Final review – MA 123F – Fall 2010

Boston University

Contents

Preface

This is meant to be a very brief review of what we covered. Look at the section titles (or the table of contents) for an outline of (most of) what we've seen. In some cases below, I simply refer you to your course notes or the textbook, in others, I give some basic comments, and in a few others, I give more extensive comments. This is meant to help you study, not to replace your course notes. In addition to studying your course notes, you should also go over your past assignments and make sure you know how to do the problems you got wrong (and also the problems you got right!). During the semester, I included practice problems on every assignment; doing those will also help you study. If you have any questions, please email me, or come to my extended office hours. Good luck!

1. Limits and continuity

1.1. **Computing limits.**

1.1.1. *"Plugging in": sum, product, quotient, and composition rules.* Many functions we dealt with are built up out of the basic functions ($x^n, e^x, \sin x, \cos x$, etc.) by addition, multiplication, division, and composition. We saw how to compute limits by breaking up complicated functions into simpler ones. We also saw when this doesn't work.

Another way we built up functions was by defining them "piecewise". Computing the limit where the function is glued together is one place where left-hand and right-hand limits come in handy.

1.1.2. *Proving a limit doesn't exist.* We saw a few ways to prove a limit doesn't exist. Different things work in different situations, but I just want to mention that this is another place where left-hand and right-hand limits can come in handy.

1.1.3. *Indeterminate forms.* This is about what happens when plugging in gives something that "doesn't make sense". The things that don't make sense are the following:

(1) $\frac{0}{0}$: You know you want to do L'Hospital's rule, but to begin you should try other methods we saw in the first half of the class (remember factoring, rationalizing, rearranging). These methods won't always work, and then is the time to whip out L'Hospital's rule:

$$\text{If } \lim_{x\to a} \frac{g(x)}{h(x)} = \frac{0}{0}, \text{ then } \lim_{x\to a} \frac{g(x)}{h(x)} = \lim_{x\to a} \frac{g'(x)}{h'(x)}.$$

Using L'Hospital's rule won't always work, which is why you should try other methods first. For example, try applying L'Hospital's rule to

$$\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x}$$

(this is case (2), but it's a nice simple example of L'Hospital's rule not helping).

(2) $\frac{\pm\infty}{\pm\infty}$: as in (1), except the statement of L'Hospital's rule we use is:

$$\text{If } \lim_{x\to a} \frac{g(x)}{h(x)} = \frac{\pm\infty}{\pm\infty}, \text{ then } \lim_{x\to a} \frac{g(x)}{h(x)} = \lim_{x\to a} \frac{g'(x)}{h'(x)}.$$

(3) $0 \cdot \infty$: rewrite as $\dfrac{0}{1/\infty}$ or $\dfrac{\infty}{1/0}$ to find yourself in situation (1) or (2).
(4) $\infty - \infty$: try rearranging the function somehow (factoring, rationalizing, inserting a common denominator, ...).
(5) 0^0: take ln of the function to bring down the exponent. You will then be in situation (3) above, so use the method there to find the limit. This gives you the limit of $\ln(f(x))$, so don't forget to raise e to your answer.
(6) ∞^0: as in (5).
(7) 1^∞: as in (5).

Warning: L'Hospital's rule *only* applies directly in cases (1) and (2)! In cases where the limit is not an indeterminate form, L'Hospital's rule will give the wrong answer; and in cases (3)–(7), it just doesn't make sense (you have to rewrite things first).

1.1.4. *Squeeze theorem.* See your notes.

1.2. **Applications.**

1.2.1. *Checking continuity.* We saw that the basic functions are continuous wherever they are defined and we saw some rules for figuring out where more complicated functions are known to be continuous. These rules can only go so far and they leave use with a few points that we need to check individually (i.e. we need to check the definition of continuity at these points). Remember that $f(x)$ is continuous at a if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a).$$

1.2.2. *Horizontal and vertical asymptotes.*

Horizontal asymptotes: remember to check both $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$ (and remember that a function can have at most 2 horizontal asymptotes).

Vertical asymptotes: Typically, try to look for where the denominator is 0 to find *possible* vertical asymptotes (remember to check that the limit at these points is actually infinite). Other vertical asymptotes can arise from the presence of $\ln(x)$ or $\tan(x)$, for example.

2. Derivatives

The derivative of a function $y = f(x)$ measures the rate of change of y with respect to x. If x represents time and y represents position, then y' is velocity; if x and y are simply coordinates, then y' gives the slope of the tangent line to the graph of $f(x)$; etc.

2.1. **Computing derivatives.** As mentioned in the limit section, most functions we deal with are defined by taking the basic functions we know ($x^n, e^x, \sin x, \cos x$, etc.) and combining them using addition, multiplication, division, and function composition to make new functions. Again as in the limit section, we learned rules for differentiating the basic functions and rules for dealing with the various combinations we can take. Two exceptions to this are the cases where $f(x)$ is defined implicitly (see section 2.1.5) or by an integral (see section 3.2.3).

2.1.1. *Basic functions.* Learn the derivatives of:

- x^n
- e^x, a^x
- $\ln x, \log_a x$
- $\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$
- $\arcsin x, \arccos x, \arctan x$

2.1.2. *Sum, product, and quotient rules.* Learn:

- $(f+g)' = f' + g'$
- $(fg)' = fg' + f'g$
- $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$

2.1.3. *Chain rule.*

$$f(g(x))' = f'(g(x)) \cdot g'(x).$$

2.1.4. *Logarithmic derivatives and $g(x)^{h(x)}$.* To compute the derivative of $f(x) = g(x)^{h(x)}$, take ln of both sides and use implicit differentiation:

$$\begin{aligned} (\ln(f(x))' &= \left(\ln\left(g(x)^{h(x)}\right)\right)' \\ \frac{1}{f(x)} \cdot f'(x) &= (h(x)\ln(g(x)))' \end{aligned}$$

so

$$f'(x) = f(x)\left(h(x)\ln(g(x))\right)'.$$

We also applied this trick to functions that were complicated ratios.

2.1.5. *Implicit differentiation.* Here, we were given some equation involving x and y. We took the derivative of both sides (remembering to use the chain rule when y is involved), and solved for y'. Remember also to check what we did about second derivatives.

2.1.6. *Non-differentiable functions.* Not all functions we deal with can be differentiated everywhere. We saw how to recognize these functions: they had corners, discontinuities, or vertical tangents. See your notes or section 2.7 of the textbook for details.

2.2. **Applications.**

2.2.1. *Linear approximation and differentials.* Remember the *linearization* of a function $f(x)$ at $x = a$ is

$$L(x) = f(a) + f'(a) \cdot (x - a)$$

and that this offers a good approximation to $f(x)$ near a.

We saw that we can phrase approximations in terms of differentials as well. Remember that the differential of a function $f(x)$ is the expression

$$df = f'(x)dx$$

and that this can be used to approximate how small changes in x affect the value of $f(x)$.

2.2.2. *Related rates.* We studied some "real world" problems and showed how a relation between quantities can be turned into a relation between their rates of change.

2.2.3. *Optimization: local and global extrema.* We saw that local extrema only occur at critical points and saw how to test which critical points are local max/mins (the first and second derivative tests). We saw that to locate global extrema on a closed interval $[a, b]$, we simply needed to list the critical points and the end points in a table and the max/min would occur at the highest and lowest values respectively. We also treated cases where we were trying to find global extrema on an interval that could be infinite and we saw how to justify that the extrema we found were actually global.

We also covered how to classify inflection points.

2.2.4. *Graph plotting.* We saw how $f'(x)$ and $f''(x)$ provide information on the shape of the graph of $f(x)$ and we used this to plot $f(x)$.

2.2.5. *Tangent lines and normal lines.* Remember that the slope of the tangent line to the graph of $f(x)$ at $x = a$ is given by $f'(a)$ and the slope of the normal line is $-1/f'(a)$.

This is enough information to find the equations of the tangent line because if you know that the slope of a line is, say, a, and (x_0, y_0) is a point on the line, then the equation of the line is

$$(y - y_0) = a \cdot (x - x_0).$$

2.2.6. *Making inequalities.* Remember that we saw that if $f(a) \geq g(a)$ and $f'(x) \geq g'(x)$ for all $x \geq a$, then $f(x) \geq g(x)$ for all $x \geq a$.

3. Integrals and antiderivatives

One interpretation of definite integrals is as the "net area" below a curve. Additionally, if you are given $r(t)$ which represents the rate of change of a quantity, then

$$\int_a^b r(t)dt$$

computes the total change of the quantity between time a and time b.

Indefinite integrals are just a shorthand for "tell me the most general antiderivative" (i.e. find an antiderivative and write " $+\ C$" after it).

3.1. **Computing integrals.**

3.1.1. *Basic approach.* The basic approach to computing definite integrals is provided by the Fundamental Theorem of Calculus, Part II, which states that if $f(x)$ is a continuous function on an interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

This gives us a method for computing definite integrals: first, find an antiderivative of $f(x)$, then plug in the limits of integration and take the difference of the two values. So, the problem of integrating is reduced to that of finding antiderivatives.

3.1.2. *Recognition.* The basic tool we use for finding antiderivatives is "recognition", i.e. to find an antiderivative of $f(x)$, we try to recognize it as the derivative of some function. This is another reason why it's important for you to remember the derivatives of the basic functions. Sometimes, the function we're trying to find an antiderivative of is not something we immediately recognize as the derivative of some other function. In some cases, all you need to do is rewrite the function (e.g. by expanding, or rewriting trigonometric functions). For more complicated situations, several techniques have been developed, we covered one of them: u-substitution.

3.1.3. *u-substitution.* u-substitution allows you to change the variable in an integral in order to rewrite what you are trying to integrate. This should lead to an expression that you can now recognize as the derivative of something.

For indefinite integrals, your answer should be in terms of the original variable, so remember to plug it back in.

For definite integrals, you have to make sure you deal with the limits of integration properly. Remember that we saw two ways of doing this: either you keep the bounds as is and make sure to plug them into the original variable, or you alter your bounds according to the new variable u.

3.1.4. *Using properties.* We covered several properties of definite (and indefinite) integrals and sometimes these can be used to solve (or help solve) the problem. You should make sure you know these properties and how they can be applied. Some examples we saw were how to integrate the absolute value of a function by breaking up the interval and how to compute derivatives of functions defined by integrals when both limits are functions of x.

3.1.5. *Via area interpretation.* Sometimes, you won't be able to find an antiderivative in order to evaluate a definite integral, but you'll be able to use the interpretation of the integral as a "net area" in order to find the answer. For example, the integral may represent the area of part of a circle, or the integral may be zero because it measures areas that cancel out (such as for odd functions integrated from $-a$ to a).

3.2. **Applications.**

3.2.1. *Position, distance, velocity, and acceleration.* Since velocity is the rate of change of position, integration can be used to determine how much a position has changed over a period of time. Similarly, acceleration is the rate of change of velocity, and so can be used to determine how much velocity has changed. And integrating acceleration twice will tell you about the position.

Remember also that we saw that there was a difference between the change in position and the distance covered. In order to compute how much distance was covered from time a to time b, you have to integrate the absolute value of velocity.

3.2.2. *Total change.* See your notes.

3.2.3. *Functions defined by integrals and their derivatives.* We saw that we can define functions in terms of integrals. For example, if $g(t)$ is a function, then

$$f(x) = \int_a^x g(t)dt$$

is another function. The Fundamental Theorem of Calculus, Part I, told us how to compute the derivative of such functions. Specifically, it states that: if $g(t)$ is continuous on the interval $[a, b]$, then

$$\frac{d}{dx}\int_a^x g(t)dt = g(x).$$

Note that you don't have to integrate $g(t)$ or anything of the sort: you just plug in x!

But remember that more complicated examples can have x replaced with some function $h(x)$ (then the chain rule is required when computing the derivative), and even a can be replaced by a function $j(x)$ (in which case, to compute the derivative, you'll also need to use a property of integrals that allows you to break up an interval; see your notes and assignment 10 for examples).

4. Extras

4.1. **Inverse trigonometric functions.** Some of you had some trouble with inverse trigonometric functions and you should make sure that you understand their definitions, domains, ranges, etc. for the final.