Assignment 2 – MA 542

Due in class: Wednesday, Jan. 27, 2010

Chapter 13: 6

Chapter 14: 4, 18, 29, 39, 45, 46

(1) Let $R = \mathbf{R}[x]$ and let $I = (x^2)$. Find a nilpotent element in R/I.

(2) Let R be a commutative ring with identity. An *idempotent* in R is an element e such that $e^2 = e$. Show that $e \in R$ is an idempotent if, and only if, there exists a $b \in R$ such that $eb = 0$ and $e + b = 1$.

(3) Let G be a finite group (written multiplicatively) whose identity we denote by e, and R a commutative ring with unity 1. The *group ring of G over R*, $R[G]$, is the following (generally non-commutative) ring with unity:

$$R[G] = \{\text{all functions } f : G \to R\}$$

with addition component-wise: $(f_1 + f_2)(g) = f_1(g) + f_2(g)$ and the following multiplication:

$$(f_1 f_2)(g) = \sum_{\substack{x,y \in G \\ xy = g}} f_1(x) f_2(y).$$

An alternate, and perhaps simpler way, to define this ring is as follows: $R[G]$ is the set of all formal linear combinations of elements in G with coefficients in R, i.e.

$$R[G] = \left\{ \sum_{g \in G} a_g \cdot g : a_g \in R \right\}$$

Let

$$a = \sum_{g \in G} a_g \cdot g \qquad \text{and} \qquad b = \sum_{g \in G} b_g \cdot g$$

be two such formal linear combinations, then addition is defined component-wise

$$a + b := \sum_{g \in G} (a_g + b_g) \cdot g$$

and multiplication is defined by first multiplying like polynomials, then using the

group structure of G, i.e.

$$ab = \left(\sum_{g\in G} a_g \cdot g\right)\left(\sum_{h\in G} b_h \cdot h\right) := \sum_{g\in G}\sum_{h\in G} a_g b_h \cdot (gh).$$

(The identification of these two definitions is via $f \mapsto \sum_{g\in G} f(g)\cdot g$).

(a) Show that the identity in $R[G]$ is the function $f : G \to R$ given by

$$f(g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise,} \end{cases}$$

(alternatively, this is the formal linear combination $1 \cdot e$).

(b) Define the following subset of $R[G]$,

$$I_G = \left\{ f : G \to R : \sum_{g\in G} f(g) = 0 \right\} = \left\{ \sum_{g\in G} a_g \cdot g : \sum_{g\in G} a_g = 0 \right\}.$$

Show that I_G is an ideal (it is called the *augmentation ideal*).

(c) Show that $I_G = (g-1)_{g\in G, g\neq e}$, i.e. I_G is the ideal generated by the formal linear combinations $(-1\cdot e + 1\cdot g)$ as g runs over the non-identity elements of G.