Lattices and Algebras: Some Connections

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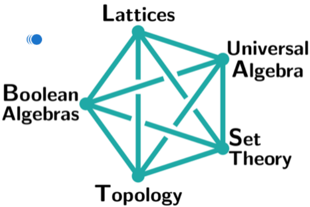


Part I

Modular **L**attices

A Very Brief History of Modular Lattices

In 1897 and 1900 Dedekind

- defined the **modular law:**

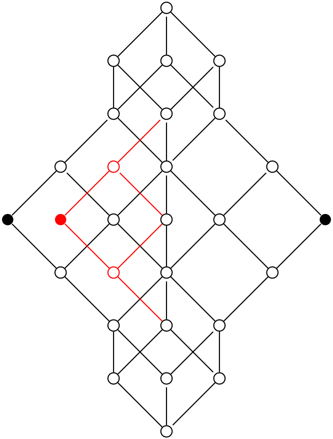
$$x \wedge (y \vee (x \wedge z)) \approx (x \wedge y) \vee (x \wedge z),$$

- showed a lattice is modular iff $\mathbf{N}_5$ is not a sublattice,

$\mathbf{M}_3$ $\mathbf{N}_5$

- showed the submodules of a module form a modular lattice,
- characterized the free modular lattice on 3 generators:

A Brief History of Modular Lattices



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Some Structure and Some Pathology

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Case $n = 3$**.** Length 3 complemented modular lattices **L** are projective planes but these are vector space lattices iff **L** satisfies Jónsson's **arguesian** identity.
Nonarguesian planes cannot be embedded in vector space lattices, and so are "pathological."

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The Embedding Problem

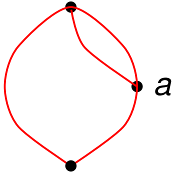
While the $\mathbf{M}_k$'s and nonarguesian projective planes have some pathology, they don't settle the following:

The Embedding Problem: *Can every modular lattice be embedded into a complemented modular lattice?*

Of course every distributive lattice can be embedded into a complemented distributive lattice (a Boolean algebra). Nevertheless the answer is No, as was shown by Hall and Dilworth the early 1940's using their now famous Hall-Dilworth gluing.

More Pathology: Hall-Dilworth Gluing

- If $[a) = \{x \in L_0 : x \geq a\}$ is a filter in a lattice $\mathbf{L}_0$ which is isomorphic to an ideal $(b] = \{y \in L_1 : y \leq b\}$ in a lattice $\mathbf{L}_1$ then we can glue these lattices together:



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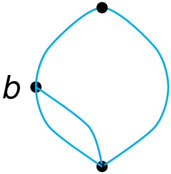

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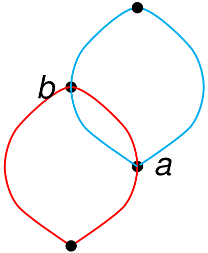
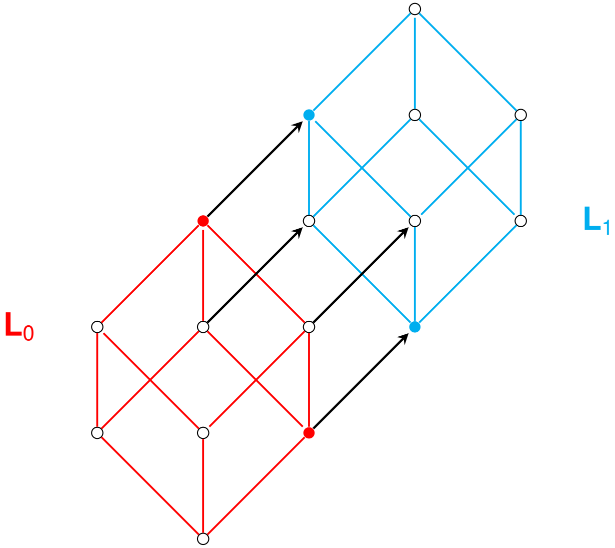


- Hall and Dilworth gave 3 examples showing that **not all modular lattices (not even all finite modular lattices) can be embedded into a complemented modular lattice**, solving one of the important problems of the time.

More Pathology: H-D Gluing Examples



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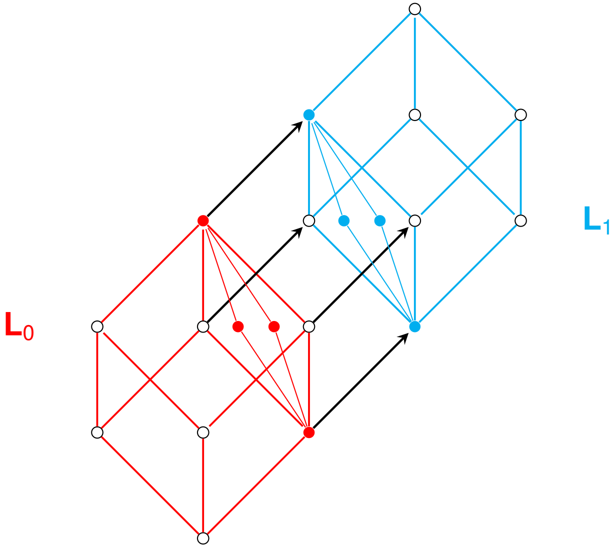

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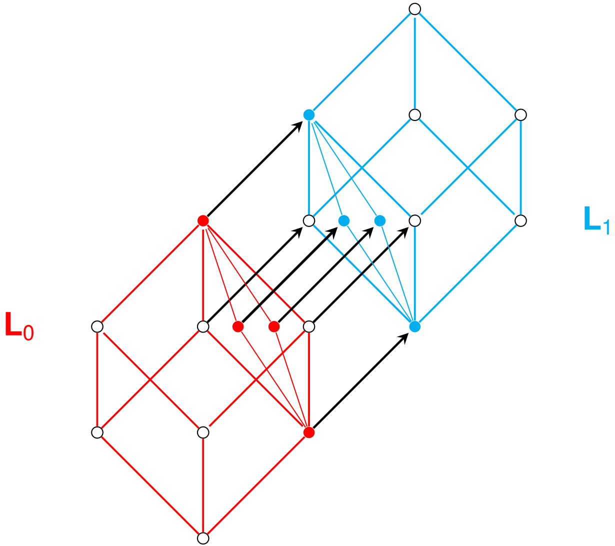

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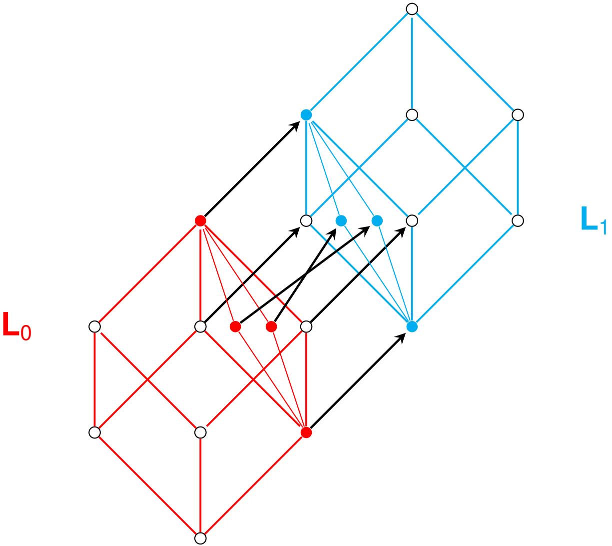

Key Observation:

- The automorphism group of $\mathbf{M}_k$ consists of all permutations of the atoms,
- and while the automorphism group of vector space lattices are big,
- not all of the permutations of $\mathbf{M}_k$ can be extended to automorphisms of higher dimensional vector space lattices.
- This is useful in constructing pathological examples.

Uses of Hall-Dilworth Gluing

- $\mathbf{L}_0$ and $\mathbf{L}_1$ the same vector space lattice but the gluing isomorphism (the arrows) does not extend to an automorphism.

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  - *The equational theory of modular lattices is not computable (i.e., nonrecursive).*
  - *Every free distributive lattice,* $\mathrm{FD}(\kappa)$*, can be embedded into a free modular lattice.*

Representations with Equivalence Relations

- A representation of **L** is an embedding into **EQV**(X), the lattice of equivalence relations on X.
- Whitman: Every lattice has such a representation.
- Jónsson:
  - *Every lattice has a* 4*-permutable representation:* $\alpha \vee \beta = \alpha \circ \beta \circ \alpha \circ \beta$*.*
  - *Every modular lattice has a* 3*-permutable representation:* $\alpha \vee \beta = \alpha \circ \beta \circ \alpha$*, **and conversely!***
  - *There are modular lattices without a* 2*-permutable representation; eg., nonarguesian projective planes.*
  - *There is a lattice equation, the **arguesian law**, stronger than the modular law, holding in 2-permutable lattices.*

Questions

1. Do all arguesian lattices have a representation by permuting equivalence relations?
2. Is the class of lattices with a representation by permuting equivalences finitely axiomizable?
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Answers:

1. No.
2. No. (Mark Haiman 1991)
3. Open.

Higher arguesian identities: Bill Lampe

$$\bigwedge_{i=0}^{n-1}(\alpha_i \vee \alpha_i') \leq \alpha_0' \vee (\alpha_0 \wedge (\alpha_1 \vee [(\alpha_0' \vee \alpha_1') \wedge \bigvee_{i=1}^{n-1} \gamma_i])) \qquad (*_n)$$

where $\gamma_i = (\alpha_i \vee \alpha_{i+1}) \wedge (\alpha_i' \vee \alpha_{i+1}')$, mod n so $\gamma_{n-1} = (\alpha_{n-1} \vee \alpha_0) \wedge (\alpha_{n-1}' \vee \alpha_0')$.

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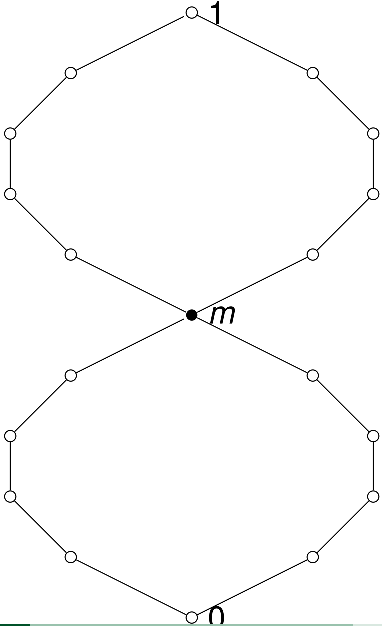
Remarks:

- When $n = 3$ this is Jónsson's arguesian identity.
- $(*_n)$ holds in any lattice representable by permuting equivalence relations. In fact,.
- The **relation** $(*_n)$ holds if α_i and α_i' permute, for each i.

Haiman's Lattices, $\mathbf{H}_n(\mathbf{F})$



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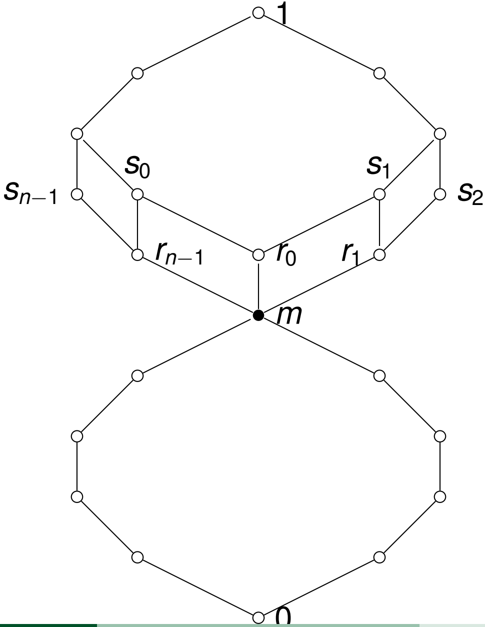

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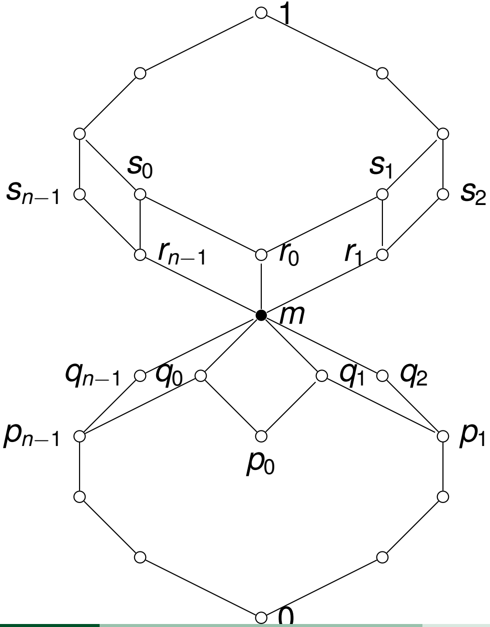

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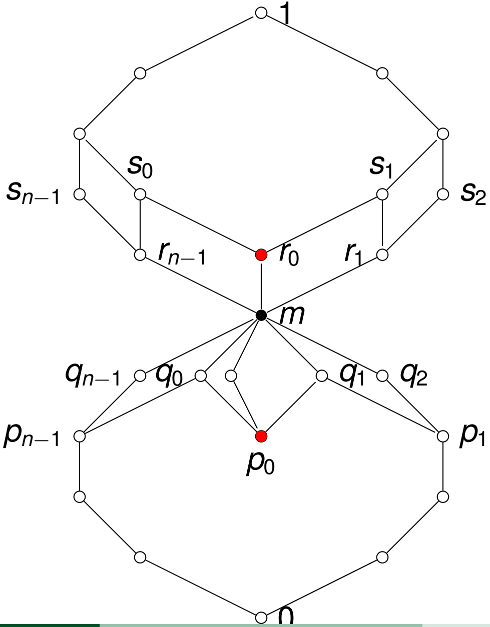

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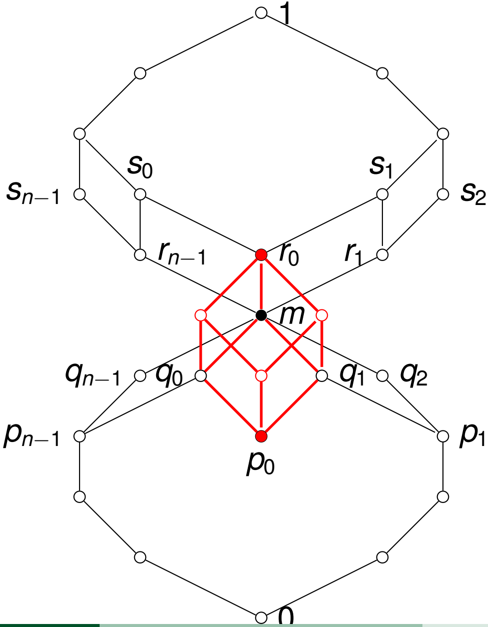

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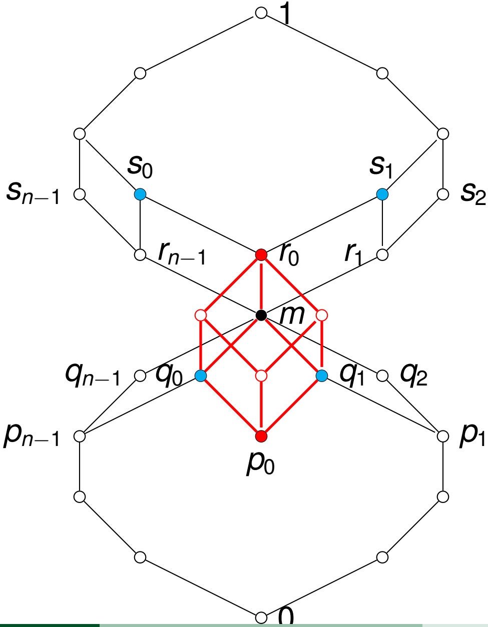

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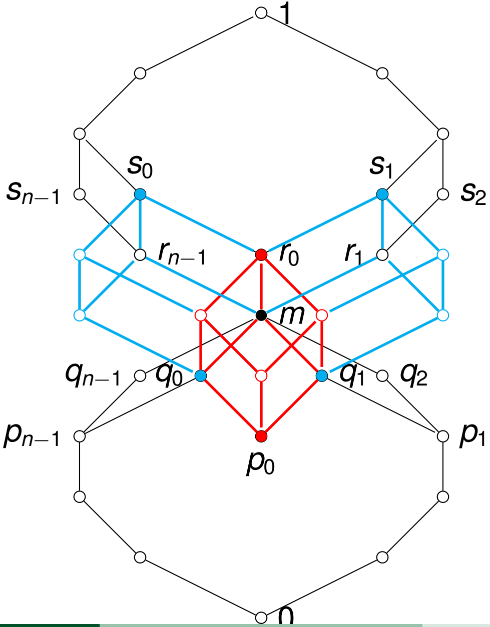

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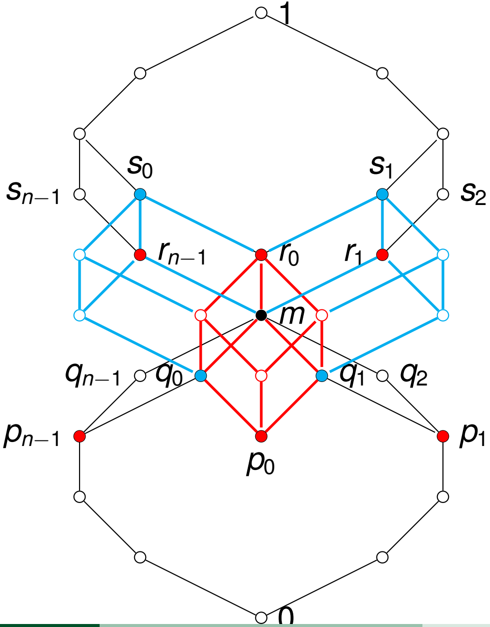

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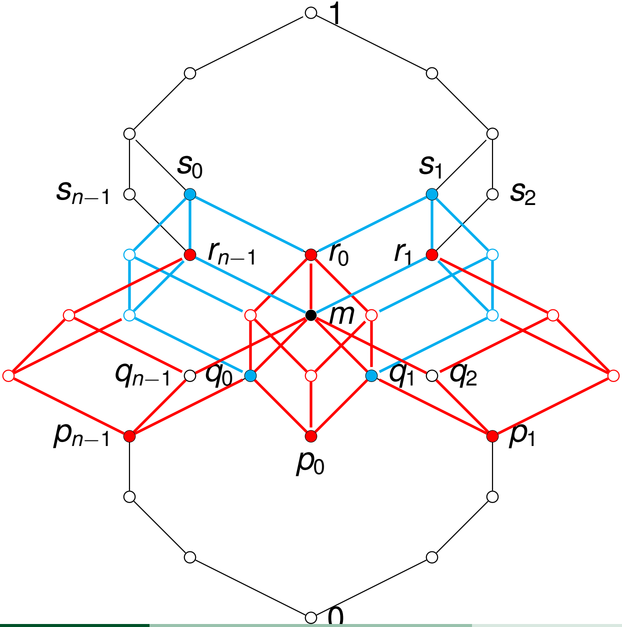

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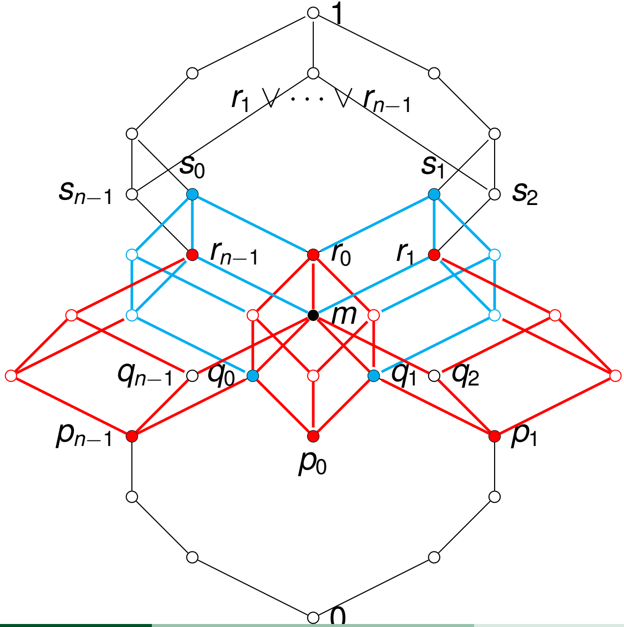

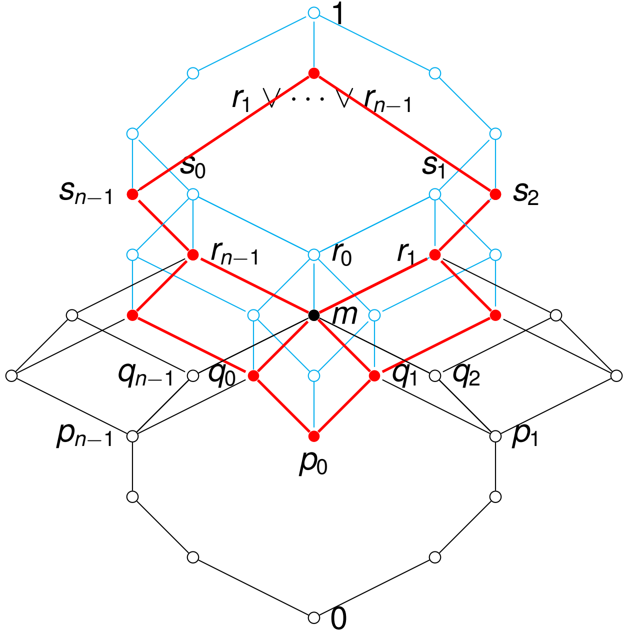
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$r_1 \vee \cdots \vee r_{n-1}$
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s_{n-1}
s_2
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p_1
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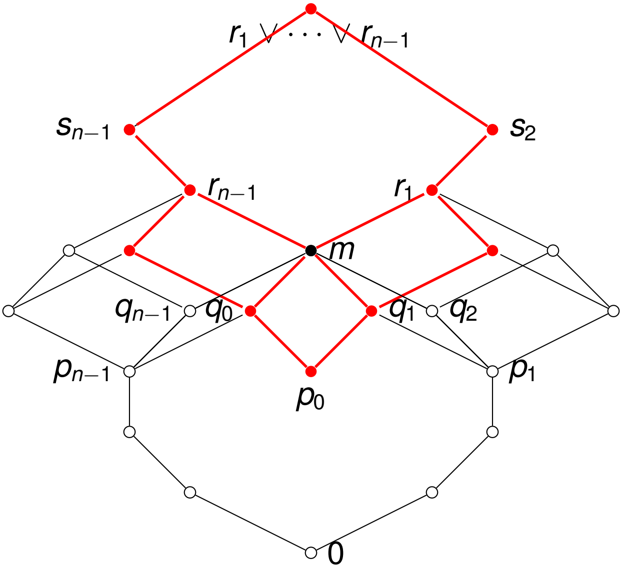
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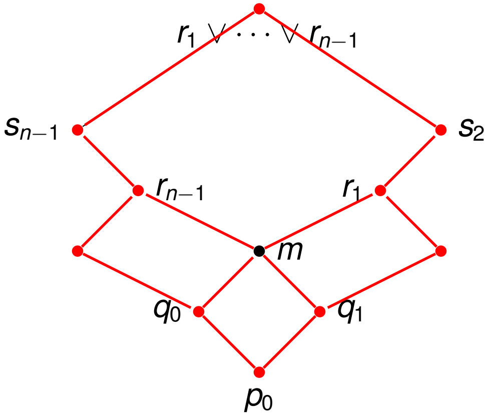

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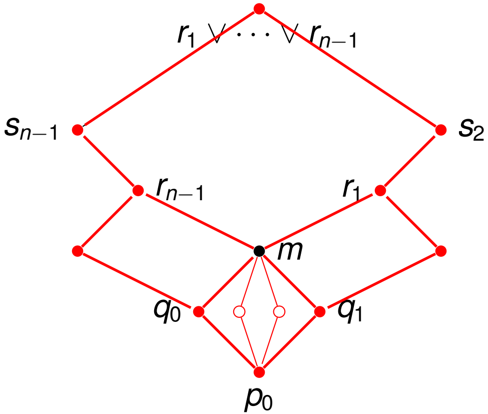
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Theorem (Haiman 1991)

The class of lattices representable with permuting equivalences is not finitely axiomatizable.

Haiman's lattices $\mathbf{H}_n(\mathbf{F})$ and the equations $(*_n)$ satisfy

- $(*_n)$ holds in any lattices of permuting equivalence relations.
- $(*_n)$ fails in $\mathbf{H}_n(\mathbf{F})$.
- Every $n-1$ generated sublattice is proper.
- Every proper sublattice is embeddable into the lattice of subspaces of a vector space over $\mathbf{F}$.
- Any nonprincipal ultraproduct of the $\mathbf{H}_n$'s is representable by permuting equivalences.

Part II

Universal **A**lgebra

Renaissance: The late 60's, 70's and 80's:

Some Highlights:

- **Mal'tsev Conditions.** Mal'tsev 1954, Jónsson 1967, Day 1969.
- **Commutator Theory.** Smith 1976, Hagemann-Herrmann 1979.
- **Representation Theory.** Grätzer-Schmidt 1963.
- **Congruence Varieties.** Nation 1973.

Renaissance: The late 60's, 70's and 80's:

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- **Mal'tsev Conditions.** Mal'tsev 1954, Jónsson 1967, Day 1969.
  - $\mathcal{V}$ is congruence permutable iff there is a term t with $t(x, x, y) \approx y \approx t(y, x, x)$.
- **Commutator Theory.** Smith 1976, Hagemann-Herrmann 1979.
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Let $\mathcal{V}$ be a variety (equational class) of algebras. Jónsson's results above imply

- If the congruence lattices of each algebra of $\mathcal{V}$ 3-permute (that is $\mathcal{V}$ is 3-permutable), then $\mathcal{V}$ is congruence modular.
- If $\mathcal{V}$ is congruence permutable, then $\mathcal{V}$ is congruence arguesian.

But in fact:

Theorem (RF and B. Jónsson 1976)

If $\mathcal{V}$ is congruence modular, then it is congruence arguesian.

Question: *Are there stronger lattice identities which are implied by congruence modularity?*
Yes we have a few odd examples.

Congruence Varieties

To give some context this question we need some definitions.

- $\mathcal{V}$ denotes a variety of algebras.
- $\mathbf{Con}(\mathcal{V}) := \{\mathbf{Con}(\mathbf{A}) : \mathbf{A} \in \mathcal{V}\}$.
- Define
  - **congruence variety of $\mathcal{V}$** is the variety of lattices generated by the congruence lattices of the members of $\mathcal{V}$:

$$\boldsymbol{HSP}\,\mathbf{Con}(\mathcal{V}) = \boldsymbol{HS}\,\mathbf{Con}(\mathcal{V})$$

  - **congruence prevariety of $\mathcal{V}$** by

$$\boldsymbol{SP}\,\mathbf{Con}(\mathcal{V}) = \boldsymbol{S}\,\mathbf{Con}(\mathcal{V})$$

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Almost never:

Theorem (RF, 1994)

If a modular congruence variety is finitely based, then it is distributive.

Incidentally, there are $2^{\aleph_0}$ modular congruence varieties.

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If a modular congruence variety is finitely based, then it is distributive.

Incidentally, there are $2^{\aleph_0}$ modular congruence varieties.

Work on extending the commutator to nonmodular varieties, primarily by Kearnes, Kiss, Szendrei and Lipparini, allows us to extend the above result to:

Theorem (RF & P. Lipparini, 2024)

If a proper congruence variety is finitely based, then it is join semidistributive.

Elements of the Proof

- $\mathbf{M}_3$ is projective for every congruence variety except the variety of all lattices,

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    ($SD_\vee$ failure intervals are modular.)

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Corollary

- *Haiman's lattices, $\mathbf{H}_n(\mathbf{F})$, lie in no proper congruence variety.*
- *The lattice of subspaces of a nonarguesian projective plane lies in no proper congruence variety.*

Elements of the Proof

Let $\mathcal{V}$ be a variety of algebras with congruence variety $\mathcal{K}$.

- If $\mathcal{V}$ is not congruence semidistributive, then there is a field **F** such that all vector space lattices over **F** lie in $\mathcal{K}$.
  - In fact they lie in $\boldsymbol{S}\,\mathbf{Con}(\mathcal{V})$.
- For each proper congruence variety $\mathcal{K}$ there is a field **F** such that any nonprincipal ultraproduct of $\{\mathbf{H}_n(\mathbf{F}) : n \geq 3\}$ lies in $\mathcal{K}$.

Using a standard argument we get:

Corollary

If a proper congruence variety is finitely based, then it is join semidistributive.

What about $\mathbf{S}\,\mathbf{Con}(\mathcal{V})$?

An idempotent term $d(x, y, z)$ is a **weak difference term** if

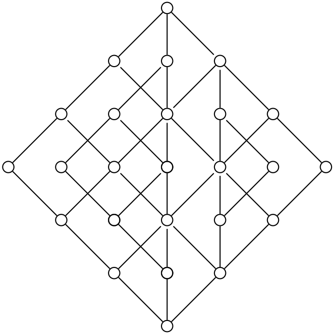
$$d(a, a, b)\ [\theta, \theta]\ b\ [\theta, \theta]\ d(b, a, a)$$

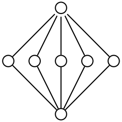
whenever θ is a congruence containing $\langle a, b\rangle$.

Sublattices of congruence lattices: **S Con**($\mathcal{V}$)

Theorem (RF and P. Lipparini, 2024)

Suppose $\mathcal{V}$ is a variety with a weak difference term and that $\mathcal{V}$ is not congruence meet semidistributive. Then every modular lattice you have ever seen a diagram of, can be embedded into a congruence lattice of a member of $\mathcal{V}$.





Sublattices of congruence lattices: $\mathbf{S}\,\mathbf{Con}(\mathcal{V})$

A lattice is 2-**distributive** if it satisfies

$$u \wedge (x \vee y \vee z) \approx (u \wedge (x \vee y)) \vee (u \wedge (x \vee z)) \vee (u \wedge (y \vee z))$$

Let $\mathcal{D}_2$ be the variety of all modular, 2-distributive lattices.

Theorem (RF and Lipparini (2024))

If $\mathcal{V}$ is a variety with a weak difference term and is not meet semidistributive, then

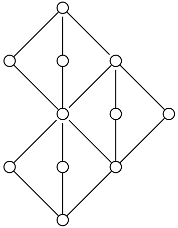
$$\mathcal{D}_2 \subseteq \mathbf{S}\,\mathbf{Con}(\mathcal{V})$$

Sublattices of congruence lattices: **S Con**($\mathcal{V}$)

Agliano, Bertalli and Fioravanti following Kearnes, Kiss and Szendrei show

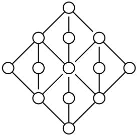
Theorem

If $\mathcal{V}$ is a variety that $\mathcal{V}$ is not congruence meet semidistributive, then all rods and snakes can be embedded into a congruence lattice of a member of $\mathcal{V}$.



Open Problems

If $\mathcal{V}$ is not congruence meet semidistributive, is $\mathbf{M}_4 \in \boldsymbol{S}\,\mathbf{Con}(\mathcal{V})$*?*

What about  ?

Does $\mathbf{Con}(\mathcal{P})$ *have a finite equational basis?* $\mathcal{P}$ is Polin's variety.

Is any proper, nontrivial congruence variety finitely based other than distributive lattices?

Thank You !!

References

P. Agliano and S. Bartali and S. Fioravanti.
On Freese's technique.
Internat. J. of Algebra and Computation, 33(8):1599–1616, 2023.

Emil Artin.
Coordinates in affine geometry.
Rep. Math. Colloquium (2), 2:15–20, 1940.

G. Birkhoff.
Lattice Theory.
Amer. Math. Soc., Providence, R. I., 1948.
rev. ed., Colloquium Publications.

P. Crawley and R. P. Dilworth.
Algebraic Theory of Lattices.
Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

Alan Day and Ralph Freese.
A characterization of identities implying congruence modularity. I.
Canad. J. Math., 32(5):1140–1167, 1980.

R. Dedekind.
Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler,
Festschrift der Herzogl. technische Hochschule zur Naturforscher-Versammlung, Braunschweig, 1897.

R. Dedekind.
Über die von drei Moduln erzeugte Dualgruppe,
Math. Annalen, 53:371–403, 1988.

References

R. Freese.
Finitely based modular congruence varieties are distributive.
Algebra Universalis, 32(1):104–114, 1994.

Ralph Freese, Christian Herrmann, and András P. Huhn.
On some identities valid in modular congruence varieties.
Algebra Universalis, 12(3):322–334, 1981.

R. Freese and B. Jónsson.
Congruence modularity implies the Arguesian identity.
Algebra Universalis, 32(1):104–114, 1994.

R. Freese and P. Lipparini.
Finitely based congruence varieties.
Algebra Universalis, 85(1), 2024.

Ralph Freese and Ralph McKenzie.
Commutator theory for congruence modular varieties, volume 125 of *London Mathematical Society Lecture Note Series*.
Cambridge University Press, Cambridge, 1987.
Online version available at: http://www.math.hawaii.edu/∼ralph/papers.html.

Ralph Freese and J. B. Nation.
Congruence lattices of semilattices.
Pacific J. Math., 49:51–58, 1973.

O. Frink.
Complemented modular lattices and projective spaces of infinite dimension.
Trans. Amer. Math. Soc., 60:452–467, 1946.

References

H. P. Gumm.
Geometrical Methods in Congruence Modular Algebras.
1983.
Memoirs Amer. Math. Soc.

M. Haiman
Arguesian lattices which are not type-1.
Algebra Univsalis., 28:128–137, 1991.

M. Hall and R. P. Dilworth
The imbedding problem for modular lattices.
Ann. of Math., 45:450–456, 1944.

C. Herrmann.
Affine algebras in congruence modular varieties.
Acta Sci. Math. (Szeged), 41:119–125, 1979.

David Hobby and Ralph McKenzie.
The structure of finite algebras, volume 76 of *Contemporary Mathematics*.
American Mathematical Society, Providence, RI, 1988.

B. Jónsson.
Representations of complemented modular lattices.
Math. Scand., 1:193–205, 1953.

References

B. Jónsson.
Arguesian lattices of dimension $n \leq 4$.
Math. Scand., 7:133–145, 1959.

B. Jónsson.
Representations of complemented modular lattices.
Trans. Amer. Math. Soc., 97:64–94, 1960.

B. Jónsson and G. S. Monk.
Representation of primary arguesian lattices.
Pacific J. Math., 30:95–139, 1969.

Keith A. Kearnes and Emil W. Kiss.
The shape of congruence lattices.
Mem. Amer. Math. Soc., 222(1046):viii+169, 2013.

Keith A. Kearnes and Ágnes Szendrei.
The relationship between two commutators.
Internat. J. Algebra Comput., 8(4):497–531, 1998.

Paolo Lipparini.
Commutator theory without join-distributivity.
Trans. Amer. Math. Soc., 346(1):177–202, 1994.

S. V. Polin.
Identities in congruence lattices of universal algebras.
Mat. Zametki, 22(3):443–451, 1977.