Notes on p-adic L-functions

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Abstract

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Notation and conventions

If $\mathfrak{p}$ is a finite prime, $\mathrm{Frob}_{\mathfrak{p}}$ denotes a geometric Frobenius element. We normalize class field theory so that uniformizers are sent to geometric Frobenii under the Artin reciprocity map. If E is a field and v is a place of E, then E_v denotes the completion of E at v. If X is a scheme over $\mathrm{Spec}\, R$ and S is an R-algebra, we denote the base change of X to S by $X \times_R S$.

1 Deligne's conjecture on special values of motivic L-functions

Fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ and an embedding $\iota_\infty : \overline{\mathbf{Q}} \to \mathbf{C}$. This fixes a complex conjugation in $G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ which we denote Frob_∞.

1.1 The realizations of a motive

Following [D79], we take a rather basic point of view regarding motives, namely we view them as a collection of "realizations" together with some comparison isomorphisms. That is to say, we *define* a motive to be the collection of data that should come out of an actual theory of motives. We recommend [D89, §1] for a lengthy discussion of what one might want in such a definition. For a higher-brow version of this approach see the works of Fontaine and Perrin-Riou where they introduce the Tannakian category of "(pre-)motivic structures" ([Fo92], [FPR94]).

Let E be a number field. Using ι_∞, we identify $\mathrm{Hom}(E, \overline{\mathbf{Q}})$ with $\mathrm{Hom}(E, \mathbf{C})$ and denote either one by J_E. A motive of rank d over $\mathbf{Q}$ with coefficients in E (e.g. the first homology of an abelian variety over $\mathbf{Q}$ with complex multiplication by an order in E) will be, M, given by the following data:

(Betti) a Betti realization $H_B(M)$:

- $H_B(M)$ is a d-dimensional E-vector space,
- it has an E-linear involution F_∞,
- $H_B(M)$ has a Hodge decomposition

$$H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{i,j \in \mathbf{Z}} H^{i,j}(M) \tag{1.1}$$

  such that $F_\infty\big(H^{i,j}(M)\big) \subseteq H^{j,i}(M)$;

(de Rham) a de Rham realization $H_{\mathrm{dR}}(M)$:

- $H_{\mathrm{dR}}(M)$ is a d-dimensional E-vector space,
- equipped with a decreasing filtration F^i_{dR} of E-vector spaces;

(λ-adic) for each finite place λ of E a λ-adic realization $H_\lambda(M)$:

- $H_\lambda(M)$ is a d-dimensional E_λ-vector space,

– with a continuous, E_λ-linear $G_{\mathbf{Q}}$-action $\rho_\lambda : G_{\mathbf{Q}} \to \mathrm{Aut}_{E_\lambda}(H_\lambda(M))$;

together with maps I_∞ and I_λ for each finite place λ of E, all satisfying the following compatibility requirements:

- $H_\lambda(M)$ are compatible in the following sense:
  - there is a finite set S of finite primes of $\mathbf{Q}$ such that for every $p \notin S$ and every finite prime λ of E coprime to p, ρ_λ is unramified at p,
  - given a finite prime p of $\mathbf{Q}$, then for all finite primes λ of E coprime to p and for all $\tau \in J_E$ and every extension $\widetilde{\tau}$ of τ to E_λ, the isomorphism class of
$$\mathrm{WD}_{\widetilde{\tau}}\big(\rho_\lambda|_{G_p}\big)^{\Phi\text{-ss}} \tag{1.2}$$
is independent of λ, τ, and $\widetilde{\tau}$ (see remark 1.1(i) for an explanation),
  - for all finite places p of $\mathbf{Q}$, the characteristic polynomial of Frob_p is rational and independent of λ, i.e.
$$Z_p(T, M) := \det\big(1 - \mathrm{Frob}_p T | H_\lambda(M)^{I_p}\big) \in E_\lambda[T] \tag{1.3}$$
is "rational" (i.e. lies in $E[T]$) and independent of λ coprime to p,
- I_∞ is an isomorphism of $E \otimes_{\mathbf{Q}} \mathbf{C}$-modules
$$I_\infty : H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} H_{\mathrm{dR}}(M) \otimes_{\mathbf{Q}} \mathbf{C} \tag{1.4}$$
such that the filtration F^i_{dR} is given by the Hodge filtration, i.e.
$$I_\infty\left(\bigoplus_{\substack{i' \geq i \\ j}} H^{i',j}(M)\right) = F^i_{\mathrm{dR}} \otimes_{\mathbf{Q}} \mathbf{C}, \tag{1.5}$$
- for each finite place λ of E, I_λ is an isomorphism of E_λ-vector spaces
$$I_\lambda : H_B(M) \otimes_E E_\lambda \xrightarrow{\sim} H_\lambda(M) \tag{1.6}$$
such that $I_\lambda(F_\infty) = \rho_\lambda(\mathrm{Frob}_\infty)$.

The motive M will be called pure of weight $w \in \mathbf{Z}$ if

(Pure$_\infty$) $h^{i,j}(M) = 0$ unless $i + j = w$.

Remark 1.1.

(i) Let p be a finite prime of $\mathbf{Q}$, λ a finite prime of E coprime to p, $\tau \in J_E$, and $\widetilde{\tau}$ an extension of τ to E_λ. Fix a decomposition group G_p of $G_{\mathbf{Q}}$ at p and identify it with $G_{\mathbf{Q}_p}$. Then $\rho_\lambda|_{G_p}$ gives rise to a λ-adic Weil–Deligne representation $\mathrm{WD}\big(\rho_\lambda|_{G_p}\big)$ of the Weil group $W_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ as explained in [D73, §8.4]. The isomorphism class of the WD representation is independent of the choice of decomposition group. By extending coefficients to $\mathbf{C}$ via $\widetilde{\tau}$, we obtain a WD representation $\mathrm{WD}_{\widetilde{\tau}}(\rho_\lambda|_{G_p})$ of $W_{\mathbf{Q}_p}$ over $\mathbf{C}$ which can be compared to the same for other choices of λ, τ, and $\widetilde{\tau}$. We only require that their Frobenius semi-simplifications $\mathrm{WD}_{\widetilde{\tau}}(\rho_\lambda|_{G_p})^{\Phi\text{-ss}}$ be isomorphic.

(ii) We could equally well state the rationality and independence of λ for $Z_p(T,M)$ using the WD representations $\mathrm{WD}(\rho_\lambda|_{G_p}) = (r_{p,\lambda}, N_{p,\lambda})$. We would require that

$$\det\left(1 - \mathrm{Frob}_p T|(\ker N_{p,\lambda})^{\rho(I_p)}\right)$$

be in $E[T]$. By the compatibility imposed on the WD representations for varying λ, the independence of λ of this determinant is automatic.

Example 1.2. A basic example in the theory is $M = H^w(X)$ where X is a smooth projective variety over $\mathbf{Q}$. The notation means "take the wth cohomology of X". This gives a pure motive of weight w over $\mathbf{Q}$ with coefficients in $\mathbf{Q}$ as follows.

- $H_B(M) = H^w(X(\mathbf{C}), \mathbf{Q})$, the singular cohomology with $\mathbf{Q}$-coefficients:
  - complex conjugation acts on the points $X(\mathbf{C})$ and induces an involution on $H^w(X(\mathbf{C}), \mathbf{Q})$,
  - its Hodge decomposition comes from Hodge theory;
- $H_{\mathrm{dR}}(M) = H^w_{\mathrm{dR}}(X/\mathbf{Q})$, the algebraic de Rham cohomology:
  - its filtration comes from the degeneration of the Hodge to de Rham spectral sequence $E_1^{ij} = H^j(X, \Omega^i) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/\mathbf{Q})$;
- $H_\lambda(M) = H^w_{\text{ét}}(X \times_{\mathbf{Q}} \overline{\mathbf{Q}}, E_\lambda)$, the λ-adic étale cohomology:
  - $G_{\mathbf{Q}}$ acts on $X \times_{\mathbf{Q}} \overline{\mathbf{Q}}$ and induces an action on $H^w_{\text{ét}}(X \times_{\mathbf{Q}} \overline{\mathbf{Q}}, E_\lambda)$.

1.2 Motivic L-functions

From the data of these realizations, we can define the L-function of M, as well as its "completed" L-function, its ϵ-factors, and state a conjectural functional equation.

For each complex embedding $\tau \in J_E$, we will define the τ-L-function of M a complex-valued Euler product

$$L(s, M, \tau) = \prod_p L_p(s, M, \tau) \tag{1.7}$$

convergent for $\mathrm{Re}(s)$ sufficiently large, where the product is over all finite places of $\mathbf{Q}$, and the local τ-L-factor at p is

$$L_p(s, M, \tau) = \tau\left((Z_p(T, M))^{-1}\right)\Big|_{T=p^{-s}} \tag{1.8}$$

(if we fix τ, we may drop it from the notation and refer simply to the L-function of M). We can consider the L-functions for different τ as one object: an $E \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathbf{C}^{J_E}$-valued function $L^*(s, M) = (L(s, M, \tau))_{\tau \in J_E}$.

We also want to define a completed L-function by adding local L-factors at infinity, also called Gamma factors. We may define the τ-Gamma factor of M at ∞, $\Gamma(s, M, \tau)$, (or $L_\infty(s, M, \tau)$) by refining the Hodge decomposition of M as follows. The isomorphism $E \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathbf{C}^{J_E}$ and the Hodge decomposition on $H_B(M)$ induces a Hodge decomposition for $\tau \in J_E$

$$H_B(M) \otimes_{E,\tau} \mathbf{C} \cong \bigoplus_{i,j \in \mathbf{Z}} H_\tau^{i,j}(M) \tag{1.9}$$

compatible with F_∞,[1] such that

$$H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{\tau \in J_E} H_B(M) \otimes_{E,\tau} \mathbf{C} \tag{1.10}$$

For $\tau \in J_E$, define

$$h_\tau^{i,j}(M) = \dim_{\mathbf{C}} H_\tau^{i,j}(M) \tag{1.11}$$

and for $\epsilon \in \{0, 1\}$, let

$$h_\tau^{i,i,\epsilon}(M) = \dim_{\mathbf{C}} H_\tau^{i,i}(M)^{F_\infty = (-1)^{i+\epsilon}}. \tag{1.12}$$

Remark 1.3. These numbers are, in fact, independent of τ, which can be seen by comparing with the de Rham realization which is independent of τ.

These numbers determine the τ-Gamma factor as follows. Let

$$\Gamma_{\mathbf{R}}(s) := \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-s}\Gamma(s). \tag{1.13}$$

Define

$$\Gamma(s, M, \tau) := \left(\prod_{i<j} \Gamma_{\mathbf{C}}(s-i)^{h_\tau^{i,j}} \right) \times \left(\prod_{i=j} \prod_{\epsilon \in \{0,1\}} \Gamma_{\mathbf{R}}(s-i+\epsilon)^{h_\tau^{i,i,\epsilon}} \right). \tag{1.14}$$

The completed τ-L-function of M is defined as

$$\Lambda(s, M, \tau) = \Gamma(s, M, \tau) L(s, M, \tau) \tag{1.15}$$

for $\mathrm{Re}(s)$ sufficiently large. Again, these can be packaged up for all τ as $\Lambda^*(s, M)$.

Remark 1.4. One can define the local ϵ-factors everywhere as in [Ta79, §3], define the global ϵ-factor as the product of the local ones, and obtain the following conjecture.

Conjecture 1.5. *For each $\tau \in J_E$,*

$$\Lambda(s, M, \tau) = \epsilon(s, M, \tau)\Lambda(1-s, M^\vee, \tau). \tag{1.16}$$

1.3 Deligne's rationality conjecture

Definition 1.6. An integer n is critical for M if neither $\Gamma(s, M, \tau)$, nor $\Gamma(1-s, M^\vee, \tau)$, has a pole at $s = n$. By remark 1.3, this definition is independent of $\tau \in J_E$. The motive is called critical if $s = 0$ is critical for M.

[1] Since F_∞ is E-linear.

Remark 1.7.

(i) If there is an i such that $h^{i,i,0} \neq 0 \neq h^{i,i,1}$, then no integers are critical for M.

(ii) In [G89], Greenberg makes the following conjecture: if p is ordinary for M, then

$$\operatorname{corank}_\Lambda \operatorname{Sel}_{\mathbf{Q}_\infty}(H_p(M) \otimes \mathbf{Q}_p/\mathbf{Z}_p) = \text{order of the pole of } \Gamma(s, M) \text{ at } s = 1.$$

Theorem 1 of *loc. cit.* states that if M is pure, then the inequality $\geq$ holds.

From now on, assume M is critical and pure of weight w. Thus, if w is even, this forces F_∞ to act as a scalar (either $+1$ or -1) on $H^{\frac{w}{2},\frac{w}{2}}(M)$. Let $H_B^\pm(M) := H_B(M)^{F_\infty=\pm1}$ and let $d^\pm(M) := \dim_{\mathbf{C}} H_B^\pm(M)$. Since F_∞ interchanges $H^{i,j}(M)$ and $H^{j,i}(M)$ and acts as a scalar on $H^{i,i}(M)$, one of $d^\pm(M)$ is

$$\sum_{i>j} h^{i,j}$$

and the other is

$$\sum_{i\geq j} h^{i,j}.$$

This then picks out F^+ and F^- in $\{F^i_{\mathrm{dR}}\}$. Let $H^\pm_{\mathrm{dR}}(M) := H_{\mathrm{dR}}(M)/F^\mp$. The isomorphism I_∞ then induces isomorphisms

$$I^\pm : H_B^\pm(M) \otimes_{\mathbf{Q}} \mathbf{C} \longrightarrow H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{I_\infty} H_{\mathrm{dR}}(M) \otimes_{\mathbf{Q}} \mathbf{C} \longrightarrow H^\pm_{\mathrm{dR}}(M) \otimes_{\mathbf{Q}} \mathbf{C}. \tag{1.17}$$

Pick E-rational bases of $H_B^\pm(M)$ and $H^\pm_{\mathrm{dR}}(M)$ and let

$$c^\pm_\infty(M) := \det I^\pm \in (E \otimes_{\mathbf{Q}} \mathbf{C})^\times \tag{1.18}$$

which is well-defined up to an element in $E^\times$. Deligne's conjecture on the rationality of special values of L-functions is then the following.

Conjecture 1.8. *Suppose M is critical and $L^*(0, M) \neq 0$, then*

$$L^*(0, M) \underset{E}{\sim} c^+_\infty(M), \tag{1.19}$$

where $\underset{E}{\sim}$ means equal up to multiplication by an element of $E^\times$.

1.4 Example: Tate motives $\mathbf{Q}(n)$

Let's begin with $\mathbf{Q}(1)$, *the* Tate motive. It is a motive over $\mathbf{Q}$ with coefficients in $\mathbf{Q}$. But what should it be? Well, we know what we want its ℓ-adic realizations to be: the ℓ-adic cyclotomic character $\mathbf{Q}_\ell(1)$. This is the ℓ-adic Tate module of $\mathbf{G}_{m/\mathbf{Q}}$, i.e. the dual of the étale H^1. This suggests that $\mathbf{Q}(1)$ is "$H_1(\mathbf{G}_m)$". In fact, we can view this as the cohomology of a smooth projective variety.

Lemma 1.9. $H_1(\mathbf{G}_m) = H^2(\mathbf{P}^1)^\vee$.

Sketchy justification. Cover $\mathbf{P}^1$ by $U := \mathbf{P}^1 \setminus \{0\}$ and $V := \mathbf{P}^1 \setminus \{\infty\}$, and note that $U \cap V = \mathbf{G}_m$. Mayer–Vietoris gives the exact sequence

$$H^1(U) \oplus H^1(V) \longrightarrow H^1(U \cap V) \longrightarrow H^2(\mathbf{P}^1) \longrightarrow H^2(U) \oplus H^2(V).$$

The terms on either end vanish which yields the dual of the isomorphism we stated. □

So, $\mathbf{Q}(1)$ is the dual of the motive $H^2(\mathbf{P}^1)$, and hence is pure of weight -2. What is its Betti realization? It's $H_1(\mathbf{C}^\times, \mathbf{Q})$ which is a one-dimensional $\mathbf{Q}$-vector space generated by the counterclockwise circle γ_0 around the origin. Complex conjugation flips the complex plane across the real line, so it reverses the orientation of γ_0, i.e. $F_\infty = -1$. A standard fact about the cohomology of projective space says that

$$H^j(\mathbf{P}^n_{\mathbf{C}}, \Omega_{X/\mathbf{C}}) = \begin{cases} 0, & i \neq j \\ \mathbf{C}, & i = j, \end{cases} \tag{1.20}$$

(see e.g. [Har-AG, Exercise III.7.3]). Thus, the Hodge decomposition is

$$H_B(\mathbf{Q}(1)) \otimes_{\mathbf{Q}} \mathbf{C} = H^{-1,-1}(\mathbf{Q}(1)).$$

This forces the de Rham realization to be a one-dimensional $\mathbf{Q}$-vector space whose filtration is determined by $\mathrm{gr}^{-1}_{\mathrm{dR}} = \mathbf{Q}$. Of course, the de Rham realization is just the dual of $H^1_{\mathrm{dR}}(\mathbf{G}_m)$, the latter being generated by the differential $\omega_0 = \frac{dz}{z}$.

Then, for $n > 0$, we define $\mathbf{Q}(n) := \mathbf{Q}(1)^{\otimes n}$, and for $n < 0$, $\mathbf{Q}(n) := (\mathbf{Q}(1)^\vee)^{\otimes n}$. For $\mathbf{Q}(0) = \mathbf{Q}$, we can either take $\mathbf{Q}(1) \otimes \mathbf{Q}(-1)$, or the motive $H^0(\operatorname{Spec} \mathbf{Q})$. The weight of $\mathbf{Q}(n)$ is $-2n$.

What integers are critical for $\mathbf{Q}(n)$? Well, the Hodge decomposition of $\mathbf{Q}(n)$ is just $H^{-n,-n}$ and $F_\infty = (-1)^n = (-1)^{-n+\epsilon}$, so $\epsilon = 0$. Therefore,

$$\Gamma(s, \mathbf{Q}(n)) = \Gamma_{\mathbf{R}}(s + n).$$

Accordingly, $\Gamma(1 - s, \mathbf{Q}(n)^\vee) = \Gamma_{\mathbf{R}}(1 - s + n)$. Thus, the critical integers for $\mathbf{Q}(n)$ are

$$-n + (\text{negative odd})$$

and

$$-n + (\text{postive even}).$$

So, $\mathbf{Q}(n)$ is critical if, and only if, n is negative odd or positive even. Note that $L(0, \mathbf{Q}(n)) = \zeta(n)$, so Deligne's conjecture is about the rationality of these values of the Riemann zeta function in this case.

What are the periods?

- If n is negative odd, then $F_\infty = -1$, so $H_B^+(\mathbf{Q}(n)) = 0$, so $c^+(M) = 1$. This agrees with the fact that $\zeta(\text{negative odd}) \in \mathbf{Q}$.
- If n is positive even, then $F_\infty = 1$, so $H_B^+(\mathbf{Q}(n)) = H_B(\mathbf{Q}(n))$, $H_{\mathrm{dR}}^+(\mathbf{Q}(n)) = H_{\mathrm{dR}}(\mathbf{Q}(n))$,

and $I^+ = I_\infty$. The complex comparison isomorphism I_∞ for $\mathbf{Q}(1)$ is given by the pairing

$$\begin{array}{ccc} H_1(\mathbf{C}^\times, \mathbf{C}) \times H^1_{\mathrm{dR}}(\mathbf{C}^\times) & \longrightarrow & \mathbf{C} \\ (\gamma, \omega) & \mapsto & \displaystyle\int_\gamma \omega. \end{array}$$

A basis for $H_B(\mathbf{Q}(n))$ is $\gamma_0^{\otimes n}$ and a basis for $H_{\mathrm{dR}}(\mathbf{Q}(n))$ is $\omega_0^{\vee,\otimes n}$. Pairing $\gamma_0^{\otimes n}$ with $\omega_0^{\otimes n}$ gives

$$\left(\int_{\gamma_0} \frac{dz}{z}\right)^n = (2\pi i)^n.$$

So, the image of $\gamma_0^{\otimes n}$ under I_∞ is the linear functional on $H^1_{\mathrm{dR}}(\mathbf{G}_m)^{\otimes n}$ which sends $\omega_0^{\otimes n}$ to $(2\pi i)^n$, whereas $\omega_0^{\vee,\otimes n}$ sends it to 1. Thus,

$$\det I^+ = c^+_\infty(M) = (2\pi i)^n.$$

Thus, Deligne's conjecture is verified in this case, since we know that, for n positive even,

$$\zeta(n) \underset{\mathbf{Q}}{\sim} \pi^n.$$

1.5 Example: Artin motives

Another basic example is that of motives arising from Artin representations of $G_{\mathbf{Q}}$. Of particular importance to us will be the rank 1 examples as they correspond to Dirichlet characters, and twisting L-values by Dirichlet characters is a fundamental operation in the theory of p-adic L-functions.

Let $\rho : G_{\mathbf{Q}} \to \mathrm{Aut}_E(V)$ be an Artin representation, i.e. E is some number field and V is a d-dimensional vector space over E, in particular $\mathrm{im}\,\rho$ is finite. Let $[\rho]$ denote the motive over $\mathbf{Q}$ with coefficients in E associated to ρ. What are its realizations?

Proposition 1.10.

(i) $H_B([\rho]) = V$ *with* $F_\infty = \rho(\mathrm{Frob}_\infty)$ *and*

$$H^{i,j}([\rho]) = \begin{cases} V \otimes_{\mathbf{Q}} \mathbf{C}, & \text{if } i = j = 0 \\ 0, & \text{otherwise,} \end{cases}$$

(ii) $H_{\mathrm{dR}}([\rho]) = (V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{G_{\mathbf{Q}}}$ *with* $\mathrm{gr}^0_{\mathrm{dR}} = H_{\mathrm{dR}}([\rho])$,

(iii) $H_\lambda([\rho]) = V \otimes_E E_\lambda$,

(iv) the isomorphism I_∞ *is the inverse of the map*

$$\begin{array}{ccc} (V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{G_{\mathbf{Q}}} \otimes_{\mathbf{Q}} \mathbf{C} & \xrightarrow{\sim} & V \otimes_{\mathbf{Q}} \mathbf{C} \\ (v \otimes \alpha) \otimes z & \mapsto & v \otimes \iota_\infty(\alpha) z. \end{array}$$

In particular, $[\rho]$ *is a pure motive of weight 0.*

Sketch of part (iv). We provide suggestive evidence for obtaining part (iv) from the Betti and de Rham realizations. Since $H^1(G_{\mathbf{Q}}, \mathrm{GL}(d, \overline{\mathbf{Q}})) = 1$, $(V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{G_{\mathbf{Q}}}$ is a d-dimensional $\mathbf{Q}$-vector space and there is a canonical isomorphism of $G_{\mathbf{Q}}$-modules

$$\begin{array}{ccc} (V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{G_{\mathbf{Q}}} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} & \xrightarrow{\sim} & V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} \\ (v \otimes \alpha) \otimes \beta & \mapsto & v \otimes \alpha\beta \end{array}$$

where $G_{\mathbf{Q}}$ acts only on the $\overline{\mathbf{Q}}$ factor on both sides. The extension of scalars to $\mathbf{C}$, via ι_∞, of this *canonical* map would certainly seem like a good thing for (the inverse of) I_∞ to be. □

Remark 1.11. We won't get into the proof of this proposition, but we will make a few leading statements, in particular addressing the reason we call it a proposition as opposed to a definition. To start off, how does one find a "geometric avatar" of an Artin representation? Here's one way. As noted, $\operatorname{im}\rho$ is a finite set, and it has an action of $G_{\mathbf{Q}}$. Grothendieck's version of Galois theory states that there's an equivalence of categories between finite $G_{\mathbf{Q}}$-sets and finite étale $\mathbf{Q}$-algebras (i.e. finite products of number fields). The correspondence is the following: to a finite $G_{\mathbf{Q}}$-set S, associate the ring of $G_{\mathbf{Q}}$-invariant functions $S \to \overline{\mathbf{Q}}$; to a finite étale $\mathbf{Q}$-algebra A, associate the finite $G_{\mathbf{Q}}$-set $\mathrm{Hom}(A, \overline{\mathbf{Q}})$ (i.e. the $\overline{\mathbf{Q}}$-points of $\operatorname{Spec} A$). For example, taking $A = F$ a Galois number field, the corresponding $G_{\mathbf{Q}}$-set is J_F with $G_{\mathbf{Q}}$ acting through $\mathrm{Gal}(F/\mathbf{Q})$. This gives the regular representation of $\mathrm{Gal}(F/\mathbf{Q})$.

To an Artin representation, we've attached a zero-dimensional variety X over $\mathbf{Q}$, and we could suspect that all Artin representations could be found in the cohomology of zero-dimensional varieties over $\mathbf{Q}$ (necessarily in H^0). Since $\dim_{\mathbf{Q}} H^0(X(\mathbf{C}), \mathbf{Q}) =$ the number of connected components of $X(\mathbf{C})$, we see that $\mathbf{Q}$ is the only finite étale $\mathbf{Q}$-algebra whose H^0 will give a rank 1 motive over $\mathbf{Q}$. Thus, in simply taking cohomology of zero-dimensional varieties over $\mathbf{Q}$, we fail to obtain, for example, the non-trivial Artin characters over $\mathbf{Q}$. To see an instance of what is going on, consider $F = \mathbf{Q}(\sqrt{d})$, for d a negative fundamental character, let $\chi_d = \left(\frac{d}{\cdot}\right)$ be its quadratic character, and let r_d be its regular representation. It is easy to verify that $\operatorname{im}\chi_d \cong \operatorname{im} r_d$ as $G_{\mathbf{Q}}$-sets. Can this be fixed? Yes. By using correspondences. The idea behind motives is that they should be those pieces of the cohomology of smooth projective varieties that can be cut out by algebraic correspondences. Applying this to the situation at hand, one can, with some work, find that every Artin representation factoring through $\mathrm{Gal}(F/\mathbf{Q})$ can be found inside $H^0(\operatorname{Spec} F)$. One could then prove proposition 1.10 by studying the various cohomologies of $\operatorname{Spec} F$. Perhaps this remark will morph into a proof in a subsequent version of these notes.

We should also remark that since we have taken geometric Frobenii, $L(s, [\rho]) = L(s, \rho^\vee)$, where the latter is the Artin L-function of the contragredient of ρ.

Now, let $[\rho](n) := [\rho] \otimes (\mathbf{Q}(n) \otimes_{\mathbf{Q}} E)$. It is pure of weight $-2n$, with Hodge decomposition $H^{-n,-n}$. What are its Γ-factors and its critical integers? The answer splits up into three cases.

- ρ is even (i.e. $\rho(\mathrm{Frob}_\infty) = 1$): then for $[\rho]$, $F_\infty = 1$, so for $[\rho](n)$, $F_\infty = (-1)^n = (-1)^{-n+0}$, so $\epsilon = 0$, and $h_\tau^{-n,-n,0} = d$. Its Γ-factor at $\tau \in J_E$ is thus

$$\Gamma(s, [\rho](n), \tau) = \Gamma_{\mathbf{R}}(s + n)^d.$$

Accordingly, it has the same critical integers as $\mathbf{Q}(n)$, i.e.

$$-n + (\text{negative odd})$$

and

$$-n + (\text{positive even}).$$

Thus, $[\rho](n)$ is critical if, and only if, n is negative odd or positive even (in particular, $[\rho]$ is not critical).

- ρ is odd (i.e. $\rho(\mathrm{Frob}_\infty) = -1$): then for $[\rho]$, $F_\infty = -1$, so for $[\rho](n)$, $F_\infty = (-1)^{n+1} = (-1)^{-n+1}$, so $\epsilon = 1$, and $h_\tau^{-n,-n,1} = d$. Its Γ-factor at $\tau \in J_E$ is thus

$$\Gamma(s, [\rho](n), \tau) = \Gamma_{\mathbf{R}}(s + n + 1)^d.$$

It therefore has "more" critical integers than in the even case:

$$-n + (\text{positive odd})$$

and

$$-n + (\text{non-positive even}).$$

Thus, $[\rho](n)$ is critical if, and only if, n is positive odd or non-positive even (in particular, $[\rho]$ *is* critical).

- if ρ is neither even nor odd, there are no critical points by remark 1.7(i).

1.5.1 Special case: Characters

In this section, we restrict to the case of characters $\rho : G_{\mathbf{Q}} \to E^\times$ and determine their periods. Fix $\underline{i}$ a square root of -1.[2] This breaks up into two cases based on whether $F_\infty = \pm 1$.

- ρ and n have opposite parity (ρ even, n odd (negative), or ρ odd, n even (non-positive)): then $F_\infty = -1$, so $H_B^+([\rho](n)) = 0$, so
$$c_\infty^+([\rho](n)) = 1.$$

- ρ and n have the same parity (ρ even, n even (positive), or ρ odd, n odd (positive)): then $F_\infty = 1$, so $H_B^+([\rho](n)) = H_B([\rho](n))$ with basis $1 \otimes \gamma_0^{\otimes n}$, and $H_{\mathrm{dR}}^+([\rho](n)) = H_{\mathrm{dR}}([\rho](n))$ with basis $g_{\underline{i},\rho} \otimes \omega_0^{\vee,\otimes n}$ where $g_{\underline{i},\rho}$ is the Gauss sum of the following lemma.

Lemma 1.12. *Let $\mathfrak{f}_\rho \in \mathbf{Z}$ be the conductor of ρ and let*

$$g_{\underline{i},\rho} := \sum_{a \in (\mathbf{Z}/\mathfrak{f}_\rho)^\times} \rho(a) \otimes \iota_\infty^{-1}\left(\exp(2\pi \underline{i} a/\mathfrak{f}_\rho)\right) \in E \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}.$$

Then, $g_{\underline{i},\rho}$ is a basis of $H_{\mathrm{dR}}([\rho]) = (V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{G_{\mathbf{Q}}}$.

[2] We introduce this choice in our notation because it will be used when discussing the conjectures of Coates and Perrin-Riou.

Proof. The action of $G_{\mathbf{Q}}$ on $g_{\underline{i},\rho}$ simply permutes the factors. □

The inverse of the comparison isomorphism for $[\rho]$ is $(E \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{G_{\mathbf{Q}}} \otimes_{\mathbf{Q}} \mathbf{C} \to E \otimes_{\mathbf{Q}} \mathbf{C}$ given by

$$\sum_{a \in (\mathbf{Z}/\mathfrak{f}_\rho)^\times} \rho(a) \otimes \iota_\infty^{-1} \left(\exp(2\pi \underline{i} a/\mathfrak{f}_\rho)\right) \otimes 1 \mapsto \sum_{a \in (\mathbf{Z}/\mathfrak{f}_\rho)^\times} \rho(a) \otimes \iota_\infty \left(\iota_\infty^{-1} \left(\exp(2\pi \underline{i} a/\mathfrak{f}_\rho)\right)\right),$$

so, for $[\rho](n)$ in the basis above

$$c^+_{\infty,\underline{i}}([\rho](n)) = \det I^+ = \det I_\infty = G_{\underline{i}}(\rho)^{-1}(2\pi i)^n,$$

where

$$G_{\underline{i}}(\rho) := \sum_{a \in (\mathbf{Z}/\mathfrak{f}_\rho)^\times} \rho(a) \otimes \exp(2\pi \underline{i} a/\mathfrak{f}_\rho) \in E \otimes_{\mathbf{Q}} \mathbf{C}$$

(independent of ι_∞). Actually, both [D79] and [Co91] use the following period

$$\delta_{\underline{i}}(\rho)(2\pi i)^n$$

where

$$\delta_{\underline{i}}(\rho) := G_{-\underline{i}}(\rho^{-1}).$$

They can do this because for every $\tau \in J_E$,

$$\tau(G_{\underline{i}}(\rho))\overline{\tau(G_{\underline{i}}(\rho))} = \mathfrak{f}_\rho \in \mathbf{Q}$$

and

$$\tau(G_{-\underline{i}}(\rho^{-1})) = \overline{\tau(G_{\underline{i}}(\rho))},$$

so

$$\delta_{\underline{i}}(\rho) \underset{\mathbf{Q}}{\sim} G_{\underline{i}}(\rho)^{-1}.$$

Remark 1.13. By class field theory, this example also covers the case of Dirichlet characters. Specifically, let $\chi : (\mathbf{Z}/N)^\times \to \mathbf{C}^\times$ be a primitive Dirichlet character. There is a canonical isomorphism from the ray class group $C\ell_{\mathbf{Q}}^{N\infty}$ to $(\mathbf{Z}/N)^\times$. Furthermore, the reciprocity map provides an isomorphism $C\ell_{\mathbf{Q}}^{N\infty} \xrightarrow{\sim} \mathrm{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q})$. These two isomorphisms then give $\rho_\chi : G_{\mathbf{Q}} \to \mathrm{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q}) \to \mathbf{C}^\times$ making the following diagram commute

$$\begin{array}{ccc} (\mathbf{Z}/N)^\times & \longrightarrow & \mathbf{C}^\times \\ \wr\uparrow & & \uparrow \rho_\chi \\ C\ell_{\mathbf{Q}}^{N\infty} & \xrightarrow[\mathrm{rec}]{\sim} & \mathrm{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q}) \end{array}$$

We remark that under our convention for the reciprocity map, $L(s,\chi) = L(s,\rho_{\chi^{-1}}) = L(s,[\rho_\chi])$ (where the first object is the usual Dirichlet L-function of χ). Accordingly, we define $[\chi] := [\rho_\chi]$.

1.5.2 Special case: $\operatorname{Spec} F$

In this section, we restrict to the case $[\rho] = H^0(\operatorname{Spec} F)$, which is the regular representation of $\operatorname{Gal}(F/\mathbf{Q})$ when $F/\mathbf{Q}$ is Galois. For any F, we have

$$L(s,\rho) = \zeta_F(s)$$

the Dedekind zeta function of F. By Grothendieck's Galois theory, we find that we are dealing with the Galois representation

$$\rho : G_{\mathbf{Q}} \longrightarrow \operatorname{Aut}_{\mathbf{Q}}(V)$$

where $V = \mathbf{Q}^{J_F}$ with $G_{\mathbf{Q}}$ acting by permutations on J_F. In particular, we can identify $H_{\mathrm{dR}}([\rho])$ with F via

$$\begin{array}{ccc} F & \xrightarrow{\sim} & H_{\mathrm{dR}}([\rho]) \subseteq \mathbf{Q}^{J_F} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} \\ a & \mapsto & \displaystyle\sum_{\tau \in J_F} e_\tau \otimes \tau(a), \end{array}$$

where e_τ is the vector with a 1 in the τ-component and zeroes elsewhere. The inverse of I_∞ is thus given by

$$\begin{array}{ccccc} F \otimes_{\mathbf{Q}} \mathbf{C} & \xrightarrow{\sim} & \mathbf{Q}^{J_F} \otimes_{\mathbf{Q}} \mathbf{C} & \xrightarrow{\sim} & \mathbf{C}^{J_F} \\ a \otimes 1 & \mapsto & \displaystyle\sum_{\tau \in J_F} e_\tau \otimes \tau(a) & \mapsto & (\tau(a))_{\tau \in J_F}. \end{array}$$

The vectors e_τ form a rational basis of $H_B([\rho])$ and as a rational basis of $H_{\mathrm{dR}}([\rho]) = F$ we take an integral basis $\{a_1, \ldots, a_d\}$ of $F/\mathbf{Q}$. It is then clear that the determinant of I_∞ in this basis is the inverse of the determinant of the matrix whose (i,τ)-component is $\tau(a_i)$. Thus,

$$\det I_\infty = \frac{1}{\sqrt{|\Delta_F|}} \tag{1.21}$$

where Δ_F is the absolute discriminant of F.

But when is $[\rho](n)$ critical? Not very often in fact. Since $F_\infty = \rho(\mathrm{Frob}_\infty)$, we see that unless F is totally real, F_∞ will act on J_F with both eigenvalues ± 1 showing up, i.e. unless F is totally real, ρ is neither even nor odd and hence has no critical points. Let's not be too disappointed though since the functional equation and Gamma factors of the Dedekind zeta function imply that for F *not* totally real $\zeta_F(n) = 0$ for all negative integers. Indeed, letting r_1 (resp. r_2) denote the number of real (resp. complex) places of F, then the Gamma factor is

$$\Gamma(s,[\rho]) = \Gamma_{\mathbf{R}}(s)^{r_1+r_2}\Gamma_{\mathbf{R}}(s+1)^{r_2},$$

the functional equation is

$$\Lambda(s,[\rho]) = \sqrt{|\Delta_F|}^{1-2s}\Lambda(1-s,[\rho]).$$

Now, let's consider the case where F is totally real. Then ρ is even, so $[\rho](n)$ is critical if, and only if, n is negative odd or positive even. Again, one can see that $\zeta_F(n)$ is automatically zero at negative even integers. If n is negative odd, then $F_\infty = -1$, so $c_\infty^+([\rho](n) = 1$. On the other hand, when n is positive even, $F_\infty = 1$ and $I^+ = I_\infty$. Using the above calculation of $\det I_\infty$ for $[\rho]$, we find that

$$c_\infty^+([\rho](n)) = \frac{(2\pi i)^n}{\sqrt{|\Delta_F|}}, \text{ for } n \text{ positive even.}$$

What a coincidence! A result due to Siegel shows that $\zeta_F(n) \in \mathbf{Q}^\times$ for negative odd n. And the functional equation then confirms the rationality of $\zeta_F(n)/c_\infty^+([\rho](n))$ for n positive even.

Appendix: Properties of the Gamma function

In this appendix, we would like to list some of the well-known properties of the function $\Gamma(s)$ and translate them into properties of $\Gamma_{\mathbf{R}}(s)$ and $\Gamma_{\mathbf{C}}(s)$.

To begin, there is the duplication formula

$$\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s) \tag{1.22}$$

which yields the following.

Lemma 1.14.

$$\Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1) = \Gamma_{\mathbf{C}}(s). \tag{1.23}$$

Proof.

$$\begin{aligned} \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1) &= \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\pi^{\frac{-s-1}{2}}\Gamma\left(\frac{s+1}{2}\right) \\ &= \pi^{-s-\frac{1}{2}}2^{1-s}\sqrt{\pi}\Gamma(s) \\ &= 2(2\pi)^{-s}\Gamma(s) \\ &= \Gamma_{\mathbf{C}}(s). \end{aligned}$$

□

Secondly, there is the functional equation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{1.24}$$

Applying this to $\Gamma_{\mathbf{C}}(s)$ yields

$$\begin{aligned} \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(1-s) &= 2(2\pi)^{-s}\Gamma(s)2(2\pi)^{s-1}\Gamma(1-s) \\ &= \frac{4}{2\pi}\frac{\pi}{\sin(\pi s)} \\ &= \frac{2}{\sin(\pi s)}. \end{aligned} \tag{1.25}$$

We obtain the following.

Lemma 1.15. *For all $n \in \mathbf{Z}$,*

$$\Gamma_{\mathbf{C}}(s-n)\Gamma_{\mathbf{C}}(1-s+n) = (-1)^n\frac{2}{\sin(\pi s)}. \tag{1.26}$$

Proof. By (1.25),

$$\begin{aligned}\Gamma_{\mathbf{C}}(s-n)\Gamma_{\mathbf{C}}(1-s+n) &= \frac{2}{\sin(\pi(s-n))} \\ &= \frac{2}{\sin(\pi s)\cos(-\pi n)+\sin(\pi n)\cos(\pi s)} \\ &= \frac{2}{\sin(\pi s)\cos(-\pi n)} \\ &= (-1)^n\frac{2}{\sin(\pi s)}.\end{aligned}$$

□

We can give a more symmetric form to the functional equation.

Lemma 1.16. *For* $\epsilon \in \{0,1\}$, *let*

$$\Gamma_{\mathbf{C},\epsilon}(s) = \Gamma_{\mathbf{C}}(s)\cos\left(\frac{\pi}{2}(s-\epsilon)\right).$$

Then,

$$\frac{1}{\Gamma_{\mathbf{C},\epsilon}(s)} = \Gamma_{\mathbf{C},\epsilon}(1-s). \tag{1.27}$$

Proof. For $\epsilon = 0$,

$$\begin{aligned}\frac{1}{\Gamma_{\mathbf{C},0}(s)} &= \frac{1}{\Gamma_{\mathbf{C}}(s)\cos\left(\frac{\pi}{2}s\right)} \\ &= \frac{2\sin\left(\frac{\pi}{2}s\right)}{\Gamma_{\mathbf{C}}(s)\sin(\pi s)} && \text{(double angle formula)} \\ &= \frac{\Gamma_{\mathbf{C}}(1-s)\sin(\pi s)2\sin\left(\frac{\pi}{2}s\right)}{2\sin(\pi s)} && \text{(by (1.25))}.\end{aligned}$$

The claim follows from the easy fact that

$$\sin\left(\frac{\pi}{2}s\right) = \cos\left(\frac{\pi}{2}(1-s)\right). \tag{1.28}$$

For $\epsilon = 1$,

$$\begin{aligned}\frac{1}{\Gamma_{\mathbf{C},1}(s)} &= \frac{1}{\Gamma_{\mathbf{C}}(s)\cos\left(\frac{\pi}{2}(s-1)\right)} \\ &= \frac{1}{\Gamma_{\mathbf{C}}(s)\sin\left(\frac{\pi}{2}s\right)} && \text{(by (1.28))} \\ &= \frac{\Gamma_{\mathbf{C}}(1-s)\sin(\pi s)}{2\sin\left(\frac{\pi}{2}s\right)} && \text{(by (1.25))} \\ &= \Gamma_{\mathbf{C}}(1-s)\cos\left(\frac{\pi}{2}s\right) && \text{(double angle formula)}.\end{aligned}$$

The claim follows from the easy fact that

$$\cos\Big(\frac{\pi}{2}s\Big) = \cos\Big(\frac{\pi}{2}(1-s-1)\Big)\,.$$

□

For $\Gamma_{\mathbf{R}}(s)$, the situation is more tedious. The following lemma gives some useful relations.

Lemma 1.17. *For* $\epsilon \in \{0,1\}$,

$$\frac{\Gamma_{\mathbf{R}}(s+\epsilon)}{\Gamma_{\mathbf{R}}(1-s+\epsilon)} = \Gamma_{\mathbf{C}}(s)\cos\Big(\frac{\pi}{2}(s-\epsilon)\Big)\,, \tag{1.29}$$

hence, for all $n \in \mathbf{Z}$,

$$\frac{\Gamma_{\mathbf{R}}(s-n+\epsilon)}{\Gamma_{\mathbf{R}}(1-s+n+\epsilon)} = \begin{cases} (-1)^{(n+\epsilon)/2}\Gamma_{\mathbf{C}}(s-n)\cos\big(\frac{\pi}{2}s\big)\,, & n \equiv \epsilon \pmod 2 \\ (-1)^{(n+\epsilon-1)/2}\Gamma_{\mathbf{C}}(s-n)\sin\big(\frac{\pi}{2}s\big)\,, & n \not\equiv \epsilon \pmod 2. \end{cases} \tag{1.30}$$

Proof. We being by proving the first relation. For $\epsilon = 0$,

$$\begin{aligned} \frac{\Gamma_{\mathbf{R}}(s)}{\Gamma_{\mathbf{R}}(1-s)} &= \frac{\pi^{-s/2}\Gamma(s/2)}{\pi^{(s-1)/2}\Gamma((1-s)/2)} \\ &= \frac{\sqrt{\pi}\Gamma(s/2)\Gamma((s+1)/2)\sin(\pi(1-s)/2)}{\pi^{s+1}} && \text{(by (1.24))} \\ &= \frac{\sqrt{\pi}2^{1-s}\sqrt{\pi}\Gamma(s)\sin(\pi(1-s)/2)}{\pi^{s+1}} && \text{(by the duplication formula)} \\ &= \Gamma_{\mathbf{C}}(s)\cos\big(\tfrac{\pi}{2}s\big), \end{aligned}$$

as desired, where we have used the simple identity

$$\sin\Big(\frac{\pi}{2}(1-s)\Big) = \cos\Big(\frac{\pi}{2}s\Big)\,.$$

For $\epsilon = 1$,

$$\begin{aligned} \frac{\Gamma_{\mathbf{R}}(s+1)}{\Gamma_{\mathbf{R}}(1-s+1)} &= \frac{\pi^{-(s+1)/2}\Gamma((s+1)/2)}{\pi^{(s-1-1)/2}\Gamma((1-s+1)/2)} \\ &= \frac{\sqrt{\pi}\Gamma(s/2)\Gamma((s+1)/2)\sin(\pi s/2)}{\pi^{s+1}} && \text{(by (1.24))} \\ &= \frac{\sqrt{\pi}2^{1-s}\sqrt{\pi}\Gamma(s)\sin(\pi s/2)}{\pi^{s+1}} && \text{(by the duplication formula)} \\ &= \Gamma_{\mathbf{C}}(s)\cos\big(\tfrac{\pi}{2}(s-1)\big) && \text{(by (1.28)).} \end{aligned}$$

Now, for $n \equiv \epsilon \pmod 2$, plug $s-n$ into (1.29) to obtain

$$\begin{aligned}\frac{\Gamma_{\mathbf{R}}(s-n+\epsilon)}{\Gamma_{\mathbf{R}}(1-s+n+\epsilon)} &= \Gamma_{\mathbf{C}}(s-n)\cos\Big(\frac{\pi}{2}(s-(n+\epsilon))\Big)\\ &= \Gamma_{\mathbf{C}}(s-n)\left(\cos\Big(\frac{\pi}{2}s\Big)\cos\Big(-\frac{\pi}{2}(n+\epsilon)\Big)-\sin\Big(\frac{\pi}{2}s\Big)\sin\Big(-\frac{\pi}{2}(n+\epsilon)\Big)\right)\\ &= \Gamma_{\mathbf{C}}(s-n)\cos\Big(\frac{\pi}{2}s\Big)\cos\Big(-\frac{\pi}{2}(n+\epsilon)\Big)\\ &= \Gamma_{\mathbf{C}}(s-n)\cos\Big(\frac{\pi}{2}s\Big)(-1)^{(n+\epsilon)/2},\end{aligned}$$

as desired.

Similarly, for $n \not\equiv \epsilon \pmod 2$,

$$\begin{aligned}\frac{\Gamma_{\mathbf{R}}(s-n+\epsilon)}{\Gamma_{\mathbf{R}}(1-s+n+\epsilon)} &= \Gamma_{\mathbf{C}}(s-n)\cos\Big(\frac{\pi}{2}(s-(n+\epsilon))\Big)\\ &= \Gamma_{\mathbf{C}}(s-n)\left(\cos\Big(\frac{\pi}{2}s\Big)\cos\Big(-\frac{\pi}{2}(n+\epsilon)\Big)-\sin\Big(\frac{\pi}{2}s\Big)\sin\Big(-\frac{\pi}{2}(n+\epsilon)\Big)\right)\\ &= \Gamma_{\mathbf{C}}(s-n)\sin\Big(\frac{\pi}{2}s\Big)\sin\Big(\frac{\pi}{2}(n+\epsilon)\Big)\\ &= \Gamma_{\mathbf{C}}(s-n)\sin\Big(\frac{\pi}{2}s\Big)(-1)^{(n+\epsilon-1)/2}.\end{aligned}$$

□

2 The conjecture of Coates and Perrin-Riou

2.1 Notation and conventions

Fix a prime p, an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, and an embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$ and fix an isomorphism $\iota : \mathbf{C} \tilde{\to} \mathbf{C}_p$ making the diagram

$$\begin{array}{ccc}\overline{\mathbf{Q}}_p & \hookrightarrow & \mathbf{C}_p\\ {\scriptstyle \iota_p}\uparrow & & \uparrow{\scriptstyle \iota}\,\wr\\ \overline{\mathbf{Q}} & \xrightarrow{\iota_\infty} & \mathbf{C}\end{array}$$

commute, where ι_∞ is the embedding fixed in §1. It will be convenient to denote by $\underline{i}$ a choice of a square root of -1, so we do.

We will be using ϵ-factors in this section, so let us fix some choices of data. Let dx_∞ be the usual measure on $\mathbf{R}$, and for finite places v, let dx_v be the Haar measure on $\mathbf{Q}_v$ giving $\mathbf{Z}_v$ measure 1. Depending on the choice $\underline{i}$, fix the additive character $\psi_{\infty,\underline{i}}$ of $\mathbf{R}$ given by

$$\psi_{\infty,\underline{i}}(x) := \exp(2\pi\underline{i}x).$$

For a finite place v of $\mathbf{Q}$, let

$$\psi_{v,\underline{i}}(x) := \exp(-2\pi\underline{i}x)$$

under the identification of $\mathbf{Q}_v/\mathbf{Z}_v$ with the v-primary subgroup of $\mathbf{Q}/\mathbf{Z}$. Then, for each place v of $\mathbf{Q}$ and each $\tau \in J_E$, one has Deligne's ϵ-factor

$$\epsilon_v(s, M, \underline{i}, \tau)$$

as defined in [D73] (see also [D79, §5] or [Ta79, §3]).

Let $\mathbf{Q}_\infty^+$ be the maximal totally real subfield of $\mathbf{Q}(\mu_{p^\infty})$ and let $\Gamma^+ := \mathrm{Gal}(\mathbf{Q}_\infty^+/\mathbf{Q})$. Let $\chi_p : G_\mathbf{Q} \to \mathbf{Z}_p^\times$ be the p-adic cyclotomic character. Recall that all algebraic $\overline{\mathbf{Q}}_p^\times$-valued characters of $G_\mathbf{Q}$ are of the form $\chi_p^n \chi$ for $n \in \mathbf{Z}$ and χ finite order. Such a character factors through Γ^+ if, and only if, χ has p-power conductor and $\chi(\mathrm{Frob}_\infty) = (-1)^n$. We denote $\mathfrak{X}_{\mathrm{alg}}^+$ the collection of such characters. If $\psi \in \mathfrak{X}_{\mathrm{alg}}^+$, let $n_\psi \in \mathbf{Z}$ and χ_ψ be such that $\psi = \chi_p^{n_\psi} \chi_\psi$ with χ_ψ finite order. We use ι_p to identify finite order characters valued in $\overline{\mathbf{Q}}_p^\times$ with those valued in $\overline{\mathbf{Q}}^\times$. Then, ψ gives rise to a motive over $\mathbf{Q}$ with coefficients in $E_\psi := \mathbf{Q}(\chi_\psi)$ defined by

$$[\psi] := (\mathbf{Q}(n_\psi) \otimes_\mathbf{Q} E_\psi) \otimes [\chi_\psi].$$

Given a motive M over $\mathbf{Q}$ with coefficients in E, its twist by ψ is the motive over $\mathbf{Q}$ with coefficients in $E(\chi_\psi)$

$$M(\psi) := (M \otimes_E E(\chi_\psi)) \otimes \left([\psi] \otimes_{E_\psi} E(\chi_\psi)\right).$$

2.2 Introductory remarks

Given a motive M that is good and ordinary at p and critical, Coates and Perrin-Riou formulate a conjecture on the existence and uniqueness of a p-adic L-function that interpolates L-values of critical twists of M by characters in $\mathfrak{X}_{\mathrm{alg}}^+$. They also describe the poles this p-adic L-function should have and the p-adic functional equation it satisfies. These conjectures are described in a series of papers (see esp. [Co91]). The general idea is to modify $c_\infty^+(M)$ to $\Omega_\infty(M)$ and to try to interpolate values

$$\frac{\Lambda_{(\infty,p)}(0, M(\psi))}{\Omega_\infty(M)}$$

where the the subscript (∞, p) on Λ indicates that the Euler factors at ∞ and p have been modified. The modification at ∞ keeps track of the periods of the Tate twists, while the modification at p does the same for the twists by finite-order p-power conductor characters χ. The modification at p also serves to regularize the p-adic distribution one obtains. One must do these things while preserving the functional equation.

Since these several modifications are made partially to take into account the change in the periods under twists, we record here the following lemma.

Lemma 2.1. *Let $\psi = \chi_p^n \chi \in \mathfrak{X}_{\mathrm{alg}}^+$ (with χ finite order). Suppose that M is a critical motive with coefficients in $E \supseteq E_\psi$ and suppose $M(\psi)$ is also critical. Then*

$$c_\infty^+(M(\psi)) \underset{E}{\sim} c_\infty^+(M) \left((2\pi i)^n \delta_{\underline{i}}(\chi)\right)^{d^+(M)}. \tag{2.1}$$

The main idea is to consider

$$R_v(s,M,\underline{i},\tau) := \frac{L_v(s,M,\tau)}{\epsilon_v(s,M,\underline{i},\tau)L_v(-s,M^\vee(1),\tau)}. \tag{2.2}$$

where v is a place of $\mathbf{Q}$ and $\tau \in J_E$. One then defines modified Euler factors $E_v(s,M,\underline{i},\tau)$ at $v=\infty,p$ such that

$$R_v(s,M,\underline{i},\tau) = \frac{E_v(s,M,\underline{i},\tau)}{E_v(-s,M^\vee(1),-\underline{i},\tau)}. \tag{2.3}$$

Noting that

$$\epsilon_v(s,M,\underline{i},\tau)\epsilon_v(-s,M^\vee(1),-\underline{i},\tau) = 1 \tag{2.4}$$

(see e.g. [Ta79, 3.4.7]), we see that

$$R_v(s,M,\underline{i},\tau) = R_v(-s,M^\vee(1),-\underline{i},\tau)^{-1}. \tag{2.5}$$

Thus, defining

$$\Lambda_{(\infty,p)}(s,M,\underline{i},\tau) := \prod_{v\in\{\infty,p\}} E_v(s,M,\underline{i},\tau) \prod_{v\notin\{\infty,p\}} L_v(s,M,\tau)$$

gives the functional equation

$$\Lambda_{(\infty,p)}(s,M,\underline{i},\tau) = \left(\prod_{v\notin\{\infty,p\}} \epsilon_v(s,M,\underline{i},\tau)\right)\Lambda_{(\infty,p)}(-s,M^\vee(1),-\underline{i},\tau).^3 \tag{2.6}$$

Mostly, the modification will define $E_v(s,M,\underline{i},\tau)=1$ or $R_v(s,M,\underline{i},\tau)$ so that $E_v(-s,M^\vee(1),-\underline{i},\tau)$ will be the opposite. This has the effect of taking into consideration the dichotomy in the periods of $\mathbf{Q}(n)$ and $[\chi]$ based on whether $F_\infty = \pm 1$.

2.3 Modification of the Euler factor at ∞

As in the case of the Gamma factors (1.14), we define $E_\infty(s,M,\underline{i},\tau)$ as a product over pieces of the Hodge decomposition. Specifically, for each factor U of the Hodge decomposition define a factor as follows

- if $U = H^{i,j}_\tau \oplus H^{j,i}_\tau$ with $i<j$, define

$$E_\infty(s,U,\underline{i},\tau) := \left(\underline{i}^{-s}\Gamma_{\mathbf{C}}(s-i)\right)^{h^{i,j}_\tau},$$

- if $U = H^{i,i}_\tau$ with $i \geq 0$, define

$$E_\infty(s,U,\underline{i},\tau) := 1,$$

- if $U = H^{i,i}_\tau$ with $i<0$, define

$$E_\infty(s,U,\underline{i},\tau) := R_\infty(s,U,\underline{i},\tau).$$

[3] The switch $\underline{i}$ to $-\underline{i}$ here is one of the main reasons to carry around this choice of $\underline{i}$.

Then,

$$E_\infty(s,M,\underline{i},\tau) = \prod_{U\subseteq H_\tau^{i,j}} E_\infty(s,U,\underline{i},\tau) \tag{2.7}$$

$$= \prod_{i<j}\left(\underline{i}^{-s}\Gamma_{\mathbf{C}}(s-i)\right)^{h_\tau^{i,j}} \prod_{i<0}\prod_{\epsilon\in\{0,1\}} \frac{\Gamma_{\mathbf{R}}(s-i+\epsilon)^{h_\tau^{i,j,\epsilon}}}{\underline{i}^{\epsilon}\Gamma_{\mathbf{R}}(1-s+i+\epsilon)^{h_\tau^{i,j,\epsilon}}} \prod_{i\geq 0} 1 \tag{2.8}$$

(compare with (1.14)).

Let

$$r(M) := \sum_{i<0} i h^{i,j}.$$

We then have the following.

Lemma 2.2. *Suppose M is critical. Then*

$$E_\infty(0,M,\underline{i},\tau) \underset{E}{\sim} (2\pi\underline{i})^{r(M)}$$

with the fudge factor independent of $\underline{i}$.

If $M(\psi)$ is also critical (and $E\supseteq E_\psi$), then

$$E_\infty(0,M(\psi),\underline{i},\tau) \underset{E}{\sim} E_\infty(0,M,\underline{i},\tau)(2\pi\underline{i})^{-n_\psi d^+(M)}.$$

Let

$$\Lambda_{(\infty)}(s,M,\underline{i},\tau) := E_\infty(s,M,\underline{i},\tau)L(s,M,\tau)$$

and modify the period as follows

$$\Omega_{\infty,\underline{i}}(M) := c_\infty^+(M)(2\pi\underline{i})^{r(M)}. \tag{2.9}$$

Write $(\delta_{\underline{i}}(\chi,\tau))_{\tau\in J_E}$ (resp. $(\Omega_{\infty,\underline{i}}(M,\tau))_{\tau\in J_E}$) for the image of $\delta_{\underline{i}}(\chi)$ (resp. $\Omega_{\infty,\underline{i}}(M)$) under $E\otimes_{\mathbf{Q}}\mathbf{C}\cong\mathbf{C}^{J_E}$. Assuming $M(\psi)$ is critical, let

$$\Lambda^{\mathrm{alg}}_{(\infty)}(M,\psi,\tau) := \frac{\Lambda_{(\infty)}(0,M(\psi),\underline{i},\tau)}{\Omega_{\infty,\underline{i}}(M,\tau)\delta_{\underline{i}}(\chi_\psi,\tau)}. \tag{2.10}$$

This is independent of the choice of $\underline{i}$. We obtain the following equivalent form of Deligne's conjecture for the motive $M(\psi)$.

Conjecture 2.3. *Suppose both M and $M(\psi)$ are critical (and $E\supseteq E_\psi$). Then, there is an element $\Lambda^{\mathrm{alg}}_{(\infty)}(M,\psi)\in E$ such that*

$$\tau\left(\Lambda^{\mathrm{alg}}_{(\infty)}(M,\psi)\right) = \Lambda^{\mathrm{alg}}_{(\infty)}(M,\psi,\tau),\ \textit{for all } \tau\in J_E. \tag{2.11}$$

2.4 Modification of the Euler factor at p

From now on, we take all our algebraic extensions of $\mathbf{Q}$ within $\overline{\mathbf{Q}}$; in particular, the coefficient field E. We will then fix $\tau = \iota_\infty|_E \in J_E$ and drop it from our notation. The embedding ι_p singles out a prime $\mathfrak{p}$ of E as well as one of $\overline{\mathbf{Q}}$. We denote by G_p the decomposition group of the latter and use ι_p to identify G_p with $G_{\mathbf{Q}_p}$.

Let λ be a finite prime of E coprime to p. There is a process, described in [D73, §8] and [Ta79, §4], that turns the λ-adic representation ρ_λ into a Weil–Deligne representation of $W_{\mathbf{Q}_p}$ over $\mathbf{C}$. Briefly, one restricts ρ_λ to G_p and then to the Weil group of $\mathbf{Q}_p$ to obtain a λ-adic representation of $W_{\mathbf{Q}_p}$.[4] Grothendieck's ℓ-adic monodromy theorem allows one to transfer this to a Weil–Deligne representation $(r_{p,\lambda}, N_{p,\lambda})$ of $W_{\mathbf{Q}_p}$ over E_λ. Picking an extension of ι_∞ to E_λ allows us to transfer this to a Weil–Deligne representation of $W_{\mathbf{Q}_p}$ over $\mathbf{C}$ which, by our definition of motive, is independent on the choice of extension (up to isomorphism). We denote this Weil–Deligne representation by (r_p, N_p). The local L-factor and ϵ-factor at p are defined in terms of (r_p, N_p). However, Deligne pointed out in [D79, Remarque 5.2.1] that $R_p(s, M, \underline{i})$ is unchanged if one replaces (r_p, N_p) with $(r_p, 0)$. Coates remarks in [Co91, Lemma 6] that, in fact, it is unchanged if one further replaces $(r_p, 0)$ with $(r_p^{\mathrm{ss}}, 0)$ (where r_p^{ss} is the semi-simplification of r_p). The modified Euler factor $E_p(s, M, \underline{i})$ is defined as a product over the irreducible pieces U of r_p^{ss} of $E_p(s, U, \underline{i})$ as follows.

By [D73, §4.20], every irreducible representation U of $W_{\mathbf{Q}_p}$ over $\mathbf{C}$ is of the form $U \cong \xi_U \otimes \omega_{s(U)}$ where $\xi_U(W_{\mathbf{Q}_p})$ is a finite group, $s(U) \in \mathbf{C}$, and ω_s is the character of $W_{\mathbf{Q}_p}$ corresponding to the character $z \mapsto |z|^s$ of $\mathbf{Q}_p^\times$ under the local reciprocity map $\mathbf{Q}_p^\times \tilde{\to} W_{\mathbf{Q}_p}^{\mathrm{ab}}$. Let Φ_p denote a choice of (geometric) Frobenius in $W_{\mathbf{Q}_p}$ (unique up to inertia I_p). Let

$$H_p(T, U) := \det(1 - \Phi_p T | U).$$

Assume the roots of this polynomial lie in $\iota_\infty(\overline{\mathbf{Q}})$.[5] Though the inverse roots $\alpha_{U,j}$ of $H_p(T, U)$ depend on the choice of Φ_p, their p-adic valuation $\mathrm{ord}_p(\iota_p \iota_\infty^{-1} \alpha_{U,j})$ is independent of this choice (since the image of inertia is finite, so the eigenvalues that occur are roots of unity, hence have p-adic valuation 0). In fact, the p-adic valuation of the $\alpha_{U,j}$ only depends on U (since $U \cong \xi_U \otimes \omega_{s(U)}$). We denote this valuation by $\mathrm{ord}_p(U)$ and impose the following assumption on the prime p and the motive M:

- assume $\mathrm{ord}_p(U) \neq -1/2$ for all irreducible U in r_p^{ss}.

This assumption won't be a problem for us, as the ordinarity hypothesis we impose will force $\mathrm{ord}_p(U) \in \mathbf{Z}$.

For $U \subseteq r_p^{\mathrm{ss}}$ irreducible, define

$$E_p(s, U, \underline{i}) := \begin{cases} 1, & \mathrm{ord}_p(U) > -1/2 \\ R_p(s, U, \underline{i}), & \mathrm{ord}_p(U) < -1/2. \end{cases}$$

[4] Technically, one does not restrict to the Weil group, rather one composes with the canonical continuous homomorphism $\varphi : W_{\mathbf{Q}_p} \to G_{\mathbf{Q}_p}$.

[5] We have assumed that $\det\left(1 - \Phi_p T | \left(\ker N_p\right)^{I_p}\right)$ is rational, however, as we are not taking the characteristic polynomial on the unramified part, we do not know that $H_p(T, U)$ is rational.

Then, define

$$E_p(s, M, \underline{i}) := \prod_{\substack{U \subseteq r_p^{\mathrm{ss}} \\ \text{irreducible}}} E_p(s, U, \underline{i}).$$

Note that $\mathrm{ord}_p(U^\vee(1)) = -1 - \mathrm{ord}_p(U)$, so (2.3) is satisfied.

2.4.1 Example: M good at p

When M is good at p (in the sense that I_p acts trivially on $H_\lambda(M)$ for all λ coprime to p), there is a formula for the modified Euler factor in terms of some simple data one usually has lying around. Let $d_p(M)$ be the number of inverse roots α of $\iota_\infty(Z_p(T, M))$ such that $\mathrm{ord}_p(\iota_p \iota_\infty^{-1} \alpha) < 0$.

Proposition 2.4. *Suppose M is good at p and let α run over the inverse roots of $\iota_\infty(Z_p(T, M))$. Then*

$$E_p(s, M, \underline{i}) = \left(\prod_{\mathrm{ord}_p(\alpha) > -1/2} (1 - \alpha p^{-s}) \right) \left(\prod_{\mathrm{ord}_p(\alpha) < -1/2} (1 - (\alpha p)^{-1} p^{-s}) \right) L_p(s, M). \tag{2.12}$$

Furthermore, if χ is a finite order character of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ of conductor $\mathfrak{f}_\chi = p^{r(\chi)}$, then

$$E_p(s, M(\chi), \underline{i}) = \delta_{\underline{i}}(\chi, \iota_\infty)^{-d_p(M)} \mathfrak{f}_\chi^{s d_p(M)} \left(\prod_{\mathrm{ord}_p(\alpha) < -1/2} \alpha \right)^{r(\chi)} L_p(s, M(\chi)). \tag{2.13}$$

2.5 The conjecture

2.5.1 Ordinarity

We now assume that M is good and ordinary at $\mathfrak{p}$. The latter condition is that there is an exhaustive separated descending G_p-stable filtration $F_{\mathfrak{p}}^j$ of $E_{\mathfrak{p}}$-subspaces of $\rho_{\mathfrak{p}}|_{G_p}$ such that I_p acts via multiplication by χ_p^j on the jth graded piece $\mathrm{gr}_{\mathfrak{p}}^j := F_{\mathfrak{p}}^j / F_{\mathfrak{p}}^{j+1}$. According to Fontaine (see [PR94, Théorème 1.5]), if $\rho_{\mathfrak{p}}$ is ordinary, then it is semistable (and, *a fortiori*, Hodge–Tate). The Hodge–Tate decomposition of $\rho_{\mathfrak{p}}$ is

$$\rho_{\mathfrak{p}} \otimes_{E_{\mathfrak{p}}} \mathbf{C}_p \cong \bigoplus_{j \in \mathbf{Z}} \mathbf{C}_p(-j)^{h_{\mathfrak{p}}^j}.$$

We call j a Hodge–Tate weight of M at $\mathfrak{p}$ if $h_{\mathfrak{p}}^j \neq 0$ (e.g. $\mathbf{Q}(1)$ has Hodge–Tate weight -1 at p, for all p).

We introduce two new assumptions which should really be part of the definition of a motive to begin with:

- assume that for all $j \in \mathbf{Z}$, $h_{\mathfrak{p}}^j = h_{\iota_\infty}^{j, w-j}$ (where w is the weight of M),
- assume $\iota_p Z_p(T, M) =$ the characteristic polynomial of the Frobenius on $D_{\mathrm{st}}(\rho_{\mathfrak{p}}|_{G_p})$.

The second assumption implies that the number for inverse roots of $\iota_p Z_p(T, M)$ with p-adic valuation j equals $h_{\mathfrak{p}}^{-j}$. Two important consequences of this are

- $\mathrm{ord}_p\,\alpha \in \mathbf{Z}$ for all inverse roots of all $H_p(T, U)$,
- $d_p(M) = d^+(M)$.

Applying the second consequence to equation (2.13), shows that a factor of $\delta_{\underline{i}}(\chi, \iota_\infty)^{-d^+(M)}$ appears in the modified Euler factor at p of $M(\chi)$, accounting for that factor occurring in the period of a twist given in equation (2.1).

2.5.2 Picking out poles

To pick out the poles that might occur in the p-adic L-function, let

$$H_{\mathfrak{p}}^{\mathrm{cyc}}(M) := H_{\mathfrak{p}}(M)^{G_{\mathbf{Q}_\infty^+}}.$$

With $G_{\mathbf{Q}}$ acting on the first factor of $H_{\mathfrak{p}}^{\mathrm{cyc}}(M) \otimes_{E_{\mathfrak{p}}} \mathbf{C}_p$, there is an isomorphism of $G_{\mathbf{Q}}$-modules

$$H_{\mathfrak{p}}^{\mathrm{cyc}}(M) \otimes_{E_{\mathfrak{p}}} \mathbf{C}_p \cong \bigoplus_{\psi \in B(M)} \psi^{e(\psi)}$$

that defines $B(M) \subseteq \mathfrak{X}_{\mathrm{alg}}^+$ and $e(\psi) \in \mathbf{Z}_{\geq 1}$.

Remark 2.5.

(i) If $\chi_p^n \chi \in B(M)$ (with χ of finite order), then $n = -w/2$.

(ii) There is a conjecture that $e(\psi)$ is the order of the pole of $L(s, M(\psi^{-1}))$ at $s = 1$ (and that $L(s, M(\psi^{-1}))$ is holomorphic outside $s = 1$).

2.5.3 Statement of the conjecture

Conjecture 2.6 (Coates–Perrin-Riou, [Co91])**.** *Suppose M is critical good and ordinary at p. Then,*

(i) for each choice of $c_\infty^+(M)$ (chosen up to $E^\times$), there is a unique pseudo-measure $\mu_{c_\infty^+(M)}$ on Γ^+ such that for all $\psi \in \mathfrak{X}_{\mathrm{alg}}^+$ satisfying

(a) $M(\psi)$ is critical,

(b) $\psi^{-1} \notin B(M)$,

(c) $\psi \notin B(M^\vee(1))$,

one has

$$\int_{\Gamma^+} \psi\, d\mu_{c_\infty^+(M)} = \frac{\Lambda_{(\infty,p)}(s, M, \underline{i})}{\Omega_{\infty,\underline{i}}(M, \iota_\infty)} \tag{2.14}$$

(ii) there is a non-zero $b \in \mathbf{Z}_p$ such that, for any choices of $\gamma_\psi, \gamma_\eta \in \Gamma^+$, one has

$$b \left(\prod_{\psi \in B(M)} \left(\psi^{-1}(\gamma_\psi) - \gamma_\psi\right)^{e(\psi)} \right) \left(\prod_{\eta \in B(M)} \left(\eta(\gamma_\eta) - \gamma_\eta\right)^{e(\eta)} \right) \mu_{c_\infty^+(M)} \in \mathbf{Z}_p[\![\Gamma^+]\!]. \tag{2.15}$$

2.5.4 p-adic functional equation

If we suppose that $\mu_{c_\infty^+(M)}$ and $\mu_{c_\infty^+(M^\vee(1))}$ exist (for some choice of $c_\infty^+(M^\vee(1))$), then they satisfy a p-adic functional equation. In order to state this, define an involution on $\mathbf{Z}_p[\![\Gamma^+]\!]$ by

$$\left(\sum_\gamma a_\gamma \gamma\right)^{\#} = \sum_\gamma a_\gamma \gamma^{-1}.$$

Let $\mathrm{rec}(\mathfrak{f}_M)$ be the Artin symbol of the conductor of $\rho_{\mathfrak{p}}$ in Γ^+ and let

$$\epsilon_{(\infty,p)}(M) := (-1)^{r(M^\vee(1))} \frac{\epsilon(0, M, \iota_\infty)}{\epsilon_\infty(0, M, \underline{i}, \iota_\infty)} \frac{\Omega_{\infty,\underline{i}}(M, \iota_\infty)}{\Omega_{\infty,\underline{i}}(M^\vee(1), \iota_\infty)}.$$

Proposition 2.7. *One has*

$$\mu_{c_\infty^+(M)} = \epsilon_{(\infty,p)}(M)\, \mathrm{rec}(\mathfrak{f}_M)^{\#} \mu^{\#}_{c_\infty^+(M^\vee(1))}. \tag{2.16}$$

2.5.5 Example: Dirichlet characters

Let $\chi : G_{\mathbf{Q}} \to E^\times$ be a finite order character of conductor N (i.e. N is the smallest positive integer such that χ factors through $\mathrm{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q})$). In this example, we will make explicit what the Coates–Perrin-Riou conjecture says about the p-adic L-function of the motive $M := [\chi]$.

First off, according to §1.5, M is critical if, and only if, χ is odd, so we restrict to this situation. Secondly, the assumption that M is ordinary at p implies that $p \nmid N$, so we fix such a prime. Let $\psi \in \mathfrak{X}^+_{\mathrm{alg}}$. When is $M(\psi)$ critical? This again follows from §1.5. By definition $\chi_\psi(\mathrm{Frob}_\infty) = (-1)^{n_\psi}$, so $\chi\chi_\psi$ and n_ψ have opposite parity. Thus, $M(\psi)$ is critical if, and only if, $n_\psi \leq 0$.

Remark 2.8. The Coates–Perrin-Riou conjecture for M will say that the $L(n_\psi, \chi\chi_\psi)$ can be interpolated p-adically for $n_\psi \leq 0$ and $\chi\chi_\psi$ of the opposite parity as n_ψ. These parity conditions ensure that $L(n_\psi, \chi\chi_\psi) \neq 0$. For $n_\psi \leq 0$ and of the same parity as $\chi\chi_\psi$, $L(n_\psi, \chi\chi_\psi) = 0$.

So, let $\psi \in \mathfrak{X}^+_{\mathrm{alg}}$ with $n_\psi \leq 0$. For M, $H^{0,0}(M) \neq 0$, so $H^{-n_\psi,-n_\psi}(M(\psi)) \neq 0$. Since $-n_\psi \geq 0$,

$$E_\infty(s, M(\psi), \underline{i}) = 1, \quad r(M) = 0$$

$$c_\infty^+(M) = 1, \qquad \Omega_{\infty,\underline{i}}(M) = 1.$$

For a finite order character η, let $\mathfrak{f}_\eta = p^{r(\eta)}\mathfrak{f}'_\eta$ where $p \nmid \mathfrak{f}'_\eta$. Since $p \nmid N$, M is good at p, so we can use proposition 2.4. Note that

$$\chi\chi_p^n(\mathrm{Frob}_p) = p^{-n}\chi(\mathrm{Frob}_p)$$

so

$$Z_p(T, M(\psi)) = 1 - \chi\chi_\psi(\mathrm{Frob}_p)p^{-n_\psi}T.$$

Let α be the inverse root $\chi\chi_\psi(\mathrm{Frob}_p)p^{-n_\psi}$ of $\iota_\infty(Z_p(T, M(\psi))$, then $\mathrm{ord}_p\,\iota_p\iota_\infty^{-1}\alpha = -n_\psi \geq 0$, so $d_p(M(\psi)) = 0$. By proposition 2.4,

$$\begin{aligned} E_p(s, M(\psi), \underline{i}) &= \begin{cases} \left(1 - \chi(\mathrm{Frob}_p)p^{-n_\psi}p^{-s}\right) L_p(s, M(\psi)), & r(\chi_\psi) = 0, \\ 1 \cdot L_p(s, M(\psi)), & r(\chi_\psi) \geq 1, \end{cases} \\ &= 1 \end{aligned} \tag{2.17}$$

Indeed, when $r(\chi_\psi) = 0$, $L_p(s, M(\psi)) = (1 - \chi(\mathrm{Frob}_p)p^{-n_\psi}p^{-s})^{-1}$, and when $r(\chi_\psi) \geq 1$, the local L-factor is 1 (since $\chi\chi_\psi$ is ramified at p). Thus,

$$\Lambda_{(\infty,p)}(s, M, \underline{i}) = L^{(p)}(s, [\chi]) = (1 - \chi(\mathrm{Frob}_p)p^{-s})L(s, [\chi]) \tag{2.18}$$

and

$$\begin{aligned} \Lambda_{(\infty,p)}(s, M(\psi), \underline{i}) &= L^{(p)}(s, [\chi](\psi)) \\ &= \begin{cases} \left(1 - \chi(\mathrm{Frob}_p)p^{-(s+n_\psi)}\right) L(s + n_\psi, [\chi]), & r(\chi_\psi) = 0, \\ L(s, M(\psi)), & r(\chi_\psi) \geq 1, \end{cases} \end{aligned} \tag{2.19}$$

where $L^{(p)}(s, M)$ is the L-function with the Euler factor at p removed.

Since $p \nmid N$, $H^{\mathrm{cyc}}_{\mathfrak{p}}(M) = 0$, so $B(M) = \emptyset$. Similarly, $B(M^\vee(1)) = \emptyset$. Thus, the following is the Coates–Perrin-Riou conjecture.

Conjecture 2.9 (Coates–Perrin-Riou conjecture for finite order characters)**.** *Let χ be an odd finite order character of conductor N and let $p \nmid N$ be a prime number. Then, there is a unique measure μ_χ on Γ^+ such that for all $\psi \in \mathfrak{X}^+_{\mathrm{alg}}$ with $n_\psi \leq 0$*

$$\int_{\Gamma^+} \psi d\mu_\chi = L^{(p)}(n_\psi, \chi\chi_\psi).$$

Remark 2.10.

(i) The conjecture as stated is for the choice of period $c^+_\infty(M) = 1$. It is clear that the full conjecture (where we don't fix a period) holds if, and only if, the version stated holds.

(ii) Though the conjecture states the interpolation property at all ψ with $n_\psi \leq 0$, the interpolation property obtained by fixing either n_ψ or χ_ψ will uniquely determine a measure. That the full interpolation property is satisfied would then follow from congruence properties of the special values (i.e. Kummer congruences for generalized Bernoulli numbers).

Since the functional equation (conjecture 1.5) for the archimedean L-function $\Lambda(s, \chi)$ is known, we get a p-adic functional equation as in proposition 2.7. Let us make explicit what it says.

Note that $M^\vee(1) = [\chi^{-1}](1)$ and χ^{-1} is also odd.

Lemma 2.11. *$M^\vee(1)(\psi)$ is critical if, and only if, $n_\psi \geq 0$.*

Proof. Note that $M^\vee(1)(\psi) = [\chi^{-1}\chi_\psi](1+n_\psi)$. Now, $\chi\chi_\psi$ and $1+n_\psi$ are of the same parity, so §1.5 says that $M^\vee(1)(\psi)$ is critical if, and only if, $1+n_\psi > 0$, as claimed. □

So, let $\psi \in \mathfrak{X}^+_{\mathrm{alg}}$ with $n_\psi \geq 0$. If n_ψ is even, then so is χ_ψ, and $\chi^{-1}\chi_\psi$ is odd, so for $M^\vee(1)(\psi)$, $h^{-1-n_\psi,-1-n_\psi,1}_{\iota_\infty} = 1$. But when n_ψ is odd, χ_ψ is too, so $\chi^{-1}\chi_\psi$ is even, and hence for $M^\vee(1)(\psi)$, $h^{-1-n_\psi,-1-n_\psi,0}_{\iota_\infty} = 1$. Thus, letting $\epsilon_\psi = 0$ or 1 if n_ψ is ood or even, respectively, we have

$$\Gamma(s, M^\vee(1)(\psi), \iota_\infty) = \Gamma_{\mathbf{R}}(s+1+n_\psi+\epsilon_\psi).$$

Since $-1-n_\psi < 0$,

$$E_\infty(s, M^\vee(1)(\psi), \underline{i}) = \frac{\Gamma_{\mathbf{R}}(s+1+n_\psi+\epsilon_\psi)}{\underline{i}^{\epsilon_\psi}\Gamma_{\mathbf{R}}(1-(s+n_\psi+1)+\epsilon_\psi)}.$$

Taking $n = -1-n_\psi$ in (1.30), and noting that ϵ_ψ and $-1-n_\psi$ have the same parity, we get

$$E_\infty(s, M^\vee(1)(\psi), \underline{i}) = (-1)^{(\epsilon_\psi-1-n_\psi)/2}\underline{i}^{-\epsilon_\psi}\Gamma_{\mathbf{C}}(s+n_\psi+1)\cos\left(\frac{\pi}{2}s\right). \tag{2.20}$$

For $M^\vee(1)$, $h^{-1,-1,1}_{\iota_\infty} = 1$, so

$$r(M^\vee(1)) = -1,$$
$$c^+_\infty(M^\vee(1)) = \delta_{\underline{i}}(\chi^{-1})2\pi i,$$

and

$$\Omega_{\infty,\underline{i}}(M^\vee(1)) = \delta_{\underline{i}}(\chi^{-1})\frac{i}{\underline{i}}.$$

We have

$$Z_p(T, M^\vee(1)(\psi)) = 1 - \chi^{-1}\chi_\psi(\mathrm{Frob}_p)p^{-1}p^{-n_\psi}T.$$

Let β be the inverse root $\chi^{-1}\chi_\psi(\mathrm{Frob}_p)p^{-(1+n_\psi)}$ of $\iota_\infty(Z_p(T, M^\vee(1)(\psi)))$, then $\mathrm{ord}_p\,\iota_p\iota_\infty^{-1}\beta = -1-n_\psi \leq -1$, so $d_p(M^\vee(1)(\psi)) = 1$. By proposition 2.4,

$$\begin{aligned} E_p(s, M^\vee(1)(\psi), \underline{i}) &= \begin{cases} \left(1-((\chi(\mathrm{Frob}_p)p)^{-1}p)^{-1}p^{-(s+n_\psi)}\right)L_p(s, M^\vee(1)(\psi)), & r(\chi_\psi)=0, \\ \delta_{\underline{i}}(\chi_\psi,\iota_\infty)^{-1}\mathfrak{f}^s_{\chi_\psi}\left(\chi(\mathrm{Frob}_p)p^{1+n_\psi}\right)^{r(\chi_\psi)}L_p(s, M^\vee(1)(\psi)), & r(\chi_\psi)\geq 1, \end{cases} \\ &= \begin{cases} \left(1-\chi(\mathrm{Frob}_p)p^{-(s+n_\psi)}\right)L_p(s+n_\psi+1, [\chi^{-1}]), & r(\chi_\psi)=0, \\ \frac{\mathfrak{f}^s_{\chi_\psi}\left(\chi(\mathrm{Frob}_p)p^{1+n_\psi}\right)^{r(\chi_\psi)}}{\delta_{\underline{i}}(\chi_\psi,\iota_\infty)}, & r(\chi_\psi)\geq 1, \end{cases} \end{aligned} \tag{2.21}$$

Thus,

$$\frac{\Lambda_{(\infty,p)}(s, M^\vee(1)(\psi), \underline{i})}{\Omega_{\infty,\underline{i}}(M^\vee(1),\iota_\infty)} = \left\{ \begin{matrix} \left(1-\chi(\mathrm{Frob}_p)p^{-(s+n_\psi)}\right) \\ \frac{\mathfrak{f}^s_{\chi_\psi}\left(\chi(\mathrm{Frob}_p)p^{1+n_\psi}\right)^{r(\chi_\psi)}}{\delta_{\underline{i}}(\chi_\psi,\iota_\infty)} \end{matrix} \right\} \frac{(-1)^{(\epsilon_\psi-1-n_\psi)/2}\underline{i}^{1-\epsilon_\psi}\Gamma_{\mathbf{C}}(s+n_\psi+1)\cos(\pi s/2)}{\delta_{\underline{i}}(\chi^{-1},\iota_\infty)i}L(s+n_\psi+1, \tag{2.22}$$

so,

$$\frac{\Lambda_{(\infty,p)}(0,M^\vee(1)(\psi),\underline{i})}{\Omega_{\infty,\underline{i}}(M^\vee(1),\iota_\infty)} = \left\{ \begin{array}{c} (1-\chi(\mathrm{Frob}_p)p^{-n_\psi}) \\ \frac{(\chi(\mathrm{Frob}_p)p^{1+n_\psi})^{r(\chi_\psi)}}{\delta_{\underline{i}}(\chi_\psi,\iota_\infty)} \end{array} \right\} \frac{(-1)^{(\epsilon_\psi-1-n_\psi)/2}2\underline{i}^{1-\epsilon_\psi}n_\psi!}{(2\pi)^{1+n_\psi}\delta_{\underline{i}}(\chi^{-1},\iota_\infty)i} L(n_\psi+1,[\chi^{-1}\chi_\psi]) \tag{2.23}$$

Unfinished: this can be related to $L(-n_\psi,[\chi\chi_\psi^{-1}])$.

3 The p-adic L-function of a Dirichlet character

3.1 Preliminaries

We collect a few basic facts concerning special values of Dirichlet L-functions.

Definition 3.1. Let χ be a primitive Dirichlet character mod N. The generalized Bernoulli numbers belonging to χ are $B_{k,\chi}$, for k a non-negative integer, given by

$$\sum_{a=1}^{N}\chi(a)\frac{te^{at}}{e^{Nt}-1}=\sum_{k=0}^{\infty}B_{k,\chi}\frac{t^k}{k!}. \tag{3.1}$$

Lemma 3.2. *Let χ be a primitive Dirichlet character mod N. Then,*

$$B_{1,\chi}=\frac{1}{N}\sum_{a=1}^{N}a\chi(a). \tag{3.2}$$

Proof. First, we find an approximate inverse to $(e^{Nt}-1)/t$. Consider

$$\begin{aligned} 1 &= \left(N+\frac{N^2t}{2}+\cdots\right)(a_0+a_1t+\cdots) \\ &= a_0N+t\left(a_1N+\frac{a_0N^2}{2}\right)\cdots . \end{aligned}$$

Thus,

$$\frac{t}{e^{Nt}-1}=\frac{1}{N}-\frac{1}{2}t+\cdots$$

and

$$\begin{aligned} \sum_{a=1}^{N}\chi(a)\frac{te^{at}}{e^{Nt}-1} &= \sum_{a=1}^{N}\chi(a)\left(\frac{1}{N}-\frac{1}{2}t+\cdots\right)(1+at+\cdots) \\ &= \sum_{a=1}^{N}\chi(a)\left(\frac{1}{N}+t\left(\frac{a}{N}-\frac{1}{2}\right)+\cdots\right). \end{aligned}$$

Since

$$\sum_{a=1}^{N} \chi(a) = 0,$$

we get the desired result. □

Recall the following definition of the L-function of χ.

Definition 3.3. The L-function of χ is defined for $\mathrm{Re}(s) > 1$ by the Dirichlet series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

where $\chi(n) = 0$ if $(n, N) > 1$. It can be analytically continued to an entire function on $\mathbf{C}$ (unless χ is the trivial character, in which case $L(s,\chi) = \zeta(s)$ has a simple pole at $s = 1$).

We remark that $L(s,\chi) = L(s,[\chi])$.
We state several standard results without proof.

Theorem 3.4 (Functional equation)**.** *Let $\epsilon \in \{0,1\}$ be given by $\chi(-1) = (-1)^\epsilon$. Then,*

$$\Lambda(s,\chi) := \Lambda(s,[\chi]) = \Gamma_{\mathbf{R}}(s+\epsilon)L(s,\chi)$$

satisfies

$$\Lambda(s,\chi) = \epsilon(s,\chi)\Lambda(1-s,\chi^{-1})$$

where

$$\epsilon(s,\chi) := \epsilon(s,[\chi]) = \frac{\tau(\chi)}{i^\epsilon N^s}$$

and

$$\tau(\chi) = \sum_{a=1}^{N} \chi(a)\exp(2\pi i a/N).$$

Theorem 3.5. *For positive integers k,*

$$L(1-k,\chi) = -\frac{B_{k,\chi}}{k} \tag{3.3}$$

and, when $k \equiv \epsilon \pmod 2$,

$$L(k,\chi) = (-1)^{1+\frac{k+\epsilon}{2}} \frac{\tau(\chi)}{2i^\epsilon N^k}(2\pi)^k \frac{B_{k,\chi^{-1}}}{k!}. \tag{3.4}$$

Proof. We indicate how to derive the second equality from the first using the functional equation of theorem 3.4. We have that

$$L(k,\chi) = \frac{\tau(\chi)}{i^\epsilon N^k} \frac{\Gamma_{\mathbf{R}}(1-k+\epsilon)}{\Gamma_{\mathbf{R}}(k+\epsilon)} L(1-k,\chi^{-1}).$$

By equation (1.30) (with $s = 0$ and $n = -k$), when $k \equiv \epsilon \pmod 2$,

$$\begin{aligned} L(k,\chi) &= \frac{\tau(\chi)}{i^\epsilon N^k}\frac{(-1)^{(-k+\epsilon)/2}}{\Gamma_{\mathbf{C}}(k)}L(1-k,\chi^{-1}) \\ &= \frac{\tau(\chi)}{i^\epsilon N^k}\frac{(-1)^{(-k+\epsilon)/2}}{2(2\pi)^{-k}(k-1)!}\left(-\frac{B_{k,\chi^{-1}}}{k}\right), \end{aligned}$$

as desired. □

3.2 Construction using Stickelberger elements

We adapt Iwasawa's construction in [Iw-LpL, §6] which essentially uses the Stickelberger elements of cyclotomic fields. Similar things are done in chapter 2 of Lang, in KubertLang, and in Rubin §3.4.

Let's start with some notation. For a positive integer N, let $G_N := \mathrm{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q})$. If a is a positive integer, let $\mathrm{rec}_N(a) \in G_N$ be defined by $\zeta_N^{-a} = \mathrm{rec}_N(a)(\zeta_N)$ (this is the reciprocity map, in particular $\mathrm{rec}_N(p) = \mathrm{Frob}_p$ for $p \nmid N$). If $M|N$, let $\mathrm{res}^M_N : G_M \to G_N$ be the map obtained by restricting the action of $\sigma \in G_M$ to $\mathbf{Q}(\mu_N)$. Given χ a primitive Dirichlet character mod N, we also denote by χ the Galois character on G_N given by $\chi(\mathrm{rec}_N(a)) = \chi(a)$, for $(a, N) = 1$ (see remark 1.13 for a discussion of our normalizations). This also gives a character, still denoted χ, on G_M, for $N|M$, via composition with res^M_N.

Definition 3.6. Let χ be a primitive Dirichlet character and let $N \geq 2$ be an integer. Define the χ-twisted (shifted) Stickelberger element of level N as

$$\theta_{N,\chi} := \sum_{\substack{a=1\\(a,N)=1}}^{N}\left(\left\{\frac{a}{N}\right\}-\frac{1}{2}\right)\chi(a)\,\mathrm{rec}_N(a) \in \mathbf{Q}(\chi)[G_N]$$

where $\{\cdot\}$ denotes the fractional part.

Remark 3.7. The usual Stickelberger element of level N is

$$\theta_N := \sum_{a=1}^{N}\left\{\frac{a}{N}\right\}\mathrm{rec}_N(a) \in \mathbf{Q}[G_N].$$

The shift by $-1/2$ we introduce is so that the $\theta_{N,\chi}$ behave well with respect to the restriction maps (see lemma 3.9). The twist by $\chi(a)$ is mostly for convenience.

The reason we are interested in these Stickelberger elements is the following result.

Lemma 3.8. *Let χ be a primitive Dirichlet character. For a Dirichlet character ψ, let $\chi\psi$ denote the primitive Dirichlet character attached to $\chi\psi$. If $N = \mathfrak{f}_{\chi\psi}$, then*

$$\psi(\theta_{N,\chi}) = B_{1,\chi\psi} = -L(0,\chi\psi). \tag{3.5}$$

Proof. We have

$$\begin{aligned}
\psi(\theta_{N,\chi}) &= \frac{1}{N}\sum_{\substack{a=1\\(a,N)=1}}^{N} a\chi(a)\psi(\mathrm{rec}_N(a)) - \frac{1}{2}\sum_{\substack{a=1\\(a,N)=1}}^{N}\chi(a)\psi(\mathrm{rec}_N(a))\\
&= \frac{1}{N}\sum_{\substack{a=1\\(a,N)=1}}^{N} a\chi(a)\psi(a) - \frac{1}{2}\sum_{\substack{a=1\\(a,N)=1}}^{N}\chi(a)\psi(a)\\
&= \frac{1}{N}\sum_{\substack{a=1\\(a,N)=1}}^{N} a\chi\psi(a) - \frac{1}{2}\sum_{\substack{a=1\\(a,N)=1}}^{N}\chi\psi(a)\\
&= B_{1,\chi\psi} - 0,
\end{aligned}$$

(I think the 3rd line is a lie) where the last equality is from lemma 3.2 and the fact that

$$\sum_{\substack{a=1\\(a,N)=1}}^{N}\chi\psi(a) = \sum_{a=1}^{N}\chi\psi(a) = 0.$$

The second equality in the statement of the lemma is theorem 3.5. □

Thus, the Stickelberger elements are related to the value at $s = 0$ of Dirichlet L-functions. We will now proceed to take a limit of twisted Stickelberger elements in order to obtain a p-adic measure that gives the p-adic L-function of χ.

Lemma 3.9. *Let χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$. Let $N \geq 2$ be a positive integer such that $\mathfrak{f}_\chi | N$ and let ℓ be a prime number. Then,*

$$\mathrm{res}^{N\ell}_{N}\,\theta_{N,\chi} = \begin{cases}\theta_{N,\chi}, & \text{if } \ell | N,\\ (1-\chi(\ell)\,\mathrm{Frob}_\ell)\theta_{N,\chi}, & \text{if } \ell \nmid N.\end{cases} \tag{3.6}$$

Proof. Note that if $N|M$ and $(bN+a, M) = 1$, then $\mathrm{res}^M_N(\mathrm{rec}_M(bN+a)) = \mathrm{rec}_N(a)$, and, since $\mathfrak{f}_\chi|N$, $\chi(bN+a) = \chi(a)$. Thus, if $\ell|N$, using the fact that $(a,N) = 1$ implies $(a, N\ell) = 1$, we have

$$\begin{aligned}
\mathrm{res}^{N\ell}_{N}\,\theta_{N,\chi} &= \sum_{\substack{a=1\\(a,N)=1}}^{N}\sum_{b=0}^{\ell-1}\left(\frac{bN+a}{N\ell}-\frac{1}{2}\right)\chi(bN+a)\,\mathrm{rec}_N(a) &(3.7)\\
&= \sum_{\substack{a=1\\(a,N)=1}}^{N}\left(\sum_{b=0}^{\ell-1}\left(\frac{bN+a}{N\ell}-\frac{1}{2}\right)\right)\chi(a)\,\mathrm{rec}_N(a).
\end{aligned}$$

Now,

$$\sum_{b=0}^{\ell-1}\left(\frac{bN+a}{N\ell}-\frac{1}{2}\right) = \frac{\ell(\ell-1)N}{2N\ell} + \frac{\ell a}{N\ell} - \frac{\ell}{2} = \frac{a}{N} - \frac{1}{2},$$

so the case $\ell|N$ is done.

Now, suppose $\ell\nmid N$. We may begin as in (3.7), but we must subtract off the terms for which $\ell|bN+a$. We obtain

$$\begin{aligned}\mathrm{res}_N^{N\ell}\,\theta_{N,\chi} &= \sum_{\substack{a=1\\(a,N)=1}}^{N}\sum_{b=0}^{\ell-1}\left(\frac{bN+a}{N\ell}-\frac{1}{2}\right)\chi(bN+a)\,\mathrm{rec}_N(a)-\sum_{\substack{c=1\\(c,N)=1}}^{N}\left(\frac{c\ell}{N\ell}-\frac{1}{2}\right)\chi(c\ell)\,\mathrm{rec}_N(c\ell)\\ &= \theta_{N,\chi}-\chi(\ell)\,\mathrm{rec}_N(\ell)\theta_{N,\chi}.\end{aligned}$$

Since $\mathrm{rec}_N(\ell)=\mathrm{Frob}_\ell$, we are done. □

References

Books and Proceedings

[ANT-Iwa] John Coates, Ralph Greenberg, Barry Mazur, and Ichirô Satake (eds.), *Algebraic number theory*, Advanced Studies in Pure Mathematics, vol. 17, Academic Press, 1989, Papers in honor of Kenkichi Iwasawa on the occasion of his 70th birthday on September 11, 1987.

[AntII] Pierre Deligne and Willem Kuyk (eds.), *Modular functions of one variable II*, Lecture Notes in Mathematics, vol. 349, Springer-Verlag, 1973, Proceedings of the International Summer School, University of Antwerp, RUCA, July 17–August 3, 1972.

[Bures] Jean-Marc Fontaine (ed.), *Périodes p-adiques*, Astérisque, vol. 223, SMF, 1994, Séminaire de Bures, 1988.

[Corv] Armand Borel and William Casselman (eds.), *Automorphic forms, representations, and L-functions*, Proceedings of the Symposium in Pure Mathematics, vol. 33, AMS, 1979, in two parts.

[GGQ] Yasutaka Ihara, Ken Ribet, and Jean-Pierre Serre (eds.), *Galois groups over* **Q**, MSRI Publications, vol. 16, Springer-Verlag, 1989, Papers from the workshop on Galois groups over **Q** and related topics held in Berkeley, California, March 23–27, 1987.

[Har-AG] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, 1977.

[Iw-LpL] Kenkichi Iwasawa, *Lectures on p-adic L-functions*, Annals of Mathematics Studies, vol. 74, Princeton University Press, 1972.

[LFA] John Coates and Martin J. Taylor (eds.), *L-functions and arithmetic*, London Mathematical Society Lecture Note Series, vol. 153, Cambridge University Press, 1991, Proceedings of the London Mathematical Society Symposium held at the University of Durham, Durham, June 30–July 11, 1989.

[Mot] Uwe Jannsen, Steven Kleiman, and Jean-Pierre Serre (eds.), *Motives*, Proceedings of the Symposium in Pure Mathematics, vol. 55, AMS, 1995, in two parts, Summer Research Conference on Motives at the University of Washington, Seattle, July 20–Aug. 2, 1991.

Articles

[Co91] John Coates, *Motivic p-adic L-functions*, in [LFA], 1991, pp. 141–172.

[D73] Pierre Deligne, *Les constantes des équations fonctionnelles des fonctions L*, in [AntII], 1973, pp. 501–597.

[D79] Pierre Deligne, *Valeurs de fonctions L et périodes d'intégrales*, in [Corv], part 2, 1979, pp. 313–346.

[D89] Pierre Deligne, *Le groupe fondamental de la droite projective moins trois points*, in [GGQ], 1989, pp. 79–297.

[Fo92] Jean-Marc Fontaine, *Valeurs spéciales des fonctions L des motifs*, Séminaire Bourbaki **1991/92** (1992), no. 751, 205–249, in Astérisque No. 206.

[FPR94] Jean-Marc Fontaine and Bernadette Perrin-Riou, *Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L*, in [Mot], part 1, 1994, pp. 599–706.

[G89] Ralph Greenberg, *Iwasawa theory for p-adic representations*, in [ANT-Iwa], 1989, pp. 97–137.

[PR94] Bernadette Perrin-Riou, *Représentations p-adiques ordinaires*, in [Bures], 1994, pp. 185–220.

[Ta79] John Tate, *Number theoretic background*, in [Corv], part 2, 1979, pp. 3–26.