Sets of equations implying congruence semidistributivity and *n*-permutability

Ralph Freese

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- Σ', the **derivative** is the augmentation of Σ by equations that say that F is independent of its i^{th} place whenever Σ implies F is weakly independent of its i^{th} place.

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- A similar theorem holds for $\mathcal{V}$ satisfying some congruence identity if

  $$\text{"}\Sigma' \text{ is inconsistent"}$$

  is replaced by

  $$\text{"}\Sigma^{(k)} \text{ is inconsistent for some } k\text{."}$$

n-permutability

- The **order derivative**, Σ^+, augments Σ by

$$x \approx F(\mathbf{w}')$$

whenever $\Sigma \models x \approx F(\mathbf{w})$, where $\mathbf{w}'$ is the same as $\mathbf{w}$ in every place except one, say i, and $\mathbf{w}'_i = x$.

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- The converse of the first statement is true if Σ is linear.
- For a finite linear, idempotent Σ one can effectively decide if Σ implies congruence n-permutability, for some n.

Semidistributivity

- The **weak derivative**, Σ^*, augments Σ by an equation expressing that F is independent of its i^{th} place whenever

$$\Sigma \models x \approx F(x, \ldots, x, y, x, \ldots, x)$$

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- The converse of the first statement is false, **even if Σ is linear**. Nevertheless
- For a finite linear, idempotent Σ one **can** effectively decide if Σ implies congruence semidistributivity.

The Variant Maltsev Condition for Semidistributivity

- A variety is congruence semidistributive iff there are terms $d_i(x, y, z)$, $i = 0, \ldots, n$, such that

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- Let Σ be these equations. Assume inductively that Σ^{*^k} implies $x \approx d_i(x, y, z)$. Then, using the above equations, one can show that $\Sigma^{*^{k+2}}$ implies $x \approx d_{i+1}(x, y, z)$.

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- So some iterated weak derivative implies $x \approx d_n(x, y, z) \approx z$ and so is inconsistent.

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