Solutions to Midterm I – MA 142

Friday, Nov. 13, 2009

1. (a) Find a basis for the following subspace of $\mathbf{R}^4$: (7 points)

$$U = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ 2 \end{pmatrix} \right\}.$$

Solution: To solve this, define a matrix A whose columns are the vectors above

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 5 & -1 \\ 0 & 0 & 1 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix},$$

then

$$U = \text{Col}(A).$$

A basis of $\text{Col}(A)$ is given by the **pivot columns** of A. To find the pivots, do row reduction:

$$\begin{pmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 5 & -1 \\ 0 & 0 & 1 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \textcircled{1} & 2 & 2 & -1 \\ 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \textcircled{1} & 2 & 2 & -1 \\ 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the first, third, and fourth columns of A are pivot columns. Therefore, a basis of $U = \text{Col}(A)$ is given by

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ 2 \end{pmatrix} \right\}.}$$

(b) What is $\dim U$? (3 points)

Solution: The dimension of a non-zero subspace U is always the **number of elements in a basis** of U. Since there are three vectors in the answer of part a), we have that

$$\boxed{\dim U = 3.}$$

(Note: since $U = \text{Col}(A)$, the dimension of U is also the **rank** of A, so it is the **number or pivots** of A, i.e. 3)

2. Let

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ -6 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

(a) What is $\det(AB^3)$? (4 points)

Solution: The determinant of a product is the product of the determinants, so

$$\det(AB^3) = \det(A)\det(B^3) = \det(A)\det(B)^3.$$

Thus, we basically just need to compute $\det(A)$ and $\det(B)$. Since A is upper-triangular and B is lower-triangular, the determinant in each case can be computed as the **product of the diagonal entries**:

$$\det(A) = 5 \cdot 2 \cdot 3 = 30$$

and

$$\det(B) = 2 \cdot 1 \cdot 1 = 2.$$

Thus,

$$\boxed{\det(AB^3) = \det(A)\det(B)^3 = 30 \cdot 2^3 = 240.}$$

(Note: taking the product AB^3 before taking determinants would take much longer (and make it more likely to make a mistake))

(b) Is B invertible? If so, what is $\det(B^{-1}A)$? (4 points)

Solution: $\boxed{B \text{ is invertible because } \det(B) = 2 \text{ is } \textbf{non-zero}.}$

To answer the second part, remember that for any invertible matrix

$$\det(B^{-1}) = \frac{1}{\det(B)}.$$

Therefore

$$\boxed{\det(B^{-1}A) = \det(B^{-1})\det(A) = \tfrac{\det(A)}{\det(B)} = \tfrac{30}{2} = 15.}$$

(Note: computing B^{-1} and taking its determinant would take much longer (and make it more likely to make a mistake))

3. Let

$$A = \begin{pmatrix} 3 & -2 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Is A diagonalizable? If so, diagonalize it. (14 points)

Solution:

Step 1: Find the eigenvalues of A (with multiplicity). These are the roots of the **characteristic**

polynomial of A:

$$\mathrm{ch}_A(x) = \begin{vmatrix} 3-x & -2 & 2 \\ 1 & -x & 1 \\ 0 & 0 & 1-x \end{vmatrix}.$$

Do Laplace expansion along the third row:

$$\mathrm{ch}_A(x) = \begin{vmatrix} 3-x & -2 & 2 \\ 1 & -x & 1 \\ 0 & 0 & 1-x \end{vmatrix} = (1-x)\begin{vmatrix} 3-x & -2 \\ 1 & -x \end{vmatrix} = (1-x)(x^2-3x+2).$$

The roots are $x = 1$ and the roots of $x^2 - 3x + 2$. For the latter, use the quadratic formula:

$$\frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 2}}{2} = 2 \text{ or } 1.$$

Therefore, the eigenvalues of A are

$$\lambda_1 = 1 \text{ with multiplicity 2 and } \lambda_2 = 2 \text{ with multiplicity 1.}$$

(Note: if we have obtained three distinct eigenvalues, we could already conclude that the matrix A is diagonalizable. To diagonalize it, we would still have to go through the following steps. Since λ_1 has multiplicity > 1, we need to check that the dimension of its eigenspace is equal to its multiplicity (i.e. equal to 2) before being able to conclude that A is diagonalizable (or not))

Step 2: Find **3 linearly independent eigenvectors** of A (if possible); two corresponding to λ_1 and one corresponding to λ_2.
$\lambda_1 = 1$: what we want to do is **find a basis of the null space** of $A - \lambda_1 I_3$, so we augment this matrix by the zero vector and row reduce:

$$A - I_3 = \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} \textcircled{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We get two free variables $x_2 = C_1$ and $x_3 = C_2$. The variable x_1 satisfies

$$x_1 = x_2 - x_3$$

so a basis for $\mathrm{Nul}(A - \lambda_1 I_3) = \lambda_1$-eigenspace is

$$\left\{ \mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(Note: since the dimension of the λ_1-eigenspace is 2 (because a basis of it is given by the 2 vectors v_1 and v_2), and the dimension of the λ_2-eigenspace is necessarily 1, we may already conclude that A is diagonalizable. We still have to do the rest of the work to diagonalize it.)
$\lambda_2 = 2$: now, we want to **find a basis of the null space** of $A - \lambda_2 I_3$, so we augment this

matrix by the zero vector and row reduce:

$$A - 2I_3 = \left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} \textcircled{1} & -2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We get one free variable $x_2 = C$. Additionally,

$$x_3 = 0,$$

and

$$x_1 = 2x_2,$$

so a basis for $\mathrm{Nul}(A - \lambda_2 I_3) = \lambda_2$-eigenspace is

$$\left\{ \mathbf{v_3} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

We found three linearly independent eigvenvectors for A so

A is diagonalizable.

Step 3: Let

$$P = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Step 4: Let

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(b) Is A invertible? (3 points)

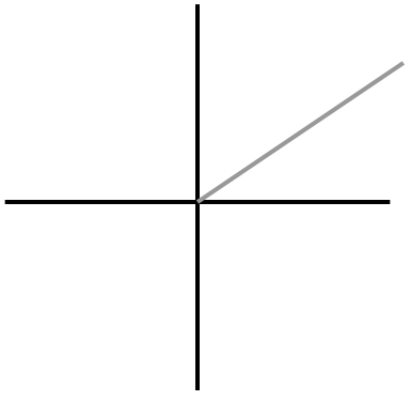
Solution: A is invertible because none of its eigenvalues are 0.

Alternatively, you could say that $\det(A) =$ **product of its eigenvalues (with multiplicity)**, so

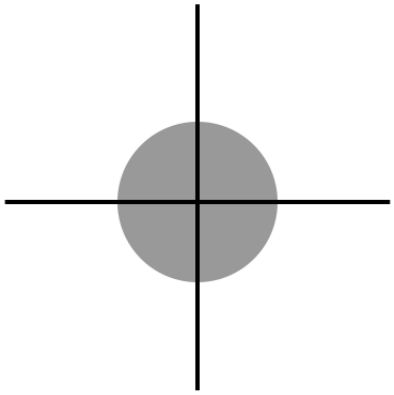
$$\det(A) = 1 \cdot 1 \cdot 2 = 2 \neq 0$$

so A is **invertible because its determinant is non-zero**.

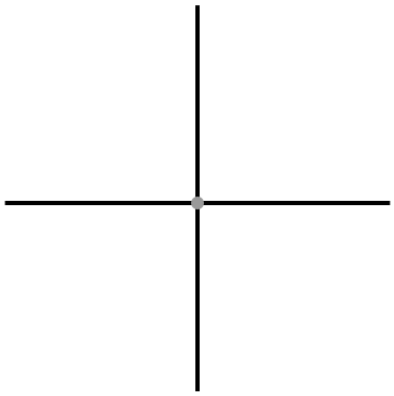
4. For each of the following four grey (■) regions, say if it is a subspace of $\mathbf{R}^2$ and explain your answer. The symbol • denotes a point. (2 points each)



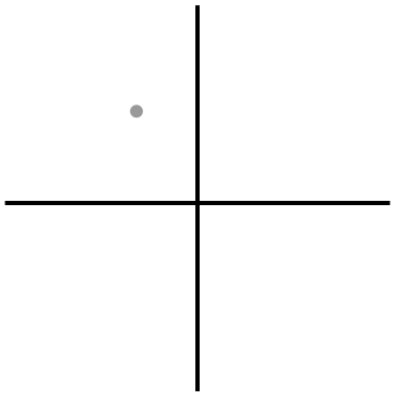
(a) No: for any $c < 0$, and any $\mathbf{u} \in U$, $c\mathbf{u} \notin U$. For example, the following is a sufficient answer: take $\mathbf{u} \in U$, then $-\mathbf{u} \notin U$.



(b) No: for any $\mathbf{u} \in U$, you can pick c large enough such that $c\mathbf{u} \notin U$ (e.g. for a $\mathbf{u}$ on the edge of the region, then $c = 1.1$ is big enough). Also, there are vectors in U that, when added together, are no longer in U.



(c) Yes: the origin is a subspace of $\mathbf{R}^2$.
i) $\mathbf{0} \in U$
ii) the only vector in U is $\mathbf{0}$, and $\mathbf{0} + \mathbf{0} = \mathbf{0} \in U$
iii) $c \cdot \mathbf{0} = \mathbf{0} \in U$.



(d) No: U doesn't satisfy any of the three conditions. Perhaps the simplest to prove is that $\mathbf{0}$ is not in U.

5. Let (10 points)

$$A = \begin{pmatrix} 1 & 3 & 2 \\ -2 & 0 & 8 \\ -1 & -3 & -1 \end{pmatrix}.$$

Is A invertible? If so, what is its inverse?

Solution: You could first compute the determinant to see if the matrix is invertible, or, since you're being asked to compute the inverse (if possible), you can put the identity matrix next to A and row reduce. The **matrix is invertible if, and only if,** $\mathrm{RREF}(A) = I_3$. The solution proceeds as follows:

$$(A \vdots I_3) = \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ -2 & 0 & 8 & 0 & 1 & 0 \\ -1 & -3 & -1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 6 & 12 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & 0 & -2 \\ 0 & 6 & 0 & -10 & 1 & -12 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\wr$$

$$(\mathrm{RREF(A)} \vdots A^{-1}) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -\frac{1}{2} & 4 \\ 0 & 1 & 0 & -\frac{5}{3} & \frac{1}{6} & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \leftsquigarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & -\frac{5}{3} & \frac{1}{6} & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right).$$

Thus, $\boxed{\mathrm{RREF(A)} = I_3, \text{ so } A \text{ is invertible}}$ and

$$\boxed{A^{-1} = \begin{pmatrix} 4 & -\frac{1}{2} & 4 \\ -\frac{5}{3} & \frac{1}{6} & -2 \\ 1 & 0 & 1 \end{pmatrix}.}$$

6. Let (6 points)

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 4 & -2 \\ -6 & 3 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 6 & -2 \\ -6 & 3 & 4 \\ 3 & 7 & -3 \end{pmatrix}.$$

Given that $\det(A) = 4$ and that B is obtained from A by doing the following three elementary row operations

(i) Replace row 2 with row 2 plus row 1 (i.e. $R_2 \rightsquigarrow R_2 + R_1$);

(ii) Switch row 2 and row 3 (i.e. $R_2 \leftrightarrow R_3$);

(iii) Scale row 1 by 2 (i.e. $R_1 \rightsquigarrow 2R_1$).

What is $\det(A^{-1}B^2)$?

Solution: Use the fact that **you know how the determinant changes when you do a row operation**. Operation (i) doesn't change the determinant. Operation (ii) multiplies the determinant

by -1. Operation (iii) multiplies the determinant by 2. Therefore,

$$\det(B) = -2\det(A) = -2 \cdot 4 = -8.$$

Since $\det(A) = 4 \neq 0$, A is invertible and

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{4}.$$

Thus,

$$\boxed{\det(A^{-1}B^2) = \tfrac{1}{\det(A)}\det(B)^2 = \tfrac{1}{4}(-8)^2 = \tfrac{64}{4} = 16.}$$

7. Let

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{pmatrix}.$$

(a) What are the eigenvalues of A (list them with their multiplicities)? (5 points)

Solution: The eigenvalues are the roots of the characteristic polynomial

$$\mathrm{ch}_A(x) = \begin{vmatrix} 3-x & 2 & 0 \\ 0 & -1-x & 0 \\ -2 & 3 & 1-x \end{vmatrix} = (1-x)\begin{vmatrix} 3-x & 2 \\ 0 & -1-x \end{vmatrix} = (1-x)(3-x)(-1-x).$$

(Laplace expansion along the third column was used to compute the first determinant). Thus, the eigenvalues are

$$\boxed{\lambda_1 = 1, \lambda_2 = 3, \text{ and } \lambda_3 = -1, \text{ each with multiplicity 1.}}$$

(b) Is A diagonalizable? (2 points)

Solution: $\boxed{A \text{ is diagonalizable since it has 3 distinct eigenvalues.}}$ (i.e. because each eigenvalue has multiplicity 1).

(c) Is A invertible? (2 points)

Solution: $\boxed{A \text{ is invertible since none of its eigenvalues are 0.}}$

Alternatively, you could say that $\det(A) =$ **product of its eigenvalues (with multiplicity)**, so

$$\det(A) = 1 \cdot 3 \cdot (-1) = -3 \neq 0$$

so A is **invertible because its determinant is non-zero**.

(d) What is the rank of A? (2 points)

Solution: $\boxed{A \text{ has rank 3 since it is invertible.}}$ (This is part of the invertible matrix theorem).