation.

Corollary 3 states that the conjugate equations of splitting lattices satisfy the main requirement for a syntactical approach as well.

Very little seems to be known about the form of these equations however. The known results are:

Theorem (McKenzie [13]): Every conjugate equation for a splitting lattice is equivalent to an equation of the form $p = q$ where q/p is a prime quotient in a finitely generated free lattice.

A lemma which McKenzie communicated to the author provides another result.

Lemma: Every conjugate equation for a non-distributive splitting lattice is equivalent to an equation of the form

$$(p \vee x) \wedge q \leq p \vee (x \wedge q)$$

where $p < q$ and x is not a variable in p nor in q.

Returning to the semantical approach for a moment, a sideline problem develops which may be of interest in its own right.

Problem 2: Is every splitting lattice the sublattice of a finite partition lattice? *

From McKenzie's results, an equivalent formulation would be:

Problem 2*: Is every homomorphic image of a sublattice of a finite partition lattice again such?

§5 THE KNOWN RESULTS IN THE FRAMEWORK OF SPLITTING LATTICES

In order to put the known results in a general setting, we need a construction from [2].

Let A be a lattice, $I = [p,q]$ a closed interval of A and define:

(i) $A[I] = (A \setminus I) \cup (I \times 2)$

(ii) $x \leq y$ (in $A[I]$) iff one of the following hold:

(1) $x,y \in A \setminus I$ and $x \leq y$ in A

(2) $x \in A \setminus I$, $y = (t,j)$ and $x \leq t$ in A

(3) $x = (s,i)$, $y \in A \setminus I$ and $s \leq y$ in A

(4) $x = (s,i)$, $y = (t,j)$, $s \leq t$ in A and $i \leq j$ in $\underset{\sim}{2}$

Intuitively $A[I]$ is formed by splitting in two every element of I. $A[I]$ is a lattice with this order relation and there is a canonical epimorphism

$$\kappa : A[I] \twoheadrightarrow A$$

which collapses the doubled elements of I. We will write $A[p]$ if $I = [p,p]$ is a trivial interval.

Now we define a class of splitting lattices as follows:

$S \in \mathcal{S}_1$ iff S is a finite subdirectly irred. lattice and $S \cong D[p]$ for some finite distributive lattice D and $p \in D$.

That $\mathcal{S}_1$ is a class of splitting lattices will be given in the next section. We will write L/S for the variety generated by a congugate equation for S.

(5.1) Theorem ([6]): For any $S \in S_1$ and any variety of algebras, K, if $\mathrm{Con}(K) \subseteq L/S$ then K is congruence modular, where $\mathrm{Con}(K)$ is the variety of lattices generated by all congruence lattices of members of K.

(5.2) Corollary: The congruence modular implications in [3], Jónsson [11], Mederly [12], and Nation [14] hold.

The corollary holds as the equations considered in all of these results all fail in some member of S_1. Therefore if the congruence variety of a variety satisfies such an equation, it must be contained in some L/S.

§6 A GENERAL APPROACH TO THE CONJECTURE

The key to this hoped-for method to prove the conjecture lies in the proof of (4.1) To prove (4.1), we used a construction that produced for an arbitrary (finite) lattice A, a lattice $\overline{A}$ and an epimorphism $\rho : \overline{A} \twoheadrightarrow A$ that "fixed up" all the places in A where Whitman's condition failed. More precisely:

if $a \wedge b \leq c \vee d$ and $\{a,b,c,d\} \cap [a \wedge b, c \vee d] = \emptyset$ in A,

then for every $\overline{a},\overline{b},\overline{c},\overline{d} \in \overline{A}$ with $\rho(\overline{x}) = x$, $x \in \{a,b,c,d\}$

we have $\overline{a} \wedge \overline{b} \nleq \overline{c} \vee \overline{d}$.

Iterating this construction produced an inverse limit which satisfies Whitman's condition. Moreover, if one starts with a suitable lattice (say FD(3)) one obtains a copy of FL(3) in the inverse limit.

Now how does this iteration relate to the conjecture? If K is a variety of algebras that is not congruence modular then Con(K) contains all finite distributive lattices. However, for each finite distributive lattice, D, $\overline{D}$ is a finite subdirect product of members of S_1 and (5.1) states that $\overline{D} \in$ Con(K). (If K not congruence modular then $S_1 \subseteq$ Con(K).) The hoped for induction to solve the conjecture rests in the following problem.

<u>Problem 3</u>: If K is not congruence modular and a (finite) lattice $A \in$ Con(K), then $\overline{A} \in$ Con(K). *

If the answer to this problem is affirmative, then the McKenzie-Burris conjecture is true. Furthermore, if the conjecture is false, partial solutions to problem 3 may help to determine the equivalence classes of the induced equivalence relation:

For varieties $V_1, V_2 \subseteq L$,

$$V_1 \equiv V_2 \text{ iff for all varieties of algebras } K,$$
$$\mathrm{Con}(K) \subseteq V_1 \text{ iff } \mathrm{Con}(K) \subseteq V_2$$

One should note at this time that problem 3 can be reduced using [5] to the following:

Problem 3*: If K is not congruence modular and a finite lattice $A \in \mathrm{Con}(K)$, then $A[I] \in \mathrm{Con}(K)$ where $I = [a \wedge b, c \vee d]$ is a closed interval of A and $\{a,b,c,d\} \cap I = \emptyset$.

It would be of interest to know if the above problem is equivalent (as it is if A is distributive) to the seemingly weaker statement.

Problem 4: If K is not congruence modular and a finite lattice $A \in \mathrm{Con}(K)$, then $A[p] \in \mathrm{Con}(K)$ for all $p \in A$. *

In conclusion, we should note that an acid test to the conjecture and to problem 3, 3* or 4 is the conjugate equation for N_6 (see fig. i), since $N_6 \cong A[p]$ for A a (finite) subdirect product of N_5 and N_5. This congugate equation is:

$$z \wedge [(x \wedge (w \vee (x \wedge y \wedge z))) \vee (y \wedge z \wedge (w \vee (x \wedge y \wedge z)))] \leq$$
$$y \vee [(x \vee (w \wedge (x \vee y \vee z))) \wedge (y \vee z \vee (w \wedge (x \vee y \vee z)))]$$

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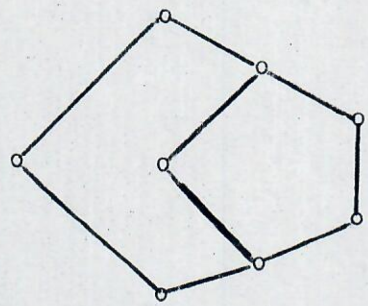

N_6

fig. (i)