SPLITTING LATTICES AND CONGRUENCE MODULARITY

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by

Alan Day*

§1 INTRODUCTION

Starting with Nation's results in [9], several papers have been written (Day [3] and [4], Jónsson [6] and Mederly [7]) showing that the modular law is a consequence of weaker lattice theoretical assumptions on the congruence variety of an arbitrary variety of algebras. These results generated a conjecture attributed by Ralph McKenzie to Stanley Burris, viz:

> McKenzie-Burris: If the congruence variety of a variety of algebras satisfies any non-trivial lattice identity, then it is already congruence modular.

The purpose of this paper, is to show that all the above mentioned results are consequences of a more general theorem.

The major theoretical tool used is the concept of a splitting lattice developed by McKenzie in [8]. The beauty of splitting lattices is that each comes paired with a (conjugate) equation so that every variety of lattices either satisfies this equation or contains the paired splitting lattice but not both. This allows one to alternate between semantical and syntactical arguments as best befits the situation at hand.

In this paper, we supply a class, S_1, of splitting lattices such that for every S in S_1, if the congruence variety (of a variety

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of algebras) satisfies the conjugate equation of S, then the variety is already congruence modular. In a sense (to be explained later) the members of S_1 are not far removed from the pentagon, N_5. They are, informally speaking, all subdirectly irreducible lattices that one can obtain by "splitting" an element of a finite distributive lattice by a method developed in [2].

In section 3 we develop the main properties of this class, S_1, and in section 4 we prove the main theorem and show that the previously known results are corollaries.

Finally, we would like to thank B. Dulley and A. McEwan of the Computer Centre at Lakehead University for indirect stimulation of this research, B. Jónsson and R. McKenzie for more direct stimulating discussions and Croy Pitzer, for use of his unpublished notes.

§2 PRELIMINARIES

We need the following facts about splitting lattices from [8] and [5]. A lattice epimorphism $f : A \twoheadrightarrow B$ is called bounded if $f^{-1}(b)$ is a closed interval for every $b \in B$. That is: there are order monomorphisms $\alpha, \beta : B \rightarrowtail A$ such that for all $b \in B$, $f^{-1}\{b\} = [\alpha(b), \beta(b)]$. Clearly α is $\vee$-preserving and β is $\wedge$-preserving. We define $\mathcal{B}$ to be the class of all finite lattices that are bounded homomorphic images of finitely generated free lattices, and $\mathcal{S}$ to be the class of subdirectly irreducible members of $\mathcal{B}$.

<u>Theorem</u> (McKenzie [8]): For every $S \in \mathcal{S}$, if $u \prec v$ are a pair of elements in S that determine the least non-trivial congruence relation on S and $f : FL(n) \twoheadrightarrow S$ is an epimorphism bounded below and above

by $\alpha,\beta : S \rightarrowtail FL(n)$ respectively, then for any variety of lattices V,

$$V \models \alpha(v) \leq \beta(u) \text{ iff } S \notin V$$

We denote the variety determined by the equation $\alpha(v) \leq \beta(u)$ by L/S. Members of S are called splitting lattices.

Corollary: For $S,T \in S$, $V(S) \subseteq V(T)$ iff $L/S \subseteq L/T$.

We will also need a construction from [2]. Let A be a lattice and $I = [p,q]$ an interval in A. Then $A[I] = (A \smallsetminus I) \cup I \times 2$ is a lattice with the partial order relation:

$x \leq y$ iff (1) $x,y \in A \smallsetminus I$ and $x \leq y$ in A

(2) $x = (a,i)$, $y \in A \smallsetminus I$ and $a \leq y$ in A

(3) $x \in A \smallsetminus I$, $y = (b,j)$ and $x \leq b$ in A

or (4) $x = (a,i)$, $y = (b,j)$, $a \leq b$ in A and $i \leq j$ in $\underset{\sim}{2}$

Moreover $\kappa : A[I] \rightarrow A$ by

$$\kappa(x) = \begin{cases} x, & x \in A \smallsetminus I \\ a, & x = (a,i) \end{cases}$$

is a lattice epimorphism.

Theorem ([5]): Let $A \in B$, and take $a,b,c,d \in A$ not satisfying Whitman's condition (i.e. $a \wedge b \leq c \vee d$ but $\{a,b,c,d\} \cap [a \wedge b, c \vee d] = \emptyset$) then for $I = [a \wedge b, c \vee d]$, $A[I] \in B$. Moreover, if $f : FL(X) \twoheadrightarrow A$ is bounded, (X finite) then there exists a bounded $g : FL(X) \twoheadrightarrow A[I]$ with $\kappa \circ g = f$. If also $f[X] \cap I = \emptyset$, g is unique and we have

$$\overline{\alpha}(a\wedge b,1) = \bigwedge\{\alpha(p) : p \notin I \text{ and } a\wedge b < p\}$$

and
$$\overline{\beta}(c\vee d,0) = \bigvee\{\beta(q) : q \notin I \text{ and } q < c\vee d\}$$

where $\overline{\alpha},\overline{\beta} : A[I] \rightarrowtail FL(X)$ are the lower and upper bound mappings for g.

§3 THE CLASS S_1

Let B be a finite Boolean algebra and take $p \in B$ doubly reducible (therefore not an atom, a coatom, nor a bound element). We write $B[p] = B[\{p\}]$ where $\{p\} = I = [a\wedge b, c\vee d]$ for some $a,b,c,d \in B$. We define:

$$\mathcal{B}_1 = \underset{\sim\sim\sim}{HSP}_{fin}\{B[p] : B \text{ finite Boolean algebra and } p \in B \text{ doubly reducible}\}$$

$$S_1 = \{S \in \mathcal{B}_1 : S \text{ is subdirectly irreducible}\}$$

Since every finite distributive lattice, D, is a sublattice of a Boolean algebra B in such a way that every element of D is doubly reducible in B, we also have that S_1 consists of subdirectly irreducible lattices of the form $D[p]$ with D a finite distributive lattice and $p \in D$.

From [8] and [5], $\mathcal{B} \subseteq \mathcal{B}_1$ and therefore every $S \in S_1$ is a splitting lattice. Moreover for every $S \in S_1$ there exists a finite Boolean algebra B and doubly reducible $p \in B$ with $S \in \underset{\sim}{V}(B[p])$ and therefore $L/S \subseteq L/B[p]$.

We are therefore interested in conjugate equations for the members of S_1 of the form $B[p]$.

<u>Theorem</u> (3.1): Let $\mathcal{B}$ be a finite Boolean algebra and take $p \in B$ doubly reducible. Let $A = \{a_1,\dots,a_n\}$ be the atoms of B not less than p and $B = \{b_1,\dots,b_m\}$ be the atoms of B less than p. Then a conjugate equation for $B[p]$ is:

$$\bigwedge_i^{1,n} (x_i \vee \bigvee_j^{1,m} y_j) \le \bigvee_j^{1,m} (u_j \wedge \bigwedge_i^{1,n} v_i)$$

where $X = \{x_i : 1 \le i \le n\} \cup \{y_j : 1 \le j \le m\}$ is a set of $n+m$ distinct variables, $u_j = \bigvee X \setminus \{y_j\}$, $1 \le j \le m$, and $v_i = \bigvee X \setminus \{x_i\}$, $1 \le i \le n$.

<u>Proof</u>: Define $f : FL(X) \to B$ by:

$$f(x_i) = a_i, \ 1 \le i \le n$$

$$f(y_j) = b_j, \ 1 \le j \le m$$

Since $B \in \mathcal{B}$, and f is surjective, f is bounded below and above by α and $\beta : B \rightarrowtail FL(X)$. Moreover to compute the values of α, and β at any member of B we need only now the α-values at the atoms and the β-values at the coatoms. These are:

$$\alpha(a_i) = x_i, \ 1 \le i \le n$$

$$\alpha(b_j) = y_j, \ 1 \le j \le m$$

and

$$\beta(a_i') = v_i, \ 1 \le i \le n$$

$$\beta(b_j') = u_j, \ 1 \le j \le m$$

Since $f[X] \cap \{p\} = \emptyset$ we have a unique lifting $g : FL(X) \to B[p]$ bounded by $\overline{\alpha}$ and $\overline{\beta}$. Moreover

$$\overline{\alpha}(p,1) = \bigwedge\{\alpha(c) : c > p\}$$

and $$\overline{\beta}(p,0) = \bigvee\{\beta(d) : d < p\}$$

However, since α and β are isotone and B is a finite Boolean algebra, we have

$$\overline{\alpha}(p,1) = \bigwedge\{\alpha(c) : p \prec c\}$$

and $$\overline{\beta}(p,0) = \bigvee\{\beta(d) : d \prec p\}$$

But $p = \bigvee_j^{1,m} b_j = \bigwedge_i^{1,n} a_i'$ and therefore if c covers p, $c = a_i \vee \bigvee_j^{1,m} b_j$ for some $i = 1,\ldots,n$ and if d is covered by p, $d = b_j' \wedge \bigwedge_i^{1,n} a_i'$ for some $j = 1,\ldots,m$. Therefore

$$\overline{\alpha}(p,1) = \bigwedge_i^{1,n} (x_i \vee \bigvee_j^{1,m} y_j)$$

and $$\overline{\beta}(p,0) = \bigvee_j^{1,m} (u_j \wedge \bigwedge_i^{1,n} v_i)$$

As mentioned in §2, this provides the splitting equation for $B[p]$.

§4 CONGRUENCE MODULARITY IMPLICATIONS

The main result of this section is the following:

<u>Theorem</u> (4.1): For any $S \in \mathcal{S}_1$ and any variety of algebras K, if $\mathrm{Con}(K) \subseteq L/S$ then K is already congruence modular.

Before presenting a proof for this theorem, let us first note its corollaries which fall into two classes.

<u>Corollary 1</u> (4.2): (Jonsson [6], Mederly [7], Day [3]). Let K be a variety of algebras whose congruence lattices satisfy one of the

following equations

(1) $(x\vee(y\wedge z))\wedge(z\vee(x\wedge y)) \le (z\wedge(x\vee(y\wedge z))) \vee (x\wedge(z\vee(x\wedge y)))$

(2) $(x\vee y)\wedge(x\vee z) \le x\vee((x\vee y)\wedge(x\vee z)\wedge(y\vee z))$

(3) for some $n \ge 2$,

$$\bigwedge_{i}^{1,n} (x\vee y_i) \le x\vee[(\bigvee_{i}^{1,n} y_i)\wedge \bigwedge_{i}^{1,n} (x\vee \bigvee_{j\ne i}^{1,n} y_j)]$$

(4) for some $n \ge 3$

$$\bigwedge_{i}^{1,n} (x_i\vee \bigwedge_{j\ne i}^{1,n} x_j) \le \bigvee_{i}^{1,n} (x_i\wedge \bigvee_{j\ne i}^{1,n} x_j)$$

(5) for some $m \ge 1$

$$x_0\wedge \bigwedge_{i<j}^{1,m+1} (x_i\vee x_j) \le x_{m+1}\vee \bigvee_{i<j}^{0,m} (x_i\wedge x_j)$$

Then K is congruence modular.

Proof: One need only show in every case that the variety of lattices defined by the equations are contained in L/S for some $S \in S_1$. In the first four cases, the equations are the precise conjugate equations for members of S_1. For (1) see fig (i), and (2), fig. (ii). For (3) these are conjugate equations for B[a] where a is an atom of the finite Boolean algebra with n+1 atoms. For (4), these are conjugate equations of the members of S_1 given in figure (iii).

Finally for (5) the equation in $x_0,\dots x_{m+1}$ fails in Q_{m+2} for every $m \ge 1$.

Corollary 2 (4.3): (Jónsson [6]). If the congruence variety of a variety of algebras satisfies a (2,2) inequality, then the variety of algebras is already congruence modular.

(Actually Jónsson proved a stronger result namely that the variety will be already congruence distributive.)

Jónsson showed that such a variety must satisfy an identity of the form (5) in Corollary 1.

The original result of this type due to Nation in [9] has always seemed to the author to be the most baffling. The following shows that it fits nicely into this general framework.

Corollary 4 (4.4): (Nation [9]). Let X be a finite set of variables, and take $S_i \subseteq X$, $i = 1,\ldots,n+1$. Furthermore assume $w \in FL(X)$ satisfies

(a) $w \nleq \bigvee S_1$

(b) $w \wedge \bigvee S_1 \nleq \bigvee_{i}^{2,n+1} (\bigvee S_1 \wedge \bigvee S_i)$

then if the congruence variety of a variety of algebras K satisfies

$$w \wedge \bigvee S_1 \leq \bigvee_{i}^{2,n+1} (\bigvee S_1 \wedge \bigvee S_i)$$

K is already congruence modular.

Proof: We need only produce a member of S_1 in which such an equation fails. As Nation noted (a) and (b) are equivalent to the following statements:

(c) $w \nleq \bigvee S_i \quad 1 \leq i \leq n+1$

(d) $S_1 \nsubseteq \bigcup_{i}^{2,n+1} S_i$

and we should note that (c) is equivalent to

$$\text{(e)} \quad \bigwedge (X \setminus S_i) \le w, \quad 1 \le i \le n+1$$

Without loss of generality, we can assume the inequality holds in $\underset{\sim}{2}$ and can define for each $i = 1,2,\ldots,n+1$, $\phi_i : FL(X) \to \underset{\sim}{2}$ by:

$$\phi_i(x) = \begin{cases} 0, & x \in S_i \\ 1, & x \notin S_i \end{cases}$$

Since $\phi_i(\bigwedge(X \setminus S_i)) = 1$ for each i we have

$$\phi_i(w) = 1 \qquad 1 \le i \le n+1$$

Therefore for $B = 2^{n+1}$ there is a unique

$$\phi : FL(X) \to B \text{ with } \pi_i \circ \phi = \phi_i, \quad 1 \le i \le n+1.$$

Now $\phi(w) = 1 = (1,1,1,\ldots,1)$

$\phi(VS_1) = (0,1,1,\ldots,1)$

Moreover for $x \in X$;

$$\phi(x) = \phi(VS_1) \text{ iff } x \in S_1 \setminus \bigcup_i^{2,n+1} S_i$$

Therefore consider $B[p]$ where $p = \phi(VS_1)$ and define $\psi : FL(X) \to B[p]$ by

$$\psi(x) = \begin{cases} (p,1), & x \in S_1 \setminus \bigcup_i^{2,n+1} S_i \\ \phi(x), & \text{otherwise} \end{cases}$$

It follows easily that

$$\psi(w \wedge VS_1) = (p,1) \text{ and}$$

$$\psi\Big(\bigvee_{i}^{2,n+1} (VS_1 \wedge VS_i)\Big) = (p,0)$$

That is, the equation fails in this $B[p]$.

<u>Proof of Theorem</u>: Take $B[p]$ and its conjugate equation as in (3.1), let K be a variety of algebras, and V be a set of $2+n(m+1)$ variables;

$$V = \{s,t\} \cup \{a_{ij} : 1 \le i \le n,\ 1 \le j \le m+1\}$$

and define $h : FL(n+m) \to con(F(V))$. $(F(V) = F_K(V))$ by:

$$h(x_i) = con([s,a_{i1}],[t,a_{i,m+1}]), \quad 1 \le i \le n$$

$$h(y_j) = con([a_{ij},a_{ij+1}] : 1 \le i \le n), \quad 1 \le j \le m$$

Then $Con(K) \subseteq L/B[u]$ implies

$$(s,t) \in \bigvee_{j}^{1,m} \Big(\Theta_j \bigwedge_{i}^{1,n} \Psi_i\Big)$$

where

$$\Psi_i = con([s,t,a_{kj} : k \ne i,\ 1 \le k \le n,\ 1 \le j \le m+1],[a_{ij} : 1 \le j \le m+1]), \quad 1 \le i \le n$$

and

$$\Theta_j = con([s,a_{ik} : 1 \le i \le n,\ 1 \le k \le j],[t,a_{ik} : 1 \le i \le n,\ j+1 \le k \le m+1]), \quad 1 \le j \le m$$

Therefore there exist terms in $|V|$-variables $s = p_0,\ldots,p_\ell = t$ satisfying:

(B1) $\quad s \; \psi_i \; p_k \qquad 1 \leq i \leq n$ and $0 \leq k \leq \ell$

(B2) $\quad p_k \; \Theta_j \; p_{k+1} \qquad k \equiv j-1 \pmod{m}$

Now we need to find substitutions $\xi : V \to A = \{a,b,c,d\}$ that will force congruence modularity.

We consider the variables of V ordered in the following sequence

$$\nu = (s,t) \circ \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_n$$

where $\alpha_i = (a_{i1},\ldots,a_{im+1})$ and $\circ$ is concatenation.

A substitution $\xi : V \to A$ is called admissable if $\xi(s) = a$, $\xi(t) = b$, and there exists an $i \in [1,n]$ such that $\xi^{-1}\{c,d\} = A_i = \{a_{ij} : 1 \leq j \leq m+1\}$. More intuitively, ξ is admissable if s and t get mapped to a and b respectively, each α_i gets filled with either a's and b's, or c's and d's, and only one α_i gets filled with c's and d's.

Now let α,β be equivalence relations on A defined by the partitions:

$$\alpha = [a][b][c,d]$$
$$\beta = [a,c][b,d]$$

We define equivalence relations $\overline{\alpha}$ and $\overline{\beta}$ on the set of all admissable substitutions, Sub, by

$$\xi \; \overline{\alpha} \; \zeta \quad \text{iff} \quad \xi(\nu_\ell) \; \alpha \; \zeta(\nu_\ell), \quad 1 \leq \ell \leq 2+n(m+1)$$

$$\xi \; \overline{\beta} \; \zeta \quad \text{iff} \quad \xi(\nu_\ell) \; \beta \; \zeta(\nu_\ell), \quad 1 \leq \ell \leq 2+n(m+1)$$

<u>Lemma</u>: $\overline{\alpha} \vee \overline{\beta} = \nabla_{Sub}$

Proof: Every $\xi \in$ Sub is clearly $\overline{\beta}$-equivalent to an admissable substitution $\hat{\xi}$ for which $\hat{\xi}^{-1}\{c,d\} = A_1$. Therefore we may restrict our attention to these admissable substitutions. Also if two such admissable substitutions differ only on A_1, they are $\overline{\alpha}$-equivalent. Therefore what is needed, is a procedure to alter the blocks $A_1,\ldots,A_n$ sequentially and maintain the desired equivalence. In effect then for $\xi,\zeta \in$ Sub with $A_1 = \xi^{-1}\{c,d\} = \zeta^{-1}\{c,d\}$, we produce a sequence $\xi = \xi_0,\xi_1,\ldots,\xi_n = \zeta$ for which

(1) $\xi_{i-1}\overline{\alpha}\vee\overline{\beta}\xi_i \quad 1 \le i \le w$

and (2) $\xi_i|A_1\cup\ldots\cup A_i = \zeta|A_1\cup A_2\ldots\cup A_i$

We illustrate this procedure by an example, to avoid the technically messy details of a formal proof.

Let $i = 3$, and $j = 3$,

$$\xi = (ab;cdcd;aaab;abba)$$

and

$$\zeta = (ab;ccdd;abaa;abbb)$$

Then:

$$\begin{aligned}\xi = \xi_0\ &\overline{\alpha}(ab;ccdd;aaab;abba)\\ &\overline{\beta}(ab;aabb;cccd;abba)\\ &\overline{\alpha}(ab;aabb;cdcc;abba)\\ &\overline{\beta}(ab;ccdd;abaa;abba) = \xi_1\\ &\overline{\beta}(ab;aabb;abaa;cddc)\\ &\overline{\alpha}(ab;aabb;abaa;cddd)\\ &\overline{\beta}(ab;ccdd;abaa;abbb) = \zeta = \xi_2\end{aligned}$$

Now since the criterion for being admissable is precisely what is needed to apply the statement (B1) to $\gamma = [ab][cd]$ we obtain the following result.

Corollary: If ξ and ζ are admissable substitutions and $p_k \in F(V)$, $0 \le k \le \ell$, as above, then

$$(F\xi(p_k), F\zeta(p_k)) \in con_{F(A)}(\alpha) \vee [con_{F(A)}(\beta) \wedge con_{FA}(\gamma)]$$

where $F\xi : F(V) \to F(A)$ is the induced homomorphism.

The required admissable substitutions now become obvious from (B2), viz: for $j = 1,\ldots,m$

$$\xi_j(x) = \begin{cases} a, & x \in \{s\} \cup \{a_{ik} : i \ge 2,\ 1 \le k \le j\} \\ b, & x \in \{t\} \cup \{a_{ik} : i \ge 2,\ j+1 \le k \le m+1\} \\ c, & x \in \{a_{11},\ldots,a_{1j}\} \\ d, & x \in \{a_{1,j+1},\ldots,a_{1,m+1}\} \end{cases}$$

These substitutions imply from (B2) that

$$(F\xi_j(p_k), F\xi_j(p_{k+1})) \in con_{F(A)}(\beta) \wedge con_{F(A)}(\gamma)$$

and therefore

$$(a,b) \in con_{F(A)}(\alpha) \vee (con_{F(A)}(\beta) \wedge con_{F(A)}(\gamma))$$

and K is congruence modular by [1].

§5 CONCLUDING REMARKS

Let us now write a "nice" splitting lattice which is not a member of S_1. The one that seems most interesting is McKenzie's N_6 (see figure iv). A conjugate equation for N_6 is:

$$z \wedge [(x \wedge (w \vee (x \wedge y \wedge z))) \vee (y \wedge z \wedge (w \vee (x \wedge y \wedge z)))] \le$$

following result.

Corollary: If ξ and ζ are admissable substitutions and $p_k \in F(V)$, $0 \leq k \leq \ell$, as above, then

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$$\leq y \vee [(x \vee (w \wedge (x \vee y \vee z))) \wedge (y \vee z \vee (w \wedge (x \vee y \vee z)))]$$

At the time of this writing, it is not known to the author whether congruence "L/N_6" implies congruence modular or whether congruence L/N_6 is a Mal'cev condition in its own right. This problem is probably the next phase in the validity of the McKenzie-Burris conjecture.

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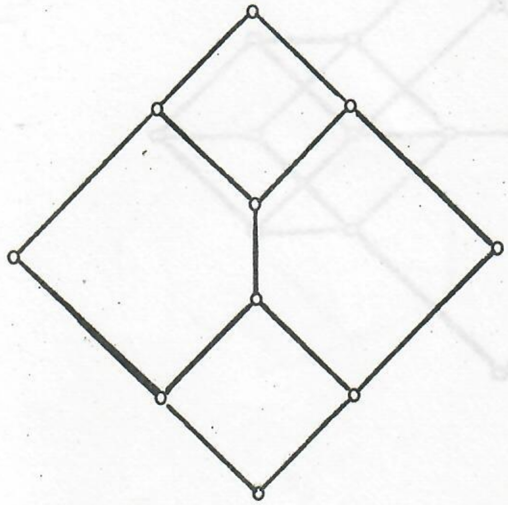


fig. (i)

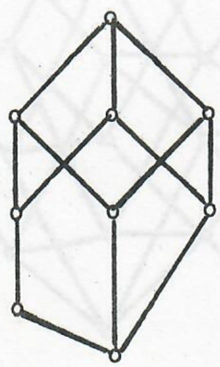
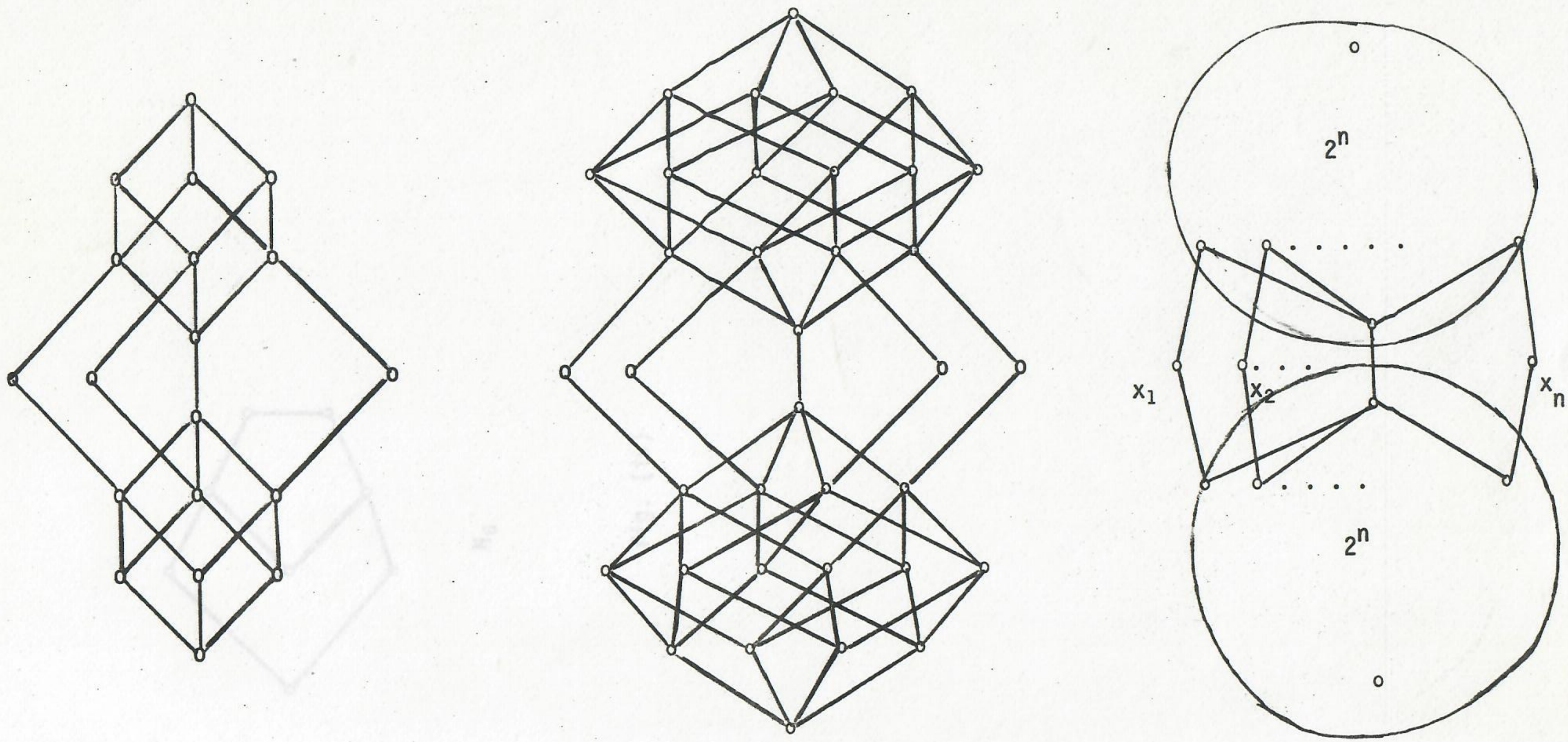
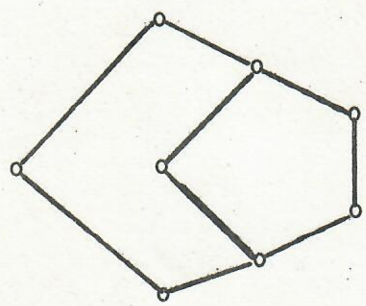


fig. (ii)



Q3 Q4 Qn

fig. (iii)



N_6

fig. (iv)