An Example With Unequal Mixed Partial Derivatives

This example is suggested by Salas and Hille in their textbook, *Calculus, 7th edition*, as problem 43 on page 941:

$$f(x,y) = \begin{cases} \dfrac{xy(y^2 - x^2)}{x^2 + y^2}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0). \end{cases}$$

We shall show that $f_{x,y}(0,0) = +1$ and $f_{y,x}(0,0) = -1$.

Here are the first derivatives:

(1) For $(x,y) \neq (0,0)$, we can use the quotient rule and simplify to obtain

$$f_x(x,y) = \frac{-x^4y - 4x^2y^3 + y^5}{(x^2 + y^2)^2}$$

(2) For $(x,y) = (0,0)$, we need to use the basic definition of derivative as a limit to get

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

(3) For $(x,y) \neq (0,0)$, we can use the quotient rule and simplify to obtain

$$f_y(x,y) = \frac{-x^5 + 4x^3y^2 + xy^4}{(x^2 + y^2)^2}$$

(4) For $(x,y) = (0,0)$, we need to use the basic definition of derivative as a limit to get

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

The computation of $f_{x,y}(0,0)$ is possible by using the limit definition of derivative:

$$\begin{aligned} f_{x,y}(0,0) &= \lim_{h \to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{-0^4 \cdot h - 4 \cdot 0^2 \cdot h^3 + h^5}{(0^2+h^2)^2} - 0}{h} \\ &= \lim_{h \to 0} \frac{h^5/h^4}{h} \\ &= 1 \end{aligned}$$

Here is the computation of $f_{y,x}(0,0)$:

$$\begin{aligned} f_{y,x}(0,0) &= \lim_{h \to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{-h^5 + 4 \cdot h^3 \cdot 0^2 + h \cdot 0^4}{(h^2+0^2)^2} - 0}{h} \\ &= \lim_{h \to 0} \frac{-h^5/h^4}{h} \\ &= -1 \end{aligned}$$