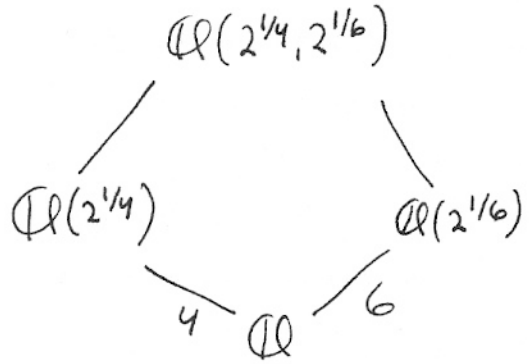
Solutions to Assignment 10 - MA542

Chapter 21:

12) Here's an example: let $F=\mathbb{Q}$, $a=2^{1/4}$, $b=2^{1/6}$. Then $[F(a):F]=4$ (since the minimal poly is $f(x)=x^4-2$) & $[F(b):F]=6$ (since the min. poly. over $\mathbb{Q}$ is $g(x)=x^6-2$)



Now, one always has

$[F(a,b):F]=[F(a,b):F(a)][F(a):F]$

so to determine $[\mathbb{Q}(2^{1/4},2^{1/6}):\mathbb{Q}]$ (degree of) we just need to determine the min. poly. of $2^{1/6}$ over $\mathbb{Q}(2^{1/4})$. Let $h(x)=x^3-2^{1/2}$.

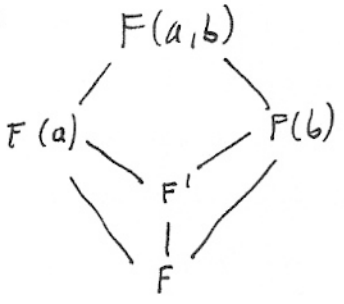
Since $2^{1/2}=(2^{1/4})^2$, $h(x)\in\mathbb{Q}(2^{1/4})[x]$.

$h(2^{1/6})=(2^{1/6})^3-2^{1/2}=2^{3/6}\cdot 2^{1/2}=2^{1/2}-2^{1/2}=0$. so the min. poly of $2^{1/6}$ over $\mathbb{Q}(2^{1/4})$ divides $h(x)$ (in $\mathbb{Q}(2^{1/4})[x]$). Hence $[\mathbb{Q}(2^{1/4},2^{1/6}):\mathbb{Q}(2^{1/4})]\le \deg h=3$

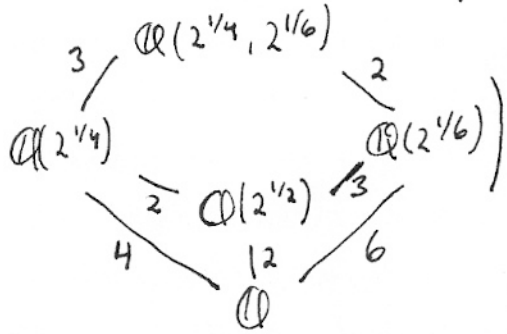
Thus, $[\mathbb{Q}(2^{1/4},2^{1/6}):\mathbb{Q}]\le 3\cdot[\mathbb{Q}(2^{1/4}):\mathbb{Q}]=12<[\mathbb{Q}(2^{1/4}):\mathbb{Q}][\mathbb{Q}(2^{1/6}):\mathbb{Q}]=24$.

(it is also necessary to show $F(a,b)\neq F(a), F(b)$: since $6\nmid 4$, $2^{1/6}\notin\mathbb{Q}(2^{1/4})$ so $F(a,b)\neq F(a)$, similarly, since $4\nmid 6$, $2^{1/4}\notin\mathbb{Q}(2^{1/6})$, so $F(a,b)\neq F(b)$).

More generally, here's the point: you have to pick a & b so that they are "related". (in the example I gave, $a^2=\sqrt{2}=b^3$). Field Theoretically, one can formalize the notion of "related" to mean that $F(a)$ & $F(b)$ have a common subfield F' that is strictly bigger than F (in the example above, $F'=\mathbb{Q}(\sqrt{2})$). One then has the following diagram:



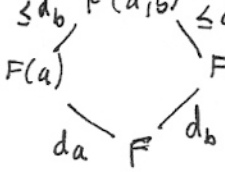
(in example above:



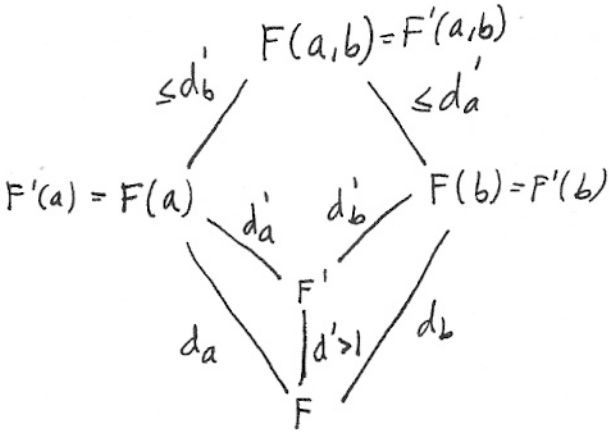
)

Why does this work? Well, given a, b, F, let $d_a=[F(a):F]$, $d_b=[F(b):F]$, Then $[F(a,b):F(b)]\le d_a$ & $[F(a,b):F(a)]\le d_b$ (because the min.poly. of a over $F(b)$ divides the min.poly. of a over F (in $F(b)[x]$) & the degree of the former is $[F(a,b):F(b)]$ while the degree of the latter is d_a. Similarly for b)

i.e. in a diagram, one always has inequalities like this:



12 cont'd) Thus, the existence of $F' \subseteq F(a)$, $F' \subseteq F(b)$ with $F' \supsetneq F$, one has the stronger inequalities of the following diagram:



So $[F(a,b):F(a)] \leq d_b' = \frac{d_b}{d'} < d_b$

Then,

$$[F(a,b):F] = [F(a,b):F(a)][F(a):F]$$
$$\leq d_b' d_a$$
$$< d_b d_a = [F(b):F][F(a):F].$$

14) Let $\alpha = \sqrt{-3} + \sqrt{2}$. Then $\alpha^2 = -3 + 2\sqrt{-6} + 2$
$= i\sqrt{3} + \sqrt{2}$

so $\alpha^2 + 1 = 2\sqrt{-6}$

so $(\alpha^2+1)^2 = 4 \cdot (-6)$

$\alpha^4 + 2\alpha^2 + 1$

so if you let $f(x) = x^4 + 2x^2 + 25 \in \mathbb{Q}[x]$

Then $f(\alpha) = 0$

Claim: $f(x)$ is the min. poly. of $\sqrt{-3} + \sqrt{2}$ over $\mathbb{Q}$

proof: we'll show that $[\mathbb{Q}(\sqrt{-3} + \sqrt{2}) : \mathbb{Q}] \geq 4$. (Then, deg of min poly of α is ≥ 4 so $f(x)$ is the min poly). We have:

$\mathbb{Q}(\sqrt{-3} + \sqrt{2})$

$|$

$\mathbb{Q}(\sqrt{2})$

$|2$

$\mathbb{Q}$

because $\sqrt{-6} = \frac{\alpha^2+1}{2} \in \mathbb{Q}(\alpha)$

so $\sqrt{-6}\alpha = i\sqrt{6}(i\sqrt{3} + \sqrt{2})$

$= -\sqrt{18} + i\sqrt{12} = -3\sqrt{2} + 2\sqrt{-3} \in \mathbb{Q}(\alpha)$

so $\frac{2\alpha - \sqrt{-6}\alpha}{5} = \frac{2\sqrt{-3} + 2\sqrt{2} + 3\sqrt{2} - 2\sqrt{-3}}{5}$

$= \sqrt{2} \in \mathbb{Q}(\alpha)$

We just need to show that $\sqrt{-3} \notin \mathbb{Q}(\sqrt{2})$ (Then $[\mathbb{Q}(\sqrt{-3} + \sqrt{2}) : \mathbb{Q}(\sqrt{2})] > 1$)

well, $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$, so $\forall \beta \in \mathbb{Q}(\sqrt{2})$, $\beta^2 \geq 0$. but $\sqrt{-3}^2 = -3 < 0$, so $\sqrt{-3} \notin \mathbb{Q}(\sqrt{2})$

so $[\mathbb{Q}(\sqrt{-3} + \sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{-3} + \sqrt{2}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] \geq 2 \cdot 2 = 4$ QED

16) Let $\alpha = \sqrt[3]{2} + \sqrt[3]{4}$. Then $\alpha^3 = 2 + 3(2^{1/3})^2 \cdot (4^{1/3}) + 3(2^{1/3})(4^{1/3})^2 + 4$

$= 6 + 3 \cdot (4^{2/3}) + 3 \cdot (2^{5/3}) = 6 + 3 \cdot 2 \cdot 2^{1/3} + 3 \cdot 2 \cdot 4^{1/3}$

$= 6 + 6(2^{1/3} + 4^{1/3}) = 6 + 6\alpha$

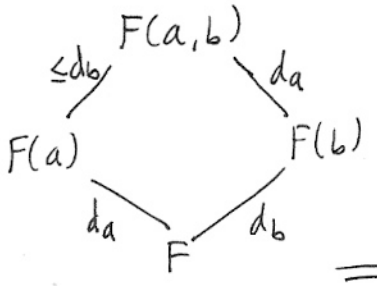
so let $f(x) = x^3 - 6x - 6 \in \mathbb{Q}[x]$. Then $f(\alpha) = 0$. $f(x)$ is irred. over $\mathbb{Q}$ by Eisenstein criterion with $p = 2$ (or even 3). So $f(x)$ is the min. poly of α over $\mathbb{Q}$.

22) See bottom of next page

32) Since $f(x)$ is an irred. poly in $F[x]$ s.t. $f(a)=0$, this implies $f(x)$ is the min poly of a over F. Similarly, $g(x)$ is the min. poly. of b over F.

Suppose $f(x)$ is irred over $F(b)$, then $f(x)$ is also the min. poly. of a over $F(b)$

Hence, $[F(a,b):F(b)] = \deg f(x)$. let $d_a := \deg f(x)$ & $d_b := \deg g(x)$. We have a diagram



. Thus, $[F(a,b):F] = d_a d_b$

so $[F(a,b):F(a)] = d_b$. So the min. poly of b over $F(a)$ has degree $d_b = \deg g(x)$.

$\Rightarrow$ $g(x)$ is the min. poly. of b over $F(a)$ (since this min. poly. must divide $g(x)$ & have the same degree). Being a min. poly. $\Rightarrow$ $g(x)$ is irred. over $F(a)$.

Reversing the roles of $g(x)$ & $f(x)$ implies the opposite implication QED.

34) since $2^{1/3} = (2^{1/6})^2$ & $2^{1/2} = (2^{1/6})^3$ one clearly has $\mathbb{Q}(2^{1/2}, 2^{1/3}) \subseteq \mathbb{Q}(2^{1/6})$

so show $\mathbb{Q}(2^{1/6}) \subseteq \mathbb{Q}(2^{1/2}, 2^{1/3})$: sufficient to show $2^{1/6} \in \mathbb{Q}(2^{1/2}, 2^{1/3})$

by closure under multiplication $\frac{2^{1/2}}{2^{1/3}} \in \mathbb{Q}(2^{1/2}, 2^{1/3})$. Now $\frac{2^{1/2}}{2^{1/3}} = 2^{1/2 - 1/3} = 2^{1/6}$ QED

35) Since $\sqrt{a} + \sqrt{b} \in \mathbb{Q}(\sqrt{a}, \sqrt{b})$ by closure under addition, one has that $\mathbb{Q}(\sqrt{a}+\sqrt{b}) \subseteq \mathbb{Q}(\sqrt{a}, \sqrt{b})$

so show $\mathbb{Q}(\sqrt{a}, \sqrt{b}) \subseteq \mathbb{Q}(\sqrt{a}+\sqrt{b})$: first show $\sqrt{a} \in \mathbb{Q}(\sqrt{a}+\sqrt{b})$

$(\sqrt{a}+\sqrt{b})^2 = a + b + 2\sqrt{ab}$ so $\sqrt{ab} = \frac{(\sqrt{a}+\sqrt{b})^2 - a - b}{2} \in \mathbb{Q}(\sqrt{a}+\sqrt{b})$

so $\sqrt{ab}(\sqrt{a}+\sqrt{b}) = a\sqrt{b} + b\sqrt{a} \in \mathbb{Q}(\sqrt{a}+\sqrt{b})$

so $a\sqrt{b} + b\sqrt{a} - a(\sqrt{a}+\sqrt{b}) = (b-a)\sqrt{a} \in \mathbb{Q}(\sqrt{a}+\sqrt{b})$

If $b \neq a$, this implies $\frac{(b-a)\sqrt{a}}{b-a} = \sqrt{a} \in \mathbb{Q}(\sqrt{a}+\sqrt{b})$. Then $(\sqrt{a}+\sqrt{b}) - \sqrt{a} = \sqrt{b} \in \mathbb{Q}(\sqrt{a}+\sqrt{b})$

so $\mathbb{Q}(\sqrt{a}, \sqrt{b}) \subseteq \mathbb{Q}(\sqrt{a}+\sqrt{b})$

Of course if $b = a$, then $\mathbb{Q}(\sqrt{a}, \sqrt{b}) = \mathbb{Q}(\sqrt{a}) \subseteq \mathbb{Q}(\sqrt{a}+\sqrt{b}) = \mathbb{Q}(2\sqrt{a})$

so, in both cases, $\mathbb{Q}(\sqrt{a}+\sqrt{b}) = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ QED

22) By assumption, $\exists\, g(x) \in F[x]$ s.t. $g(f(a)) = 0$. Let $h(x) = g(f(x))$. Clearly, $h(x) \in F[x]$.

$h(a) = g(f(a)) = 0$ so a is alg. over F QED

Questions:

Solutions to Asst 10 - MA542

(1) (a) $\alpha = e^{2\pi i/3}$, $F = \mathbb{Q}$: $\alpha^3 = 1$ so α satisfies $x^3 - 1$. but this is reducible (x=1 is a root)

$\frac{x^3-1}{x-1} = x^2+x+1$. claim: $\boxed{f(x) = x^2+x+1}$ is the min poly of α over $\mathbb{Q}$

proof: if not $\exists g(x) = x + b \in \mathbb{Q}[x]$ s.t. $g(\alpha) = 0$

but then $\alpha = -b \in \mathbb{Q}$. contradiction. QED

(b) $\alpha = i\sqrt{2}$, $F = \mathbb{Q}(\sqrt{2})$: claim: $\boxed{f(x) = x^2 + 2}$ is the min poly of α over $\mathbb{Q}(\sqrt{2})$

proof: $f(i\sqrt{2}) = -2 + 2 = 0$

if $f(x)$ is not the min. poly. $\exists g(x) = x + b \in \mathbb{Q}(\sqrt{2})[x]$

s.t. $g(i\sqrt{2}) = 0$, then $i\sqrt{2} = -b \in \mathbb{Q}(\sqrt{2})$

but $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$ & $i\sqrt{2} \notin \mathbb{R}$. QED.

(c) $\alpha = i\sqrt{2}$, $F = \mathbb{Q}$: claim: $f(x) = x^2 + 2$ is the min. poly. of α over $\mathbb{Q}$

proof: same as in part (b) (basically) QED

(d) $(a+1)^2 = a^2 + 1$, $(a+1)^3 = (a+1)(a^2+1) = a^3 + a^2 + a + 1 = a^2$ (since $a^3 = a+1$)

so $(a+1)^3 + (a+1)^2 + 1 = 0$. claim: $\boxed{f(x) = x^3 + x^2 + 1}$ is the min. poly of $\alpha = a+1$ over $\mathbb{Z}/2\mathbb{Z}$.

proof: suffices to show $f(x)$ is irred over $\mathbb{Z}/2\mathbb{Z}$. It has no roots & is of deg ≤ 3 so it's irred over $\mathbb{Z}/2\mathbb{Z}$. QED

④