Solutions to Midterm I – MA 142

Friday, Oct. 2, 2009

1. (a) Determine the reduced row echelon form of the matrix

$$A = \begin{pmatrix} 4 & -4 & 8 & 8 & -8 \\ 1 & 0 & -1 & 2 & 4 \\ -2 & 3 & -7 & -4 & 10 \\ 0 & 3 & -9 & 0 & 18 \end{pmatrix}.$$

Show your work.

Solution: Other steps are possible, but no other answer is possible. Use row reduction:

$$\begin{pmatrix} 4 & -4 & 8 & 8 & -8 \\ 1 & 0 & -1 & 2 & 4 \\ -2 & 3 & -7 & -4 & 10 \\ 0 & 3 & -9 & 0 & 18 \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & -1 & 2 & 4 \\ 4 & -4 & 8 & 8 & -8 \\ -2 & 3 & -7 & -4 & 10 \\ 0 & 3 & -9 & 0 & 18 \end{pmatrix}$$

$$\overset{R_2 \mapsto R_2 - 4R_1 \quad R_3 \mapsto R_3 + 2R_1}{\downarrow\downarrow}$$

$$\text{Ans: } \begin{pmatrix} 1 & 0 & -1 & 2 & 4 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{\substack{-\frac{1}{4}R_2 \leftrightarrow R_2 \\ \leftsquigarrow \\ R_3 + \frac{3}{4}R_2 \leftrightarrow R_3 \\ R_4 + \frac{3}{4}R_2 \leftrightarrow R_4}}{} \begin{pmatrix} 1 & 0 & -1 & 2 & 4 \\ 0 & -4 & 12 & 0 & -24 \\ 0 & 3 & -9 & 0 & 18 \\ 0 & 3 & -9 & 0 & 18 \end{pmatrix}$$

(b) Describe the set of solutions of the following system of linear equations:

$$\begin{cases} 4x_1 - 4x_2 + 8x_3 + 8x_4 = -8 \\ x_1 - x_3 + 2x_4 = 4 \\ -2x_1 + 3x_2 - 7x_3 - 4x_4 = 10 \\ 3x_2 - 9x_3 = 18. \end{cases}$$

Solution: Other forms of the answer are possible (by selecting different variables to be free and/or expressing the answer differently).

Write down the augmented matrix of the system and row reduce. The augmented matrix is the matrix A of part a), so you have already found the reduced row echelon form:

$$\left(\begin{array}{cccc:c} 1 & 0 & -1 & 2 & 4 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The third and forth columns have no pivot, so they correspond to free variables

$$x_3 = C_1, \quad x_4 = C_2.$$

Solving the equations given by the first two rows gives

$$x_1 = 4 + C_1 - 2C_2, \quad x_2 = 6 + 3C_1.$$

Answer: $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 0 \\ 0 \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(c) Is the set

$$\left\{ \begin{pmatrix} 4 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ -1 \\ -7 \\ -9 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \\ -4 \\ 0 \end{pmatrix} \right\}$$

linearly independent? Why or why not?

Solution: It turns out I slightly messed up on this question. The first and fourth vectors listed are multiples of each other, so you can immediately conclude that the set is linearly dependent. Only a few of you noticed this though. I meant for you to have to find the reduced row echelon form of the matrix, M, whose columns are given by these vectors and check if each column has a pivot. These vectors are the first four columns of the matrix A of part a), so in fact the reduced row echelon form of M is the first four columns of the reduced row echelon form of A. Since the third (and fourth) column does not have a pivot, this also shows that this set of vectors is linearly dependent.

2. Let

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 1 & -2 \\ 2 & 1 & 3 \end{pmatrix}.$$

For the following, compute the expression if it is defined, otherwise write "not defined".

(a) $A(B + C)$

(b) A^2

(c) B^2

(d) $A(B^{\mathrm{T}})$

Solution:

(a) $A(B+C) = \begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix} \left(\begin{pmatrix} 3 & 2 & 6 \\ 1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 1 & -2 \\ 2 & 1 & 3 \end{pmatrix} \right) = \begin{pmatrix} 5 & 7 & 10 \\ 12 & -4 & -2 \end{pmatrix}.$

(b) $A^2 = \begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -21 & 22 \end{pmatrix}$

(c) B^2 is not defined since the sizes do not match up.

(d) $A(B^{\mathrm{T}})$ is not defined: A has 2 columns, but B^{T}, which is the *transpose* of B, has 3 rows.

3. Let $S : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ be the linear transformation sending

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ to } \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$$

and let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be counterclockwise rotation about the origin by $\pi/6$ radians.

(a) Determine the standard matrix of S.

(b) Determine the standard matrix of T.

(c) Determine the standard matrix of the composition $S \circ T$.

(d) Is $S \circ T$ one-to-one?

Solution:

(a) $S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, so the standard matrix of S is

$$A_S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) The matrix of such a rotation through angle θ is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Here, this gives the standard matrix of T as

$$A_T = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

If you didn't remember the rotation matrix above, you could figure out what happens to the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ using trigonometry.

(c) The standard matrix of the composition of two linear transformations is the product of the standard matrices. Here this gives

$$A_S A_T = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Alternatively, you could have computed what happens to the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under the composition.

(d) A linear transformation is one-to-one if, and only, if, each column of its standard matrix has

a pivot. Doing one step of row reduction on the standard matrix of $S \circ T$ from part c) yields

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} \end{pmatrix}.$$

Thus, each column has a pivot (since $\frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} = \frac{2}{\sqrt{3}} \neq 0$), so $S \circ T$ is one-to-one.

4. Suppose $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ is a linear transformation such that

$$\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\ T\begin{pmatrix} 1 \\ 5 \end{pmatrix} &= \begin{pmatrix} -3 \\ 3 \\ 21 \end{pmatrix}. \end{aligned}$$

What is the standard matrix of T?

Solution: $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is given, so you only need to determine $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5}\left(\begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

and T is linear, you can compute

$$\begin{aligned} T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= T\left(\frac{1}{5}\left(\begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right) \\ &= \frac{1}{5}\left(T\begin{pmatrix} 1 \\ 5 \end{pmatrix} - T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ &= \frac{1}{5}\left(\begin{pmatrix} -3 \\ 3 \\ 21 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}. \end{aligned}$$

Thus, the standard matrix of T is

$$\begin{pmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{pmatrix}.$$

5. Let

$$C = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}.$$

(a) Find a non-zero 2×2 matrix, M, such that $MC = 0$.

(b) Find a non-zero matrix, N, of any size you wish, such that $CN = 0$.

Solution: Other solutions are possible.

(a) You could write

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and compute

$$MC = \begin{pmatrix} a - 3b & -3(a - 3b) \\ c - 3d & -3(c - 3d) \end{pmatrix}.$$

So, $MC = 0$ will be true whenever $a = 3b$ and $c = 3d$. An example would be

$$M = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b) Since you can choose the size of the matrix you want, you may as well choose a 2×1 matrix (i.e. a column vector in $\mathbf{R}^2$). Then, you just have to find a non-trivial solution to

$$C\mathbf{v} = 0$$

and take $N = \mathbf{v}$. An example of such an answer would be

$$N = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Alternatively, you could use the answer from part a), the properties of the transpose and the fact that, in this particular case, $C = C^{\mathrm{T}}$. The argument is as follows:

$$MC = 0$$

so

$$\begin{aligned} 0 &= (MC)^{\mathrm{T}} \\ &= C^{\mathrm{T}} M^{\mathrm{T}} \\ &= C M^{\mathrm{T}}, \end{aligned}$$

thus you can take $N = M^{\mathrm{T}}$.