MA123F - Solutions to midterm - Fall 2010

(1) (a) $f(x)$ is increasing when $f'(x)>0$
$f(x)$ is decreasing when $f'(x)<0$

$f'(x)=6x^2-4x-2$. [To find where $f'(x)>0$ & $f'(x)<0$, it is easier to find where $f'(x)=0$ & plug in values of x in the points in between]

$$f'(x)=0=6x^2-4x-2=2(x-1)(3x+1) \quad \text{so } x=1 \text{ or } \tfrac{-1}{3}$$

$f'(-1)=6+4-2=8>0$

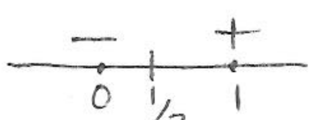
$f'(0)=-2<0$

$f'(2)=24-8-2=14>0$

so $f(x)$ is increasing on $(-\infty,-\frac{1}{3})$ & on $(1,\infty)$
& $f(x)$ is decreasing on $(-\frac{1}{3},1)$

(b) $f(x)$ is concave up when $f''(x)>0$
$f(x)$ is concave down when $f''(x)<0$

$f''(x)=12x-4$. $f''(x)=0=12x-4$. so $x=\frac{1}{3}$



$f''(0)=-4<0$

$f''(1)=12-4=8>0$

so $f(x)$ is concave up on $(\frac{1}{3},\infty)$
$f(x)$ is concave down on $(-\infty,\frac{1}{3})$

Additional remarks: Some of you solved $f'(x)=0$ & $f''(x)=0$, but then plugged values into $f(x)$. The point of plugging in values is to test whether $f'(x)$ (respectively $f''(x)$) is positive or negative, so $f(x)$ has nothing to do with this. Keeping clear in your mind what your goal is should help to avoid such mistakes.

(2) (a) [Typically, such problems involve using the derivative to find the slope of the tangent line & using the equation of a line given as $y-y_1=(\text{slope})(x-x_1)$ where (x_1,y_1) is a point on the line.]

Slope of tangent: $f'(x)=3x^2-1$ so slope $=f'(2)=12-1=\boxed{11}$

Point on the line: $(2, f(2))=(2, 8-2-1)=\boxed{(2,5)}$

so the equation is $y-5=11(x-2)$ i.e. $\boxed{y=11x-17}$

(b) Slope of normal $=\dfrac{-1}{\text{slope of tangent}}$ so slope $=\dfrac{-1}{f'(0)}=\dfrac{-1}{-1}=\boxed{1}$

Point on the line: $(0, f(0))=\boxed{(0,-1)}$

so the equation is $y--1=1\cdot(x-0)$ i.e. $\boxed{y=x+1}$

(c) Lines are parallel exactly when they have the same slope.

Slope of $3y-6x+1=0$: solve for y: $3y=6x-1$ so $y=2x-\frac{1}{3}$

so slope $=2$

Set $f'(x)=2$: $3x^2-1=2$ so $3x^2=3$ so $\boxed{x=\pm 1}$

(3) (a) $$\frac{d}{dx}\left(x^7+\frac{3}{x^7}+x^{1/7}-\frac{x^{-1/7}}{2}\right)=\frac{d}{dx}(x^7)+3\frac{d}{dx}(x^{-7})+\frac{d}{dx}(x^{1/7})-\frac{1}{2}\frac{d}{dx}(x^{-1/7})$$
$$=7x^6+3\cdot(-7)x^{-8}+\frac{1}{7}x^{-6/7}-\frac{1}{2}\cdot\left(\frac{-1}{7}\right)x^{-8/7}$$
$$=\boxed{7x^6-\frac{21}{x^8}+\frac{1}{7x^{6/7}}+\frac{1}{14x^{8/7}}}$$

(b) $$\frac{d}{dx}\left(e^x(x^3+x^2+x+1)\right)=e^x\frac{d}{dx}(x^3+x^2+x+1)+\left(\frac{d}{dx}(e^x)\right)(x^3+x^2+x+1)$$
$$=e^x(3x^2+2x+1)+e^x(x^3+x^2+x+1)$$
[Product rule]
$$=\boxed{e^x(x^3+4x^2+3x+2)}$$

(c) $$\frac{d}{dx}\left(\frac{3x-2}{2x+1}\right)=\frac{(2x+1)\frac{d}{dx}(3x-2)-(3x-2)\frac{d}{dx}(2x+1)}{(2x+1)^2}=\frac{(2x+1)\cdot 3-(3x-2)\cdot 2}{(2x+1)^2}$$
[Quotient rule]
$$=\frac{\cancel{6x}+3-(\cancel{6x}-4)}{(2x+1)^2}=\boxed{\frac{7}{(2x+1)^2}}$$

(3) (d) $\frac{d}{dx}\left(\sqrt{e^{x^2}+x}\right) = \frac{1}{2\sqrt{e^{x^2}+x}} \cdot \frac{d}{dx}\left(e^{x^2}+x\right) = \frac{1}{2\sqrt{e^{x^2}+x}}\left(\left(\frac{d}{dx}\left(e^{x^2}\right)\right)+1\right)$

$$= \frac{1}{2\sqrt{e^{x^2}+x}}\left(e^{x^2}\frac{d}{dx}(x^2)+1\right) = \boxed{\frac{2xe^{x^2}+1}{2\sqrt{e^{x^2}+x}}}$$

(e) $\frac{d}{dx}(2^{1/x}) = 2^{1/x}\ln(2)\frac{d}{dx}\left(\frac{1}{x}\right) = \boxed{\frac{-2^{1/x}\ln(2)}{x^2}}$

(f) $\frac{d}{dx}\left(e^{x+e^x}\right) = e^{x+e^x} \cdot \frac{d}{dx}(x+e^x) = \boxed{e^{x+e^x}(1+e^x)}$

Additional remarks:

- When you have $\frac{d}{dx}\left(\frac{3}{x^7}\right)$, there's no need to do a quotient rule. Note that $\frac{3}{x^7}=3x^{-7}$ so $\frac{d}{dx}\left(\frac{3}{x^7}\right)=3\frac{d}{dx}(x^{-7})$ so this is just taking the derivative of a monomial.
- Similarly, if you have $\frac{d}{dx}\left(\frac{x^{-1/7}}{2}\right)$, no quotient rule is needed; simply use the constant mulitple rule $\frac{d}{dx}\left(\frac{x^{-1/7}}{2}\right)=\frac{1}{2}\frac{d}{dx}(x^{-1/7})$.
- Another mistake that showed up several times involved the chain rule in part (d). People set up $\boxed{\begin{array}{l} y=e^u, \\ u=e^v+x \\ v=x^2 \end{array}}$ but this leads to some problems since u is now written as a function of both v & x.

$\frac{du}{dv} \neq e^v+1$ because $\frac{dx}{dv} \neq 1$ (since $v=x^2$, $\frac{dx}{dv}=\frac{1}{2x}$)
(try implicit differentiation for this)

What is true is that $\frac{du}{dx}=e^v\frac{dv}{dx}+1$ & $\frac{dv}{dx}=2x$.

If you've written a function in terms of more than one variable, be careful with your derivatives.

(4)(a) $f(x)$ is increasing from 3 to ∞, which rules out all graphs except (D) (The other graphs all show functions that take negative values somewhere between 3 & ∞).

[Note: other reasons also apply.]

(b) $f(x)$ is concave down from $-\infty$ to -2, which rules out all graphs except (II) (The other graphs all show functions that take positive values somewhere between $-\infty$ & -2).

[Note: again, there are other reasons that disqualify the other graphs.]

(5)(a) both x^5 & e^x are continous everywhere, in particular at $x=1$.

Thus, $\lim_{x\to 1} x^5 = 1^5 = 1$ & $\lim_{x\to 1} e^x = e^1 = e$. By the difference law,

$$\lim_{x\to 1} x^5 - e^x = \boxed{1-e}$$

(b) both $e^x - 2$ & x^2-7 are continuous everywhere. Since x^2-7 is non-zero at $x=0$, the quotient law (for continuity) shows that $\frac{e^x-2}{x^2-7}$ is continuous at $x=0$. Thus $\lim_{x\to 0} \frac{e^x-2}{x^2-7} = \frac{e^0-2}{0^2-7} = \frac{1-2}{-7} = \frac{-1}{-7} = \boxed{\frac{1}{7}}$

(c) Plugging in yields $\frac{0}{0}$, so more work must be done. [Since the top & bottom are polynomials, getting $\frac{0}{0}$ necessarily means the top & bottom have a common factor. Cancelling out this factor will always give a conclusive result (which could be exists or does not exist & equals ∞, $-\infty$, or neither).]

factor $\frac{x-4}{x^2-x-12} = \frac{\cancel{x-4}}{\cancel{(x-4)}(x+3)}$ so when $x \neq 4$ $\frac{x-4}{(x-4)(x+3)} = \frac{1}{x+3}$

Thus, $\lim_{x\to 4} \frac{x-4}{x^2-x-12} = \lim_{x\to 4} \frac{1}{x+3} = \boxed{\frac{1}{7}}$

(d) $\lim_{x\to 0^+} \sqrt{x} = 0$, $\lim_{x\to 0^+} \ln(x) = -\infty$ so the quotient law in the "Table of limits involving infinity" says that $\lim_{x\to 0^+} \frac{\sqrt{x}}{\ln(x)} = \frac{0}{-\infty} = \boxed{0}$

(5)(e) Plugging in yields $\frac{0}{0}$, so ~~mork~~ work must be done. [With square roots, you want to "rationalize". This will yield a common factor on the top & bottom. Cancelling this factor will give a conclusive result.]

rationalize: (if $x \neq 2$)

$$\frac{\sqrt{x-2}}{\sqrt{x^2+12}-4} \cdot \frac{(\sqrt{x^2+12}+4)}{(\sqrt{x^2+12}+4)} = \frac{\sqrt{x-2}\,(\sqrt{x^2+12}+4)}{x^2+12-16}$$

[Leave this factor here: it is causing the zero on top, so it will need to be cancelled by something on the bottom.]

$$= \frac{\sqrt{x-2}\,(\sqrt{x^2+12}+4)}{x^2-4}$$

$$= \frac{\sqrt{\cancel{x-2}}\,(\sqrt{x^2+12}+4)}{\cancel{(x-2)}(x+2)}$$

$$= \frac{\sqrt{x^2+12}+4}{\sqrt{x-2}\,(x+2)}$$

so $\lim\limits_{x\to 2^+} \frac{\sqrt{x-2}}{\sqrt{x^2+12}-4} = \lim\limits_{x\to 2^+} \frac{\sqrt{x^2+12}+4}{\sqrt{x-2}\,(x+2)}$

$\lim\limits_{x\to 2^+} \sqrt{x^2+12}+4 = \sqrt{2^2+12}+4 = \sqrt{16}+4 = 8$

& $\lim\limits_{x\to 2^+} \sqrt{x-2}\,(x+2) = 0$. The "Table of limits involving infinity"

says that $\lim\limits_{x\to 2^+} \frac{\sqrt{x-2}}{\sqrt{x^2+12}-4} = \frac{8}{0} = \boxed{\infty}$ i.e. $\boxed{\text{limit DNE } \& = \infty}$

(since the bottom if >0 when $x>2$)

(f) Plugging in yields $\infty - \infty$, so more work must be done. [This is a more difficult question that you may need to play around with. The quickest approach is to rationalize.]

rationalizing: $\sqrt{x^2+3} - x = (\sqrt{x^2+3}-x)\cdot\frac{(\sqrt{x^2+3}+x)}{(\sqrt{x^2+3}+x)}$

$$= \frac{\cancel{x^2}+3-\cancel{x^2}}{\sqrt{x^2+3}+x} = \frac{3}{\sqrt{x^2+3}+x}$$

$\lim\limits_{x\to\infty} 3 = 3$ & $\lim\limits_{x\to\infty} \sqrt{x^2+3}+x = \infty+\infty = \infty$

so $\lim\limits_{x\to\infty} \sqrt{x^2+3}-x = \frac{3}{\infty} = \boxed{0}$

(6) Vertical asymptotes:

these happen at $x=a$ when either (or both) $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

[Typically, one starts by determining where the denominator is 0. This will give you a set of possible vertical asymptotes: the numerator could also be zero in which case work has to be done to determine whether or not there's a vertical asymptote. The best way to ensure you don't get tricked is to actually compute $\lim_{x\to a^-} f(x)$ & $\lim_{x\to a^+} f(x)$. Also beware that other things can give you vertical asymptotes, such as $\ln(x)$ at $x=0$ & $\tan(x)$ at $\pi/2$ (and $-\pi/2, -3\pi/2, \ldots, 3\pi/2, 5\pi/2, \ldots$).]

Step 1: Check where the denominator $=0$:

if $x \le 0$, $f(x) = \dfrac{1+2x^2}{4-x^2} = \dfrac{1+2x^2}{(2-x)(2+x)}$ so denominator $=0$ at $x=-2$

[some of you wrote $x=+2$, but $x=2$ is far away from $x \le 0$, so it doesn't matter here.]

if $x>0$, $f(x) = \dfrac{x^2-4x+3}{x(x^2-3x+2)} = \dfrac{x^2-4x+3}{x(x-2)(x-1)}$ so denominator $=0$ at $x=2$, $x=1$, & $x=0$.

[Though $x=0$ is not in $x>0$, it is right next to it, so taking $\lim_{x\to 0^+} f(x)$ will "see" this $x=0$.]

Step 2: look for functions that have vertical asymptotes.

Here, there are no $\ln(x)$, nor $\tan(x)$, or anything else of the sort, so there's nothing to do in step 2.

Step 3: By taking limits, verify whether or not $x=-2, 0, 1, 2$ are vertical asymptotes.

$x=-2$: $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \dfrac{1+2x^2}{(2-x)(2+x)} = \dfrac{1+8}{0} = \dfrac{9}{0} = \infty$ (if $x<-2$, $(2-x)(2+x)>0$)

so $\boxed{x=-2 \text{ is a vertical asymptote}}$

(6)(cont'd)

$x=0$: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2-4x+3}{x(x^2-3x+2)} = \frac{3}{0} = \infty$ (if $x>0$, $x(x^2-3x+2)>0$)

so $\boxed{x=0 \text{ is a vertical asymptote}}$

$x=1$: $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-4x+3}{x(x^2-3x+2)} = \frac{0}{0} = ?$

so factor $= \lim_{x\to 1} \frac{\cancel{(x-1)}(x-3)}{x\cancel{(x-1)}(x-2)} = \lim_{x\to 1} \frac{(x-3)}{x(x-2)} = \frac{1-3}{1\cdot(1-2)} = 2 \neq \pm\infty$

so $x=1$ is not a vertical asymptote.

$x=2$: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{(x-1)(x-3)}{x(x-1)(x-2)} = \frac{-1}{0} = \infty$ (if $x<2$, $x(x-1)(x-2)<0$)

so $\boxed{x=0 \text{ is a vertical asymptote}}$

Horizontal asymptotes:

these happen at $y=b$ exactly when either $\lim_{x\to\infty} f(x)=b$ or $\lim_{x\to-\infty} f(x)=b$

- $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1+2x^2}{4-x^2} = \frac{\infty}{-\infty} = ?$ so factor out the highest power on the top & bottom.

  [because if x is "close" to $-\infty$ it is negative.]

$\lim_{x\to-\infty} \frac{1+2x^2}{4-x^2} = \lim_{x\to-\infty} \frac{\cancel{x^2}(1/x^2+2)}{\cancel{x^2}(4/x^2-1)} = \frac{0+2}{0-1} = -2$

so $\boxed{y=-2 \text{ is a horizontal asymptote.}}$

[Some people then checked $\lim_{x\to\infty} \frac{1+2x^2}{4-x^2}$, but since $f(x) \neq \frac{1+2x^2}{4-x^2}$ if $x>0$ this has nothing to do with $\lim_{x\to\infty} f(x)$, which is what we're interested in.]

- $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(x^2-4x+3)}{x(x^2-3x+2)} = \frac{\infty}{\infty} = ?$ so factor out highest power on top & bottom

$= \lim_{x\to\infty} \frac{x^2(1-\frac{4}{x}+3/x^2)}{x^3(1-3/x+2/x^2)} = \frac{1-0+0}{\infty\cdot(1-0+0)} = \frac{1}{\infty} = 0$ so $\boxed{y=0 \text{ is a horizontal asymptote}}$

(7)(a) $\boxed{\text{domain} = (-\infty, \infty)}$

e^x is continuous everywhere, so is $\cos(x)$.

Then, by the product law, $\boxed{e^x \cos(x) \text{ is continuous everywhere.}}$

(b) $\boxed{\text{domain} = (-\infty, \infty)}$

e^x & $\cos x$ are continuous everywhere. The composition of continuous functions is continuous, so $e^{\cos(x)}$ is continuous everywhere.

x^2 is continuous everywhere, so by the sum law $\boxed{e^{\cos x} + x^2 \text{ is cont. everywhere}}$

(c) $\boxed{\text{domain: } x \neq 4,\ x \neq -2}$

e^x is cont. everywhere & so is x^2-2x-8. The quotient law says $\frac{e^x}{x^2-2x-8}$ is continuous wherever $x^2-2x-8 \neq 0$, i.e. $\boxed{\text{everywhere on the domain.}}$

(d) domain of $\ln(x) > 0$. $x^2 > 0$ as long as $x \neq 0$. so domain of $\ln(x^2)$ is $x \neq 0$.
domain of 2^x is $(-\infty, \infty)$. So domain of $\frac{2^x}{\ln(x^2)}$ is $x \neq 0$ & $\ln(x^2) \neq 0$

$\ln(x^2) = 0$ when $x^2 = 1$, ie, when $x = \pm 1$. So $\boxed{\text{domain of } \frac{2^x}{\ln(x)} \text{ is } x \neq 0, \pm 1}$

$\ln(x)$ is cont on $x > 0$ & x^2 is cont. everywhere. Composition of two functions is cont. so $\ln(x)$ is cont. on its domain.

2^x is cont. everywhere, so quotient law says $\frac{2^x}{\ln(x^2)}$ is cont. where $\ln(x^2) \neq 0$ (i.e. $x \neq \pm 1$). So $\boxed{\frac{2^x}{\ln(x^2)} \text{ is continuous on its domain}}$

(e) For every x, the value $f(x)$ makes sense, so the $\boxed{\text{domain is } (-\infty, \infty)}$.

- If $x < -1$, the sum & quotient laws show that $1 + \frac{x^2+2x+1}{x+1}$ is continuous
- If $-1 < x < 1$, $f(x)$ is continuous because x^2 is.
- If $x > 1$, $\ln(x)$ is cont. so $f(x)$ is, too.

It remains to check the points where $f(x)$ is glued together, i.e. $x = \pm 1$

check $x=-1$: • $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 1+\frac{x^2+x+1}{x+1} = 1+\frac{0}{0} = ?$

so factor: $= \lim_{x\to -1^-} 1+\frac{(x+1)^{\cancel{2}}}{\cancel{x+1}} = \lim_{x\to -1^-} 1+x+1 = 1$

• $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x^2 = (-1)^2 = 1$

• $f(-1) = (-1)^2 = 1$ → all equal, so $f(x)$ is cont. at $x=-1$.

check $x=1$: • $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$

• $f(1) = \ln(1) = 0$ → not equal, so $f(x)$ is <u>not</u> cont. at $x=1$.

$\boxed{f(x) \text{ is cont. on } (-\infty,1)\cup(1,\infty)}$

[When $f(x)$ is defined "piecewise" as in This question, you'll always have to check for continuity at The points That are glued together using The <u>definition of</u> continuity (ie. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$).

You also have to check for other discontinuities as you usually would.]