

A real number system is a set \mathbb{R} with two operations $+$ and \cdot and an order relation $<$ which satisfy the following axioms:

- (1) For all x and y in \mathbb{R} , $x + y = y + x$ and $x \cdot y = y \cdot x$ (commutative laws).
- (2) For all x, y and z in \mathbb{R} , $x + (y + z) = (x + y) + z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associative laws).
- (3) For all x, y and z in \mathbb{R} , $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ (multiplication distributes over addition). Addition does NOT distribute over multiplication because $x + (y \cdot z) \neq (x + y) \cdot (x + z)$ for most x, y and z . However, addition DOES distribute over multiplication in Boolean algebras which are important in logic, set theory, electrical engineering and computer science.
- (4) There is some $w \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$, $x + w = x$ [a single thing w does this for all x]. It is a theorem that, if w' also has this property ($x + w' = x$ for all $x \in \mathbb{R}$) then $w = w'$. This means that any w making this equation true is unique and we can give it a special name. It's special name is 0, the additive identity.
- (5) There is some $t \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$, $x \cdot t = x$ [a single thing t does this for all x]. It is a theorem that, if t' also has this property ($x \cdot t' = x$ for all $x \in \mathbb{R}$) then $t = t'$. This means that any t making this equation true is unique and we can give it a special name. It's special name is 1, the multiplicative identity.
- (6) $0 \neq 1$.
- (7) For all $x \in \mathbb{R}$ there is some $q \in \mathbb{R}$ such that $x + q = 0$ [here, q is allowed to be different for different x 's, because the "there is some q " comes AFTER the "for all x ". It is a theorem that if, for a particular x , $x + q = 0$ and $x + q' = 0$ then $q = q'$ [this is proved without using subtraction—which hasn't been introduced yet in this list of axioms and theorems!]. Once we know any q making $x + q = 0$ is unique we can give it a name (that depends upon x , since q is assumed to be doing this job only for the specified x): we give any such q the name $-x$, the additive inverse for x .
- (8) For $x \in \mathbb{R}$ except $x = 0$, there is some $r \in \mathbb{R}$ such that $x \cdot r = 1$ [here, r is allowed to be different for different x 's, because the "there is some r " comes AFTER the "for all x ". It is a theorem that if, for a particular x , $x \cdot r = 1$ and $x \cdot r' = 1$ then $r = r'$ [this is proved without using division—which hasn't been introduced yet in this list of axioms and theorems!]. Once we know any r making $x \cdot r = 1$ is unique we can give it a name (that depends upon x , since r is assumed to be doing this job only for the specified x): we give any such r the name x^{-1} , the multiplicative inverse for x .
- (9) For all x, y and z in \mathbb{R} , if $x < y$ and $y < z$ then $x < z$ (the transitive law).
- (10) For all x and y in \mathbb{R} , exactly one of the following is true: $x < y$, $x = y$, $y < x$. (the trichotomy law).
- (11) For all x, y and z in \mathbb{R} , if $x < y$ and $x + z < y + z$.
- (12) For all x, y and z in \mathbb{R} , if $0 < z$ and $x < y$ then $x \cdot z < y \cdot z$.
- (13) For all subsets $S \subset \mathbb{R}$, if S is non-empty and S has an upper bound

then S has a least upper bound (the completeness axiom).

Remark 1. *The first 8 axioms above define what is known as a field. They are critical to modern mathematics, computer science and some kinds of engineering. A fun field to explore is a field with only two elements, 0 and 1, with the following addition and multiplication rules: $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, $1 + 1 = 0$, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, and $1 \cdot 1 = 1$. In this field of two elements, $-1 = 1$.*

Remark 2. *The first 12 axioms above define what is known as an ordered field. The rational numbers, \mathbb{Q} , are an example [the rational numbers are ratios of integers a and b with $b \neq 0$; the integers are the whole numbers $0, \pm 1, \pm 2, \dots$].*

Remark 3. *The 13th axiom essentially makes any real number system unique. The 13th axiom forces numbers such as $\sqrt{2}$ to be in \mathbb{R} (these numbers are not in \mathbb{Q}). Once it is proved carefully that there is really only one real number system, we begin to talk about the THE real number system.*

Theorem 1. *There are 12 ways to add (multiply) three real numbers, and they are equal to each other.*

Proof. This follows from repeated applications of the commutative and associative laws.

$$\begin{aligned}
 x + (z + y) &= x + (y + z), && \text{by the commutative laws,} \\
 &= (x + y) + z, && \text{by the associative laws,} \\
 &= (y + x) + z, && \text{by the commutative laws,} \\
 &= y + (x + z), && \text{by the associative laws,} \\
 &= y + (z + x), && \text{by the commutative laws,} \\
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 &= (x + z) + y, && \text{by the commutative laws,} \\
 &= x + (z + y), && \text{by the associative laws.}
 \end{aligned}$$

One way to see that there are 12 possibilities: one can permute the variables x , y , and z in $3!$ ways; for each of these permutations, there are two ways to put in the parentheses.

Version 2 Of Theorem 1. *There are 120 ways to write the sum (and the product) of four real numbers, with parentheses in place so that each addition is applied to only two objects. Examples include $(a + b) + (c + d)$, $a + (b + (c + d))$, $a + ((b + c) + d)$, $((a + b) + c) + d$, $(a + (b + c)) + d$, etc. Try listing all 120 ways to add (multiply) four real numbers and use the associative and commutative laws to show that they are all equal.*

Hint: With four numbers, there are 12 ways to write $a + ()$ where b , c and d are added somehow inside the parentheses before a is added last. Theorem 1 can be quoted here to say that these 12 expressions are all equal to each other, without reproving Theorem 1 for this special case. By using Theorem 1 cleverly, you can get 8 groups of 12 expressions, with all expressions in each group of 12 being equal to each other. That leaves two problems: what about the 24 expressions of the

form $(x + y) + (w + v)$ where $x, y, w,$ and v are a, b, c and d in some order, and how do we prove that expressions from different groups of 12 or from this last group of 24 are equal to each other?

Version 3 of Theorem 1. *Think of all the ways to add n real numbers, a_1, \dots, a_n where n is a positive integer. The answer is $c_n \cdot n!$ where c_n is the number of ways to put in parentheses so that each addition refers to two numbers, and $n!$ is the number of ways of putting a_1, \dots, a_n in a specific order from left to right. We have $c_1 = 1, c_2 = 1, c_3 = 2,$ and $c_4 = 5.$ In general, for $n > 1$*

$$c_n = \sum_{j=1}^{n-1} c_j \cdot c_{n-j}.$$

Thus, $c_5 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14$ and $c_6 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42.$ $n! = n(n-1)(n-2) \cdot (3)(2)(1).$ Thus, one can add 6 real numbers in $42 \cdot 720 = 30240$ ways. The commutative laws and associative laws can be used to show that all 30240 expressions are equal to each other! That is the general (hard to prove) theorem: you can add (multiply) n real number in any order and obtain an expression equal to any other order of addition (multiplication).

Theorem 2 (additive identities are unique). *Suppose that w and w' both have the property that, for all $x \in \mathbb{R}, x + w = x$ and $x + w' = x.$ Then $w = w'.$*

Theorem 3 (multiplicative identities are unique). *Suppose that t and t' both have the property that, for all $x \in \mathbb{R}, x \cdot t = x$ and $x \cdot t' = x.$ Then $t = t'.$*

Theorem 4 (additive inverses are unique). *Suppose that for some $x \in \mathbb{R}$ there are q and q' in \mathbb{R} such that $x + q = 0$ and $x + q' = 0.$ Then $q = q'.$*

Theorem 5 (multiplicative inverses are unique). *Suppose that for some $x \in \mathbb{R}$ there are r and r' in \mathbb{R} such that $x \cdot r = 1$ and $x \cdot r' = 1.$ Then $r = r'.$*

Theorem 6. $-0 = 0$ and $1^{-1} = 1.$

Theorem 7. For all x in $\mathbb{R}, -(-x) = x.$ For nonzero x in $\mathbb{R}, (x^{-1})^{-1} = x.$

Theorem 8 (Cancellation Laws). For all $x, y,$ and z in $\mathbb{R}, x + z = y + z$ if and only if $x = y.$ For all x, y and nonzero z in $\mathbb{R}, x \cdot z = y \cdot z$ if and only if $x = y.$

Theorem 9. For all $x \in \mathbb{R}, 0 = x \cdot 0.$ Note that this means that 0 definitely does NOT have a multiplicative inverse: because $1 \neq 0,$ there is no $t \in \mathbb{R}$ such that $0 \cdot t = 1.$ Finally, this also implies that, for nonzero $x \in \mathbb{R}, x^{-1} \neq 0.$

Theorem 10. For all $x \in \mathbb{R}, -x = (-1) \cdot x.$

Theorem 11. For all x and y in $\mathbb{R}, -(x \cdot y) = (-x) \cdot y = x \cdot (-y)$ and $x \cdot y = (-x) \cdot (-y).$ In particular, $(-1) \cdot (-1) = 1,$ from which it follows that $(-1)^{-1} = -1.$

Theorem 12. For nonzero $x \in \mathbb{R}, -x$ is nonzero and $(-x)^{-1} = -(x^{-1}).$

Theorem 13. For x and y in $\mathbb{R},$ if $x \cdot y = 0$ then $x = 0$ or $y = 0.$

Theorem 14. For x and y in $\mathbb{R}, -(x + y) = (-x) + (-y).$ For nonzero x and y in $\mathbb{R}, x \cdot y \neq 0$ and $(x \cdot y)^{-1} = (x^{-1}) \cdot (y^{-1}).$

Definition of Subtraction. $x - y$ is defined to be $x + (-y).$

Theorem 15. *The following follow from the definition of subtraction:*

- (1) *For all $x \in \mathbb{R}$, $x - x = 0$, $x - 0 = x$, and $0 - x = -x$.*
- (2) *For all x and y in \mathbb{R} , $-(x - y) = y - x$.*
- (3) *For all x , y and z in \mathbb{R} , $x - (y - z) = (x + z) - y = z - (y - x)$ and $(x - y) - z = x - (y + z) = (x - z) - y$.*
- (4) *Finally, for all x , y and z in \mathbb{R} , $z \cdot (x - y) = (z \cdot x) - (z \cdot y)$.*
- (5) *For all x and y in \mathbb{R} , $x - y$ is the unique element $w \in \mathbb{R}$ such that $y + w = x$.*