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Polin's Non-Modular Congruence Variety

Recently, S. V. Polin constructed a non-modular congruence variety and via a rather circuitous route, I received a rough outline of his extremely neat construction. In the resurrection of Polin's proof from this note, a rather obvious generalization was obtained, one that Polin probably knows as well, which produces countably many distinct non-modular congruence varieties each of which contains no non-distributive modular lattices as in the case with the variety of semilattices. Since the join of congruence varieties is again a congruence variety, a result due to either Ralph Freese or Ralph McKenzie, there are continuumly many non-modular congruence varieties.

The main reason for this manuscript is the possibly long delay before Polin's result will become published. This possibility exists if for no other reason than the fact that Polin's non-finite basis result was known by some people for almost two years and is still, to my knowledge, not in print. The importance of this result seems to demand immediate circulation. The following is the notes from a seminar on Polin's results presented at Vanderbilt, 1977.

§1. The Derived Class $k[\mathcal{B}]$.

All varieties considered here will be varieties of bounded semilattices with operators. That is algebras with one binary, two nullary, and arbitrarily many unary operations (called operators) and where the first three make the algebra a bounded (meet) semilattice. The observation that

$\mathcal{B}$, the variety of all Boolean algebras can be defined this way using $(\cdot, 0, 1, ')$ is the starting point of Polin's construction.

For any Boolean algebra $A \in \mathcal{B}$, $(A, \geq)$ can be considered as a category and therefore for any variety $\mathcal{k}$ one has functors from $(A, \geq)$ into $\mathcal{k}$. If $\mathcal{k}$ is a variety of bounded semilattices with operators then a new class of algebras, also bounded semilattices with operators, can be neatly produced.

For $A \in \mathcal{B}$ and functor $S: (A, \geq) \to \mathcal{k}$ ie a pair $((S(a): a \in A), (\xi^a_b: a,b \in A, a \geq b))$ where $S(a) \in \mathcal{k}$ and for $a \geq b$, $\xi^a_b: S(a) \to S(b)$ satisfying

(1) $\xi^a_a = 1_{S(a)} \quad (a \in A)$

(2) $\xi^a_b \circ \xi^b_c = \xi^a_c \quad (a,b,c \in A, a \geq b \geq c)$

we define:

$L = L(S, A) = \cup \{\{a\} \times S(a): a \in A\}$ and operations on L:

$$(a, s)\cdot(b, t) = (a\cdot b, \xi^a_{ab}(s)\cdot\xi^b_{ab}(t))$$

$$0 = (0, 0_{S(0)})$$

$$1 = (1, 1_{S(1)})$$

$$(a, s)^+ = (a', 1)$$

$(a, s)^o = (a, 0)$ and for all unary f in $\mathcal{k}$

$$\overline{f}\,(a, s) = (a, f(s))$$

$$k[\mathcal{B}] = \underset{\sim}{I}\{L(S, A) : A \in \mathcal{B}\}$$ a class of algebras of type (2, 0, 0, unary)

(1-1) Lemma: $k[\mathcal{B}]$ is a class of bounded semilattices with operators.

Proof: Clearly multiplication is commutative and idempotent. For associativity

$$[(a, s)(b, t)](c, u) = (ab, \xi^a_{ab}(s)\xi^b_{ab}(t))(c, u)$$

$$= (abc, \xi^{ab}_{abc}(\xi^a_{ab}(s)\xi^b_{ab}(t))\cdot\xi^c_{abc}(u))$$

$$= (abc, \xi^{ab}_{abc}\xi^a_{ab}(s)\cdot\xi^{ab}_{abc}\xi^b_{ab}(t)\cdot\xi^c_{abc}(u))$$

$$= (abc, \xi^a_{abc}(s)\cdot\xi^b_{abc}(t)\cdot\xi^c_{abc}(u))$$

$$= (a, s)[(b,t)\ (c, u)].$$

Note that $(a, s) \leq (b, t)$ iff $a \leq b$ and $s \leq \xi^b_a(t)$ and therefore $(0, 0)$ is the zero element and $(1,1)$ is the unit element of $L(S, A)$.

(1-2) Lemma: $\underset{\sim}{P}(k[\mathcal{B}]) \subseteq k[\mathcal{B}]$

Proof: Easily $\Pi(L(S_i, A_i) : i \in I) \simeq L(\prod_{i\in I} S_i, \prod_{i\in I} A_i)$

(1-3) Lemma: $\underset{\sim}{S}(k[\mathcal{B}]) \subseteq k[\mathcal{B}]$

Proof: Let $M \leq L(S, A)$ and define:

$$B = \{a \in A : M \cap \{a\} \times S(a) \neq \phi\}$$

$$T(b) = \{s \in S(b) : (b, s) \in M\} \quad (b \in B)$$

$$\zeta_c^b: T(b) \to T(c) \quad (b, c \in B, b \geq c)$$

$$\text{by} \quad \zeta_c^b(s) = \xi_c^b(s).$$

Standard computation shows that $B \leq A$ and for each $b \in B$ $T(b) \leq S(b)$. Now for $b \in B$, $s \in T(b)$, $b \geq c$ and $c \in B$ we have (b, s) and $(c, 1) \in M$. Therefore $(b, s)\cdot(c, 1) = (bc, \xi_{bc}^b(s)\cdot\xi_{bc}^c(1)) = (c, \xi_c^b(s)) \in M$. Clearly $M = \cup\{\{b\} \times T(b): b \in B\}$ hence $M \cong L(T, B)$.

§2. Representation of Congruences.

Let $\theta \in Con(L(S, A))$ and define:

$$\theta_A = \{(a, b) \in A^2: (a, 1)\theta(b, 1)\}$$

$$\theta_a = \{(s, t) \in S(a)^2: (a, s)\theta(a, t)\} \quad (a \in A).$$

(2-1) Lemma: $\theta_A \in Con(A)$ and $\theta_a \in Con(S(a))$ for all $a \in A$.

Proof: Easy computation.

(2-2) Lemma: $(a, s)\theta(b, t)$ iff $a\theta_A b$ and $\xi_{ab}^a(s)\theta_{ab}\xi_{ab}^b(t)$.

Proof: Assume $(a, s)\theta(b, t)$ then firstly

$$(a, 1) = (a, s)^{++}\theta(b, s)^{++} = (b, 1).$$

Secondly by meeting with $(c, 1)$ we obtain

$$(ac, \xi_{ac}^a(s))\theta(bc, \xi_{bc}^b(t))$$

and by letting c take on the values a and b

$$(ab,\ \xi^{b}_{ab}(\theta))\theta(a,\ s)\theta(b,\ t)\theta(ab,\ \xi^{a}_{ab}(s)).$$

Conversely if $(a,\ 1)\theta(b,\ 1)$ and $(ab,\ \xi^{a}_{ab}(s))\theta(ab,\ \xi^{b}_{ab}(t))$ then meeting with $(c,\ u)$ gives

$$(ac,\ \xi^{c}_{ac}(u))\theta(bc,\ \xi^{c}_{bc}(u)).$$

Using $(c,\ u) \varepsilon \{(a,\ s),\ (b,\ t)\}$ we obtain

$$(a,\ s)\theta(ab,\ \xi^{a}_{ab}(s))\theta(ab,\ \xi^{b}_{ab}(t))\ (b,\ t).$$

(2-3) Lemma: For $a \geq b$, if $s\theta_a t$ then $\xi^{a}_{b}(s)\theta_b\xi^{a}_{b}(t)$.

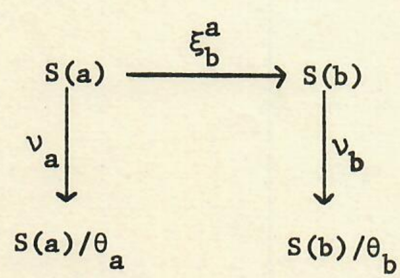
Moreover if $a\theta_A b$ then the reverse implication also holds.

Proof: If $(a,\ s)\theta(a,\ t)$ then by meeting with $(b,\ 1)$ we obtain

$$(b,\ \xi^{a}_{b}(s)) = (ab,\ \xi^{a}_{ab}(s))\theta(ab,\ \xi^{a}_{ab}(t)) = (b,\ \xi^{a}_{b}(t))$$

Conversely if $a\theta_A b$ then since $\xi^{a}_{ab}(s) = \xi^{a}_{b}(s)\theta_{ab}\xi^{a}_{b}(t) = \xi^{a}_{ab}(t)$ we have by (2-2) $(a,s)\theta(a,t)$.

(2-4) Corollary: For $\theta \ \varepsilon \ \mathrm{Con}(L(S,\ A))$ and $a \geq b$ consider the diagram



Then there exists a unique $\zeta^a_b: S(a)/\theta_a \to S(b)/\theta_b$ such that $\nu_b \circ \xi^a_b = \zeta^a_b \circ \nu_a$. If moreover $a\theta_A b$ then ζ^a_b is a monomorphism.

Definition: For algebra $L(S, A) \in k[\mathcal{B}]$, a congruence representation is a member $(\psi; (\psi a)a \in A)$ of $\mathrm{Con}(A) \times \Pi(\mathrm{Con}(S(a)): a \in A)$ satisfying:

(R1) $a \geq b$ and $s\psi_a t$ imply $\xi^a_b(s)\psi_b\xi^a_b(t)$

(R2) $a \geq b$, $a\psi b$ and $\xi^a_b(s)\psi_b\xi^a_b(t)$ imply $s\psi_a t$.

We let $\mathrm{Rep}(L(S, A))$ be the set of all congruence representations of $L(S, A)$.

(2-5) Lemma: For $(\psi; (\psi a)a \in A) \in \mathrm{Rep}(L(S, A))$ then the relation $\hat{\psi}$ defined by

$$(a, s)\hat{\psi}(b, t) \text{ iff } a\psi b \text{ and } \xi^a_{ab}(s)\psi_{ab}\xi^b_{ab}(t)$$

is a congruence on $L(S, A)$.

Proof: $\hat{\psi}$ is clearly reflexive and symmetric. If $(a, s)\hat{\psi}(b, t)\hat{\psi}(c, u)$ then firstly we have $a\psi b\psi c$ and hence $a, b, c, ab, ac, bc,$ and abc belong to the same ψ class. Secondly we have $\xi^a_{ab}(s)\psi_{ab}\xi^b_{ab}(t)$ and $\xi^b_{bc}(t)\psi\xi^c_{bc}(u)$. Since $ab, bc \geq abc$ we have by (R1): $\xi^a_{abc}(s) = \xi^{ab}_{abc}(\xi^a_{ab}(s))\psi_{abc}\xi^{ab}_{abc}\ \xi^b_{abc}(t)) = \xi^b_{abc}(t)$ and similarly $\xi^b_{abc}(t)\psi_{abc}\xi^c_{abc}(u)$. Therefore since $\psi_{abc} \in \mathrm{Con}(S(abc))$ $\xi^{ac}_{abc}(\xi^a_{ac}(s)) = \xi^a_{abc}(s)\psi_{abc}\xi^c_{abc}(u) = \xi^{ac}_{abc}(\xi^c_{abc}(u))$ and $ac \geq abc$ together with $ac\psi abc$ implies by (R2) $\xi^a_{ac}(s)\psi_{ac}\xi^c_{ac}(u)$. Therefore $\hat{\psi}$ is transitive.

Similar deductions show that $\hat{\psi}$ is compatible with meet and the unary operations.

(2-6) Theorem: The functions $\theta \to \overline{\theta} = (\theta_A; (\theta_a)_{a\varepsilon A})$ and $(\psi; (\psi_a)_{a\varepsilon A}) \to \hat{\psi}$ determine inverse lattice isomorphisms between $(\mathrm{Con}(L(S, A)), \subseteq)$ and $(\mathrm{Rep}(L(S, A)), \leq)$ where the order on $\mathrm{Rep}(L(S, A))$ is the product of the inclusion orders.

Proof: That $\theta \to \overline{\theta}$ and $\psi \to \hat{\psi}$ preserve the order relations is trivial, hence we need only show the mutual inverse part.

$$
\begin{aligned}
(a, s)\overline{\theta}(b, t) &\text{ iff } a\theta_A b \text{ and } \xi^a_{ab}(s)\theta_a\xi^b_{ab}(t) \\
&\text{ iff } (a, s)\theta(b, t) \text{ by } (2.2) \\
a(\hat{\psi})_A b &\text{ iff } (a, 1)\hat{\psi}(b, 1) \\
&\text{ iff } a\psi b \text{ and } 1 = \xi^a_{ab}(1)\psi_{ab}\xi^b_{ab}(1) = 1 \\
&\text{ iff } a\psi b \\
s(\hat{\psi})_a t &\text{ iff } (a, s)\hat{\psi}(a, t) \\
&\text{ iff } a\psi a \text{ and } s = \xi^a_a(s)\psi_a\xi^a_a(t) = t \\
&\text{ iff } s\psi_a t.
\end{aligned}
$$

(2-7) Corollary: Arbitary meets in $\mathrm{Rep}(L(S, A))$ are computed component wise.

We will now equate a congruence on $L(S, A)$ with its congruence representation.

(2-8) Corollary: $(\theta; (\theta_a)_{a\varepsilon A}) \varepsilon Con(L(S, A))$ implies $(\Delta; (\theta_a)_{a\varepsilon A}) \varepsilon Con(L(S, A))$.

Proof: (R2) is trivially satisfied in the second relation and (R1) follows from the first.

(2-9) Corollary: $(\theta; (\Delta_a)_{a\varepsilon A}) \varepsilon Con(L(S, A))$ iff for all $a,b \varepsilon A$, $a \geq b$ and $a\theta b$ imply $\xi_b^a: S(a) \rightarrow S(b)$ is a monomorphism.

Proof: (R1) is trivially verified and the stated condition is necessary and sufficient for (R2) to hold.

§3. $k[\mathcal{B}]$ is a variety:

By the results in section 1 we need only show that $k[\mathcal{B}]$ is closed under $\underset{\sim}{H}$. Therefore let $(\theta; (\theta_a)_{a\varepsilon A})$ be a congruence on $L(S, A)$. By (2-8) $\Phi = (\Delta; (\theta_a)_{a\varepsilon A})$ is also a congruence on $L(S, A)$.

Define $T: (A, \geq) \rightarrow k$ $T(a) = S(a)/\theta_a$ and let $\zeta_b^a: T(a) \rightarrow T(b)$ be the unique map such that $\nu_b \circ \xi_b^a = \zeta_b^a \nu_a$ as in 2.4. Now define

$f: L(S, A) \rightarrow L(T, A)$ by $f(a, s) = (a, s/\theta_a)$

$$f(a, s) = f(b, t) \text{ iff } (a, s/\theta_a) = (b, t/\theta_b)$$

$$\text{iff } a = b \text{ and } s\theta_a t$$

$$\text{iff } (a, s)\Phi(b, t)$$

Therefore Ker $f = \hat{\Phi}$ and easy calculations show f is a homomorphism. Moreover for $L(T, A)$ we have by (2-9) that $\psi = (\theta; (\Delta_a)_{a\varepsilon A}) \varepsilon \mathrm{Con}(L(T, A))$. Since $(f \times f)^{-1}(\psi) = \Phi$ we need only show that there exists $g: L(T, A) \to L(U, B)$ such that Ker $g = \psi$.

Let $B = A/\theta$. For each $\bar{a} \varepsilon B$, $\bar{a} = a/\theta$ we have a $\geq$-directed system of monomorphisms

$$\mathcal{D}(\bar{a}) = (T(x) \overset{\zeta^x_y}{\rightarrowtail} T(y): x,y\theta a,\ x \geq y)$$

satisfying the usual compatibility conditions. Therefore define for each $\bar{a} \varepsilon B$

$$U(\bar{a}) = \varinjlim \mathcal{D}(\bar{a}) = (\bigcup_{x\psi a} \{x\} \times T(x))/\Sigma(\bar{a})$$

where $(x, s)\Sigma(\bar{a})(y, t)$ iff $\zeta^x_{xy}(s) = \zeta^y_{xy}(t)$. Now suppose $\bar{a} \geq \bar{b}$ in B This is equivalent to: $\forall x$ $x\Theta a$ implies $xb\Theta b$. We define $\eta^{\bar{a}}_{\bar{b}} : U(\bar{a}) \to U(\bar{b})$ by

$$\eta^{\bar{a}}_{\bar{b}}((x, s)/\Sigma(\bar{a})) = (xb, \zeta^x_{xb}(s))/\Sigma(\bar{b}).$$

If $(x, s)\Sigma(\bar{a})(y, t)$ then $\zeta^x_{xy}(s) = \zeta^y_{xy}(t)$ therefore $\zeta^{xb}_{xyb}(\zeta^x_{xb}(s)) = \zeta^x_{xyb}(s) = \zeta^{xy}_{xyb}(\zeta^x_{xy}(s)) = \zeta^{xy}_{xyb}(\zeta^y_{xy}(t)) = \zeta^{yb}_{xyb}(\zeta^y_{yb}(t))$.

That is $(xb, \zeta^x_{xb}(s))\Sigma(\bar{b})(yb, \zeta^y_{yb}(t))$ and $\eta^{\bar{a}}_{\bar{b}}$ is well defined. We must now show the $\eta^{\bar{a}}_{\bar{b}}$ satisfy our functorial requirements. For $x\theta a$, $\eta^{\bar{a}}_{\bar{a}}((x, s)/\Sigma(\bar{a})) = (xa, \zeta^x_{xa}(s))/\Sigma(\bar{a})$. But $(x, s)\Sigma(\bar{a})(xa, \zeta^x_{xa}(s))$ since

$\zeta^{x}_{xa}(s) = \zeta^{xa}_{xa}(\zeta^{x}_{xa}(s))$. Therefore $\eta^{\bar{a}}_{\bar{a}} = 1_{U(\bar{a})}$.

Now if $\bar{a} \geq \bar{b} \geq \bar{c}$ we can compute:

$$\eta^{\bar{b}}_{\bar{c}}(\eta^{\bar{a}}_{\bar{b}}((x, s)/\Sigma(\bar{a}))) = \eta^{\bar{b}}_{\bar{c}}((xb, \zeta^{x}_{xb}(s))/\Sigma(\bar{b}))$$

$$= (xbc, \zeta^{xb}_{xbc}(\zeta^{x}_{xb}(s)))/\Sigma(\bar{c})$$

$$= (xbc, \zeta^{x}_{xbc}(s))/\Sigma(\bar{c})$$

and $$\eta^{\bar{a}}_{\bar{c}}((x,s)\Sigma(\bar{a})) = (xc, \zeta^{x}_{xc}(s))/\Sigma(\bar{c}).$$

But $\zeta^{xc}_{xbc}(\zeta^{x}_{xc}(s)) = \zeta^{xbc}_{xbc}(\zeta^{x}_{xbc}(s))$, hence $\eta^{\bar{b}}_{\bar{c}} \circ \eta^{\bar{a}}_{\bar{b}} = \eta^{\bar{a}}_{\bar{c}}$ for $\bar{a} \geq \bar{b} \geq \bar{c}$ in B.

Now define $g: L(T, A) \to L(U, B)$ by

$$g(a, s) = (a, s)/\Sigma(\bar{a})$$

$$g(a, 1) = g(b, 1) \text{ iff } (a, 1)\Sigma(\bar{a})(b, 1)$$

$$\text{iff } a\theta b$$

and $$g(a, s) = g(a, t) \text{ iff } (a, s)\Sigma(\bar{a})(a, t)$$

$$\text{iff } s = t .$$

Therefore Ker $g = \psi$ and an easy check shows that g is a homomorphism. Thus we have shown

(3-1) Theorem: $\mathcal{k}[\mathcal{B}]$ is a variety of bounded semilattices with operators.

§4. The relation between Con(k) and Con($k[B]$).

(4.1) Lemma: The mapping $\theta \to \theta_A$ is a lattice homomorphism from Con(L(S, A)) onto Con(A).

Proof: We have by (2-7) that this map preserves (arbitrary) meets. Now for $\theta, \psi \in$ Con(L(S, A))

$$
\begin{aligned}
a(\theta \vee \psi)_A b \quad & \text{iff} \quad (a, 1)\theta \vee \psi(b, 1) \\
& \text{iff} \quad \exists (a, 1) = (c_0, u_0) \ldots\ldots (c_n, u_n) = (b, 1) \\
& \qquad\qquad (c_i, u_i)\theta(c_{i+1}, u_{i+1}) \quad (i \text{ even}) \\
& \qquad\qquad (c_i, u_i)\psi(c_{i+1}, u_{i+1}) \quad (i \text{ odd}) \\
& \text{iff} \quad \exists\, a = c_0, \ldots\ldots c_n = b \\
& \qquad\qquad (c_i, 1)\theta(c_{i+1}, 1) \quad (i \text{ even}) \\
& \qquad\qquad (c_i, 1)\psi(c_{i+1}, 1) \quad (i \text{ odd}) \\
& \text{iff} \quad a\theta_A \vee \psi_A b
\end{aligned}
$$

(4-2) Lemma: For $\theta \in$ Con(A), $R(\theta) = \{\psi \in \text{Con}(L(S, A)) : \psi_A = \theta\} \in \text{Con}(k)$.

Proof: By (2-7) $\bigwedge R(\theta) \in R(\theta)$. Therefore we can factor L(S, A) by $\bigwedge R(\theta)$ and hence without loss of generality assume $\theta = \Delta_A$. But for any system $(\theta_a)_{a\varepsilon A}$ in $\prod_{a \varepsilon A} \text{Con}(S(a))$ that satisfies (R1), $(\Delta; (\theta_a)_{a\varepsilon A}) \in \text{Con}(L(S, A))$

Moreover if $(\theta_a)_{a\varepsilon A}$ and $(\psi_a)_{a\varepsilon A}$ satisfy (R1) then so does $(\theta_a \vee \psi_a)_{a\varepsilon A}$. Therefore $R(\Delta_A)$ is a sublattice of $\prod_{a\varepsilon A} Con(S(a))$ and hence is in $Con(k)$. We now define a sequence of lattices

$$C_1 = \underset{\sim}{2}$$

$$C_{n+1} = 1 * C_n$$

where $A*B$ is the lattice determined by the disjoint union of the posets $(A, \leq)$ and $(B, \leq)$ with a zero and unit adjoined. For example $C_2 = N_5$, the five element non-modular lattice and in general $C_k = N_{k+3}$ in McKenzie's notation (Reverse extrapolation of McKenzie's notation would make $\underset{\sim}{2} = N_4$, hence the renumbering).

By various methods one can show that for each $n \geq 1$, C_n is a projective subdirectly irreducible lattice.

(4-3) Lemma: For any $n \geq 1$ and any variety k, $C_n \varepsilon Con(k)$ implies $C_n \leq Con(S)$ for some $S \varepsilon k$.

Proof: $C_n \varepsilon \underset{\sim}{H}\,\underset{\sim}{S}\,\underset{\sim}{P}\{Con(S): S \varepsilon k\}$ implies since C_n is projective that $C_n \varepsilon \underset{\sim}{S}\,\underset{\sim}{P}\{Con(S): S \varepsilon k\}$. Now since C_n is subdirectly irreducible it follows that $C_n \varepsilon S\{Con(S): S \varepsilon k\}$.

(4-4) Lemma: For each $n \geq 1$, $(L: C_n) = \{L \varepsilon L: C_n \not\leq L\}$ is a proper variety of lattices.

Proof: Such a class is always closed under $\underset{\sim}{S}$. C_n projective says it is closed under $\underset{\sim}{H}$, and C_n subdirectly irreducible makes it closed under $\underset{\sim}{P}$. Moreover it is a proper subvariety as it does not contain C_n.

(4-5) Theorem: Let k be a variety of bounded semilattices with operators such that for some $n \geq 1$. $C_n \in \mathrm{Con}(k) \subseteq (\mathcal{L}: C_{n+1})$; then $k[\mathcal{B}]$ is again such a variety with $C_{n+1} \in \mathrm{Con}(k[\mathcal{B}]) \subseteq (\mathcal{L}: C_{n+2})$.

Proof: By (4-3) we have $C_n \leq \mathrm{Con}(S)$ for some $S \in k$. Define $S: (2, \geq) \to k$ by $S(1) = S(0) = S$ and $\xi_0^1 = 1_S$.

Now for each $\theta \in \mathrm{Con}(S)$ we have $\overline{\theta} \in \mathrm{Con}(L(S, \underset{\sim}{2}))$ where $\overline{\theta}_{\underset{\approx}{2}} = \Delta_{\underset{\approx}{2}}$ $\overline{\theta}_1 = \theta$ and $\theta_0 = \nabla_S$. By (4-2) $(\{\overline{\theta}: \theta \in \mathrm{Con}(S)\}, \subseteq)$ is isomorphic to $(\mathrm{Con}(S), \subseteq)$. But $\Sigma_{\underset{\sim}{2}} = \nabla$, $\Sigma_1 = \Sigma_0 = \Delta$ also is a congruence on $L(S, \underset{\sim}{2})$; moreover $\Sigma \vee \overline{\theta} = \nabla_{L(S,2)}$ and $\Sigma \wedge \overline{\theta} = \Delta_{L(S,\underset{\sim}{2})}$ for all $\theta \in \mathrm{Con}(S)$. Therefore $1{*}C_n \leq 1{*}\mathrm{Con}(S) \leq \mathrm{Con}(L(S, 2))$.

Now suppose $C_{n+s} = 1{*}C_{n+1} \leq \mathrm{Con}(L(S, A))$ for some $L(S, A)$ in $k[\mathcal{B}]$. Since $(\ )_A: \mathrm{Con}(L(S, A)) \to \mathrm{Con}(A)$ is a lattice homomorphism and Boolean algebras are congruence distributive, $\theta_A = \psi_A = \overline{\Phi}$ for all $\theta, \psi \in C_{n+1}$. But then

$$C_{n+1} \leq R(\Phi) \in \mathrm{Con}(k).$$

This is a contradiction.

(4-6) Corollary: There exists a countable sequence of non-modular congruence varieties which contain no non-distributive modular lattices.

Proof: Let $k_1 = \mathcal{B}$ and $k_{n+1} = k_n[\mathcal{B}]$. Then by (4.5), we have $C_n \in \mathrm{Con}(k_n) \subseteq (L: C_{n+1})$. Also, each k_n has semilattices as a reduct and therefore $M_3 \notin \mathrm{Con}(k_n)$ by the result for semilattices due to Papert.

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