Solutions to Asst. 12 - MA542

Questions:

1a) Remember that every field of characteristic 0 contains $\mathbb{Q}$. If $\varphi \in \mathrm{Aut}(F)$, then $\varphi(1)=1$ (because $\varphi(1)=\varphi(aa^{-1})=\varphi(a)\varphi(a)^{-1}=1, \forall a \in F^\times$).

let $m \in \mathbb{Z} \subseteq F$, then $\varphi(m)=\varphi(\overbrace{1+1+\cdots+1}^{m \text{ times}}) = \overbrace{\varphi(1)+\varphi(1)+\cdots+\varphi(1)}^{m \text{ times}} = 1+\cdots+1 = m$

so $\forall \frac{m}{n} \in \mathbb{Q}$, $\varphi(\frac{m}{n}) = \frac{\varphi(m)}{\varphi(n)} = \frac{m}{n}$ QED

b) Remember that every field of characteristic $p>0$ contains $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

Again, $\varphi \in \mathrm{Aut}(F) \Rightarrow \varphi(1)=1$. Every element of $\mathbb{F}_p$ is $\underbrace{1+1+\cdots+1}_{m \text{ times}}$ for some $m \in \mathbb{Z}$

& $\varphi(1+1+\cdots+1) = \varphi(1)+\varphi(1)+\cdots+\varphi(1) = 1+1+\cdots+1$ QED

2) The min. poly. of $a=\sqrt[4]{2}$ over $\mathbb{Q}$ is $f(x)=x^4-2$ (irred. by E.B. w/ $p=2$)

If $\varphi \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[4]{2}))$, then $\varphi(\sqrt[4]{2})$ must be another root of $f(x)$ in $\mathbb{Q}(\sqrt[4]{2})$.

The roots of $f(x)$ (in $\overline{\mathbb{Q}}$) are $\sqrt[4]{2}, -\sqrt[4]{2}, i\sqrt[4]{2}$, & $-i\sqrt[4]{2}$

Since $\mathbb{Q}(\sqrt[4]{2}) \subseteq \mathbb{R}$, only the first two roots are in $\mathbb{Q}(\sqrt[4]{2})$

so $\varphi(\sqrt[4]{2}) = \pm\sqrt[4]{2}$.

Since $\{1, \sqrt[4]{2}, \underbrace{\sqrt{2}}_{(\sqrt[4]{2})^2}, \underbrace{\sqrt[4]{2^3}}_{(\sqrt[4]{2})^3}\}$ is a basis of $\mathbb{Q}(\sqrt[4]{2})$ over $\mathbb{Q}$, any $\varphi \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[4]{2}))$ is determined by $\varphi(\sqrt[4]{2})$ (indeed, every element $\beta \in \mathbb{Q}(\sqrt[4]{2})$ can be written uniquely as $\beta = a_0 + a_1\sqrt[4]{2} + a_2(\sqrt[4]{2})^2 + a_3(\sqrt[4]{2})^3$ with $a_i \in \mathbb{Q}$, then $\varphi(\beta) = a_0 + a_1\varphi(\sqrt[4]{2}) + a_2\varphi(\sqrt[4]{2})^2 + a_3\varphi(\sqrt[4]{2})^3$)

so there are at most two elements in $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[4]{2}))$, one sending $\sqrt[4]{2} \mapsto \sqrt[4]{2}$ & one sending $\sqrt[4]{2} \mapsto -\sqrt[4]{2}$. The first one is the identity map. It can be checked that the second one is also an automorphism. So $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[4]{2})) \cong \mathbb{Z}_2$.

(In particular, $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not Galois since $[\mathbb{Q}(\sqrt[4]{2}):\mathbb{Q}] = 4 > \#\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[4]{2})) = 2$.

This is because it does not contain all roots of $f(x)$, i.e. it is not a normal ext'n)