Solutions to Assignment 1 (MA542)

Chapter 12:

8) Show $\forall a,b \in R$, $ab = ba$

proof: wlog $a \neq 0$ (if $a=0$, $ab = 0 = ba$ $\forall b$)

$\forall b, \forall a \neq 0$, $aba = aba$

$a(ba) = (ab)a \Rightarrow ba = ab$ QED

20) let R commutative ring with identity & let $M_n(R)$ = $n \times n$ matrices with coefficients in R

Claim: $(M_n(R))^\times = \{ g \in M_n(R) : \det g \in R^\times \} =: GL_n(R)$

proof: ($\supseteq$): suppose $\det g \in R^\times$. Using the classical formula for $g^{-1} = \frac{1}{\det(g)} Ad(g)$ → classical adjoint (aka adjugate)

shows that $g^{-1} \in M_n(R)$ (since $\det(g)$ is invertible & $Ad(g) \in M_n(R)$)

($\subseteq$): Given $g \in M_n(R)$, $\det g \in R$. Given $g, h \in M_n(R)$, $\det(gh) = \det(g)\det(h)$

so if $gh = I$ ($n \times n$ identity matrix), then $\det(g)\det(h) = 1$ so $\det(g) \in R^\times$ QED

Hence, $(M_2(\mathbb{Z}))^\times = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \in \mathbb{Z}^\times = \{\pm 1\} \} =: GL_2(\mathbb{Z})$

23) Hint: use "absolute value" & complex conjugate: If $(a+bi)(c+di) = 1$, then $(a-bi)(c-di) = 1$

so $(a+bi)(c+di)(a-bi)(c-di) = 1$

$= (a^2+b^2)(c^2+d^2)$

so $a^2+b^2 \in \mathbb{Z}^\times = \{\pm 1\}$ & c^2+d^2

so $a,b,c,d \in \{\pm 1, 0\}$

Checking these possibilities yields $(\mathbb{Z}[i])^\times = \{\pm 1, \pm i\}$

24) ($\supseteq$): Suppose $x = (u_1, \dots, u_n) \in \bigoplus_{i=1}^n R_i$ is such that $u_i \in R_i^\times$. Then $\exists u_i^{-1} \in R_i$ s.t. $u_i u_i^{-1} = 1$

Then $x^{-1} = (u_1^{-1}, \dots, u_n^{-1}) \in \bigoplus_{i=1}^n R_i$ & $xx^{-1} = 1$

($\subseteq$): Suppose $x \in \bigoplus_{i=1}^n R_i$ & $\exists y \in \bigoplus_{i=1}^n R_i$ s.t. $xy = 1$. Write $x = (x_i)_{i=1}^n$, $y = (y_i)_{i=1}^n$

Then $xy = (x_i y_i)_{i=1}^n = (1, 1, \dots, 1)$ so $x_i y_i = 1$ $\forall i$ so $x_i \in R_i^\times$ $\forall i$ QED

26) $4 \cdot 2 = 8 \equiv 2 \pmod 6$, $3 \cdot 5 = 15 \equiv 7 \pmod 8$, $9 \cdot 3 = 27 \equiv 12 \pmod{15}$

Questions:

(1) ($\Rightarrow$): suppose $a \in R^\times$, you can take $b = 1$ or $b = a^{-1}$, in each case ab is a unit

($\Leftarrow$): if $ab = u \in R^\times$, then $abu^{-1} = 1$ so $a^{-1} = bu^{-1}$ so $a \in R^\times$ QED

(2) (a) Suppose $x^n = 0$ then $(1-x)(1+x+x^2+\dots+x^{n-1}) = 1 - x^n = 1$ so $(1-x)^{-1} = \sum_{i=0}^{n-1} x^i$

(b) $(u-x)(u^{n-1} + u^{n-2}x + \dots + ux^{n-2} + x^{n-1}) = u^n - x^n = u^n$ so $(u-x)^{-1} = u^{-n} \sum_{i=0}^{n-1} u^i x^{n-1-i}$

(3) Let $f,g,h \in S$ and let $x \in X$. Let's verify the axioms

(1) $(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$ because it is true in R

$\Rightarrow f+g = g+f$ (since the above is true $\forall x \in X$)

(2) $((f+g)+h)(x) = (f+g)(x)+h(x) = [f(x)+g(x)]+h(x) = f(x)+(g(x)+h(x))$

$= f(x)+(g+h)(x) = (f+(g+h))(x)$

true $\forall x \Rightarrow (f+g)+h = f+(g+h)$

(3) let $0: X \to R$ be $0(x)=0$ $\forall x \in X$. Then $(0+f)(x) = 0(x)+f(x) = 0+f(x) = f(x)$

true $\forall x \Rightarrow 0+f=f$

(4) let $-f: X \to R$ be $(-f)(x) = -f(x)$ for $x \in X$. Then

$(f+(-f))(x) = f(x)+(-f)(x) = f(x)+-f(x) = 0$

true $\forall x$ so $f+(-f)=0$

(5) $(f\cdot(g\cdot h))(x) = f(x)(g\cdot h)(x) = f(x)(g(x)h(x)) = (f(x)g(x))h(x)$

$= (f\cdot g)(x)h(x) = ((f\cdot g)\cdot h)(x)$

true $\forall x$ so $f\cdot(g\cdot h) = (f\cdot g)\cdot h$

(6) $(f\cdot(g+h))(x) = f(x)(g+h)(x) = f(x)(g(x)+h(x)) = f(x)g(x)+f(x)h(x)$

$= (f\cdot g)(x)+(f\cdot h)(x)$ true $\forall x \Rightarrow f\cdot(g+h) = f\cdot g + f\cdot h$

$((f+g)\cdot h)(x) = (f+g)(x)h(x) = (f(x)+g(x))h(x) = f(x)h(x)+g(x)h(x)$

$= (f\cdot h)(x)+(g\cdot h)(x)$ true $\forall x \Rightarrow (f+g)\cdot h = f\cdot h + g\cdot h$

so S is a ring

Note: . If R has 1, you can check that $1(x)=1$ $\forall x$ is an identity in S

. If R is commutative, you can check that S is as well.